The Various Uses of Graphs [and Discussion]
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Cambridge Joint Board Examinations. I think the Committee will naturally press that upon the Joint Board, and I should think they would consider it very favourably. It is a little unfortunate, I think, that, although some examining bodies have recognised the principle, they have not chosen the particular propositions that we should like them to have chosen—they have not chosen the propositions, or at least not all the propositions, that the Committee recommended, and they have put on the list some that we did not ask for; but still it is their acknowledgment of the principle which may be of great use in the future, and we hope that they will come to recognise which are the ones which we want to be on that list.

If there are no other proposals about this Report, then I shall put it to the meeting that the Report be approved. (Carried.)

THE VARIOUS USES OF GRAPHS.

It cannot be said too often that each portion of a mathematical curriculum must be chosen with some definite purposes in mind, and that these must be the controlling influence in the scheme of development. Such purposes may be roughly divided into two classes: the first includes all desires to illustrate or elucidate general ideas and principles, and the second includes all desires to relate one part of the subject to another, or to some other domain of thought. It is not here implied that possible fulfilment of some such purposes enforces the inclusion of any particular branch of mathematics; its difficulty may be too great for the students in question, and in any case only a limited amount can be included in a given time-table. But it is very definitely intended that no branch shall be included merely because it happens to be sufficiently easy or to tickle the fancy. There is ample real and human interest to be found without sacrifice of purpose; and while undue difficulty is to be deprecated, it must always be remembered that definite achievement resulting from continued effort is of the very essence of the learning of mathematics.

The object of this paper is to consider the purposes which may be served by the inclusion of graphs in the curriculum, and in particular to show that they fall into each of our two classes; that is, some of them are concerned with the presentation of general ideas and principles, while others are concerned with the interrelation of various subjects. The uses of graphs extend far beyond the approximate solution of equations, finding of maxima and minima, and the like. Properly developed, they provide the best introduction to the ideas of functionality and elementary analysis; and further, they show the relation of algebra to these ideas in a manner peculiarly clear and convincing.

When one looks at a graph of any particular phenomenon for the first time, what is the first consideration which enters the mind? Despite the laborious efforts of examiners and writers of books, a man who understands what a graph is does not first look at the details of axes and scales. He places it right way up—I admit that this involves a momentary consideration of the directions of the axes—and examines its general shape, thus gaining at one glance an outline of the whole sequence of events which is being considered. The graph is not the only way of gaining this information, any more than the formula $\frac{4}{3}\pi r^3$ is the only way of learning how to find the volume of a sphere; but it is the most vivid and enlightening method, just as the formula is more vivid and enlightening than an involved verbal statement of arithmetical processes. We may say of a certain plant that it grew slowly at first, then more rapidly, then more slowly, and finally reached a maximum height, but the appropriate curve gives this information more rapidly and effectively.

Hence the first use of a graph is to give information by its general shape, and so the first aim in teaching should be to enable the pupils to say what various shapes indicate about given familiar phenomena, and to sketch shapes corresponding to events whose sequence is described. For example, they should be able to interpret a graph as illustrating the changes in weight of some individual, and to sketch an illustrative graph when they are told that a man had an accident necessitating an immediate amputation, following which he had a long illness, and recovered completely. No axes should be marked, and no numerical idea need be involved.

Although it is not of great difficulty, this discussion of shape is of the utmost importance, and deserves full and careful treatment. Apart from a vital application in the drawing of graphs, to be mentioned later, it forms the true origin for the concept of a function, for the sequence of changes is exhibited as one whole and appears as a distinct entity, namely, a function. Of course, in the illustrations which are best suited for young pupils, the function idea is not exhibited with great precision; a boy's height is a function of a good many other variables besides his age. But the essentials are clearly set forth, and that is sufficient. There is a well-known remark of Klein, to the effect that it is unwise, and even impossible, to be entirely rigorous in teaching the young. Like most sayings of equal truth and authority, this can be quoted to justify all manner of processes which are indefensible under any circumstances; but we are certainly on safe ground in quoting it to support the development of the function concept from familiar phenomena, without any meticulous examination of detail in the early stages.

We may regard graphical expression as a language, just as we regard algebraic expression as a language. An individual graph is a sentence in that language, just as an individual formula is a sentence in algebraic language, and this graphical sentence is a descriptive account of the properties of some function. Just as we now begin to teach a foreign language by developing some power of expression before examining the niceties of accidence and syntax, so should we teach graphical language by developing an understanding of the shapes of graphs before examining the numerical niceties of axes, scales, and the like. This understanding of shape implies an ability to perceive increase and decrease, maxima and minima, discontinuities and the like, and to distinguish between greater and lesser rates of change by regarding the steepness of the curve. No numerical estimates are involved; the work is essentially qualitative, not quantitative. To lead up to the qualitative idea that steepness corresponds to rate of growth by involved calculations of gradients is very much akin to proving that two sides of a triangle are greater than the third side by the assumption of a result which is no more obvious than this. It is from the spontaneous connection of steepness with rate that the measurement of rate by means of gradient should be developed.

Here I must digress for a moment to refer to the standard directions of the axes. Judging from the experience of most examiners, there appear to be many teachers who regard deviation from the accepted practice in the same light as the use of varying letters in the figures for geometrical propositions. Now this idea is wholly erroneous. If anyone chooses to state his intention, he can perfectly well write the number consisting of 3 tens and 5 units as 53, or $\substack{3\\5}$, or $\substack{5\\3}$, and so for other numbers; and if his statement is remembered, his calculations will be intelligible. But the essence of arithmetical convenience is that we all choose the same plan; and so also with the essence of graphical convenience. There is no essential virtue in the method actually adopted in either case; to think that there is, is to make the mistake of the old lady who praised Noah for giving all the animals their right names. But there is a very essential virtue in the universal use of the same method, for it is the only way of enabling one individual to gain information from the work of another, or to review his own past performances, with any degree of facility.

All qualitative information having been obtained from the shape of a graph, the next step is to examine its quantitative properties in order to obtain any numerical information which may be desired. Without touching on consideration of gradients or areas, which are beyond the scope of quite elementary work such as is under discussion, the graph can be used to obtain the following information concerning the magnitude considered:

When it has a given value.

When it is greater or less than a given value.

When it lies between two given values.

The range of values included during the period considered.

When it is greatest and least, and the corresponding values.

When it changes from increasing to decreasing, and *vice versa*.

When it is growing more and more rapidly, and when more and more slowly.

The time taken to increase or decrease by a specified amount.

Unless all this information can be obtained from a graph, its quantitative uses remain unrealised, and most of its stimulus to thought and investigation is lost. This is largely the case at present. The excessive and absurd concentration on the one matter of finding when the magnitude has a given value has obscured all other considerations, many of them far more important, and most students have little idea of the range of expression of this very simple language.

In what has already been said there is, of course, no allusion to graphs of algebraic expressions; we have been only concerned with curves illustrating definite and familiar phenomena. And further, only the uses of the curves have been considered; the methods of construction remain to be explained.

The problem of drawing a graph, however it may arise, is always presented in the same form. Information is given concerning a number of discrete points, and the curve is then drawn as accurately as may be possible. If the information arises from statistics concerning some phenomenon, it may be that no more can be obtained; but if it is obtained from an algebraic expression its amount is unlimited, for we can compute as many values of this expression as may be desired.

The various points having been plotted, it remains to draw the curve, and it is here that the preliminary consideration of shape assumes vital importance. Its bearing on the completion of the curve is best illustrated by an example.

Suppose that the depth of water at the end of a pier is observed at noon on each day for a fortnight, the first being taken at low tide, and the results indicated on a graph. A set of 14 points will be obtained which suggest a curve in the shape of an arch. The unthinking student will promptly complete this curve, and imagine that he has a graph showing the depth of water at any time during the fortnight; but in this he is entirely wrong. The tide rises and falls twice in a period of rather more than 24 hours, and the true graph is a series of undulations which pass through the marked points in approximately this manner.

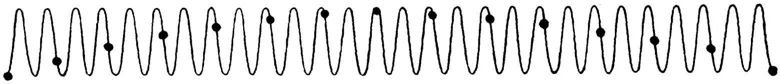

FIG. 1.

Here we see fully exposed the fatuous nature of the customary direction to "plot the points and sketch a smooth curve through them." If followed, it leads to hopeless error. The only scientific method is to sketch the curve through the points, *having regard to what is known concerning its general shape.*

Of course, in many cases it is impossible to predict any general shape, and then this direction must be followed as a counsel of despair, provided that the points do suggest any definite curve. But this should be done with an admission of large assumption, and attention should at once be directed to obtaining further information concerning intermediate points. At the same time, information of a general character is often a considerable guide to shape. If four census returns of the population of a town give four points situated as in Fig. 2, a certain curve is suggested; but if it is also known that there was a sudden influx of population between the second and third return, the curve must be drawn as in Fig. 3.

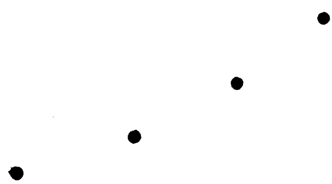

FIG. 2.

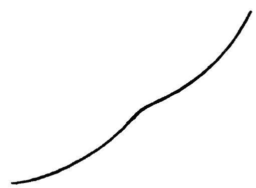

FIG. 3.

It is mental exercises of this kind which produce a man who can think mathematically concerning phenomena, rather than one who can merely perform, without any understanding, certain comparatively trivial devices of the engineer. The automatic drawing of smooth curves for little or no obvious purpose except to find some special value is very nearly as deadening as the most formal type of algebraic exercise; and it is far more dangerous, for it induces a feeling of competence and reality which has little justification.

It will now be seen that graph drawing of this type does really induce the idea of functionality; the general shape of the curve indicates the general nature of the function, and its numerical details fix with more or less precision the numerical values of the function for different values of the variable. It is in this connection that the term *function* and the functional notation should be introduced, no unnecessary formality of definition being required. The pupil should be able to say whether one number is or is not a function of another; for example, that the number of hairs in a man's head is a function of his age, but is not a function of the number of hairs in his wife's muff. And further, when the details of a particular function are specified by means of a graph, easy equations should be solved, for example,

$$f(x)+3=f(5), \quad f(10)=xf(5), \quad f(10)=3f(x).$$

Such work gives a reasonable amount of precision and completeness to ideas which have now been formed.

Of the construction of graphs of algebraic expressions little need be said here. But it may be suggested, that given a familiarity with graphs and their shapes such as has been described, some convincing proof of the shapes of simple algebraic graphs is not so hard as might be imagined. In the case of expression of the first degree, for example, $3x-4$, we have

$$\{3(x+1)-4\}-\{3x-4\}=3,$$

and hence the rise in the graph for a unit step to the right is the same *wherever that step is taken.* Few will then doubt that a straight line is the only possibility.

For a quadratic function, say x^2-3x+5, we proceed similarly. The calculation

$$\{(x+1)^2-3(x+1)+5\}-\{x^2-3x+5\}=2x-2=2(x-1)$$

shows that the rise for a unit step to the right increases steadily as the step commences further and further to the right of $x=1$; the curve must therefore ascend more and more rapidly, without any undulations.

I touch on the graphs of algebraic expressions thus lightly, because the real crux of the use of graphs in mathematics lies in the relation of these expressions and their graphs to the functional ideas already evolved, and the discussion of this relation is the real purport of this paper.

It cannot be said too often, or with too much emphasis, that the concept of a function is something quite different from, and much larger than, the concept of its algebraic expression. Important as is this consideration to the pure mathematician, it is of even more importance to the teacher of elementary mathematics in schools, for the functional relations which are the concern of most adults—I do not mean vocational concern—are seldom such as can be expressed in algebraic terms. Population, mortality, expectation of life, the value of money—none of these can be summed up in a formula, and it is with such types that the ordinary adult is really concerned.

And yet it appears to be often thought—and almost invariably taught—that *function* and *algebraic expression* are equivalent terms. In almost every text-book we find $f(x)$ merely standing for some algebraic type, the general idea of dependence being entirely ignored. The pupil may well think, as he often does think, that function is another word for formula or expression, that the whole matter is something which is dragged into algebra without apparent purpose, and rather a nuisance at that.

To enforce the distinction between function and formula, let me give one illustration. The weight of a body is a definite function of its distance from the centre of the earth; but this one function is represented by different formulae according as the point is above or below the surface of the earth. If above, the formulae is A/x^2, and if below, Bx, x being the distance from the centre of the earth.

If we say that there are two functions giving the weight, one for points inside the earth and the other for points outside, we are merely adding another word to the two, expression and formula, which already exist for one entity. The law of dependence defined by variation of weight with distance from centre is something quite different from either. It is a law, not the expression of a law; and this it is which is denoted by the term function.

The essence of algebraic expressions is that they form the ideal method of functional statement. The graph forms the first method; its want of precision is compensated by the vivid and terse manner in which it conveys the idea of dependence, and the nature of any particular dependence. But these having been acquired, no one doubts that the algebraic expression of dependence is the final type, alike in exactness, convenience, and extent. A formula is more precise and handy than any graph, and it reaches much farther.

This use of algebra is easily exemplified by simple illustrations, such as the dependence of the area of a circle on its radius. The area should have already been considered as a function of the radius, and the general nature of this function is therefore familiar; for example, since successive increases of the radius by one inch give larger and larger increases in the area, the graph is a curve which becomes steeper and steeper as we pass to the right; it can therefore be sketched from a few special values obtained by measurement of definite circles. But as soon as the formula for the area is obtained, any ordinary pupil can see that it provides the ideal expression of the dependence of area on radius. It is as much superior to the graph as the graph is itself superior to a table of values for specified radii.

Illustrations in which different formulae are necessary for different values of the variable, as in the case already mentioned of the weight of a body, should not be omitted. They emphasise the distinction between a function and its algebraic expression, a distinction which is essential to any rational comprehension of a function as an entity in itself.

The student is now in possession of two means of expressing functional dependence, graphical and algebraic. The first is illuminating as regards general nature, and fairly accurate when sufficient data are known; the second is ideal in its precision and compactness. It may be described as the coping stone of the first. But only certain very special functions are capable of this ideal expression; the functions of everyday life cannot be stated precisely by formulae. Nevertheless, formulae can be constructed to represent any function with an accuracy which is only limited by our detailed knowledge of individual values of the function, so that there is no such severance between functions which have an exact algebraic expression and those which have not, as might be supposed.

The method of adjusting formulae to given statistics is of course well known, and it is only necessary to mention one case. Suppose that a certain

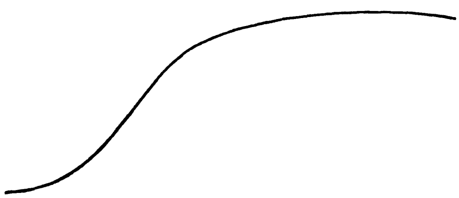

FIG. 4.

graph is known to have the shape in the figure. Such shapes are given by the expressions

$$y=\frac{bn}{(m+n)t^m}x^n, \quad 0<x<t,$$

$$y=b-\frac{bm}{m+n}x^{-n}, \quad x>t,$$

$x=t$ being the point of inflexion, that is, in words suitable for pupils, the point where the curve first becomes less steep. The constants m, n, t, b can be adjusted to suit the observations in question by solving very easy equations, and it will be found in practice that the formulae give satisfactory results if the order of accuracy of the given data is not very high.

All other simple types can be dealt with on similar lines. The work provides excellent drill in easy algebra of a useful type, and the choice of appropriate formulae enforces valuable forms of thought.

One particular case deserves further mention, in that on no account should it be omitted. The student should draw in one figure the graphs of $\sin x$, x, $x-\frac{x^3}{6}$, $x-\frac{x^3}{6}+\frac{x^5}{120}$, ... and realise how each successive term in the formulae draws the graph round into another wave, the process being obviously capable of indefinite extension; and similar work should be done for $\cos x$. The results can be summed up thus:

The formula x gives the function $\sin x$ to 2 decimal places from $x=0$ to $x=$

" $x-\frac{x^3}{6}$ " " " " "

and so on. There are few processes which give more insight into the inter-relation of functions and formulae than this, and the idea of obtaining as many waves as may be desired, to any accuracy that may be desired, is also most valuable.

It may be asked whether a student should be given such formulae without any clue to their origin. In this case there is ample justification, for their use and meaning are clear, and the statement that they are obtained by the aid of other parts of mathematics, which he may hope to learn himself, gives an outlook for the future. Many boys, and not only the most able, are impressed by such a sense of vista.

It will be seen that I have suggested that the whole function concept should be evolved from general, and not algebraic, considerations, algebraic expression being regarded as the ideal method of expressing functional relations, not as the normal way of conceiving a function. The same applies to the processes of the Calculus. These should be developed, and their notation explained, on similar lines, the algebraic processes appearing as the ideal form of statement for ideas already familiar. It is one more illustration of the fact that the proper methods of developing pure mathematics in abstract form, if only they are expressed through familiar entities and ideas, are also the proper methods of developing mathematical ideas in education.

Many advances have been made in teaching practice during the last fifteen years, and mathematical education has, to some extent, benefited in consequence. But potential gain in all branches has been in large part sacrificed by lamentable want of any considered theory, and not least so in the treatment of elementary analysis. We are often told that the idea of functionality and the processes of the Calculus can be amply illustrated by study of very simple algebraic expressions such as x^n. Now I will venture to say, and I have endeavoured to show it in this paper, that this view is based on a most vicious misconception of the true meanings of functional dependence and algebraic expression; that it is mistaken educationally is well known to many who have experience of its results. The genesis of the error is easily perceived. The old method of development from the abstract, typified for example in the text-books of Todhunter, has not really been abandoned. It is still seen in this attempt to develop functional ideas on one simple abstract function, namely, x^n. These ideas should be evolved from concrete illustrations, and then should find their abstract expression in algebraic formulae. To commence with formulae, and illustrate them by concrete examples, is still to put the cart before the horse, and this is what we are now doing.

Moreover, it is at least doubtful whether a boy gains any real power of thinking about rates and total effects by learning to differentiate and integrate x^2 and apply the processes to simple problems. To learn to think about such processes he must evolve them from familiar concrete illustrations of all kinds; in this way only can he become familiar with them. And when they are familiar tools, which can be applied to any matter in hand with some confidence, their algebraic expression may be developed as the apex of the whole structure. But if this is done, it must be done with some thoroughness. It is mockery to say that a boy can realise the practical power and abstract beauty of algebraic expression as applied to functional processes by one very simple case. We might as well say that he can appreciate the structure and beauty of Latin by learning to ask for his bare needs in that language. Familiarity with functional ideas must be acquired by ample exercise in the study of definite functions, which are defined by familiar concrete relations; and familiarity with their algebraic expression must be acquired by ample exercise in that expression; no other course is possible.

I have spoken somewhat strongly because I feel, as more and more teachers are coming to feel, that we are now drifting on to the rocks for want of any definite policy. In the old days, some boys did learn to do something, and no one knows how largely the number might have been increased by more efficient teaching on similar lines. But that efficiency has been destroyed —as was well said recently in the *Times*, one defect in modern systems of education is the substitution of the teacher's skill for the pupil's labour in self-training—and nothing stands in its place. What is needed is, first, a power of mathematical thought about common things developed in all directions as far as possible, and secondly, some real knowledge of and control over the abstract methods employed by mathematicians to express that thought;

and the two can only be acquired in this order. At present, we reverse the order, reduce the second to mere triviality, and so ruin the first, and the result is well known to many. Boys cannot do the simplest piece of algebra with any efficiency, and the gain in intelligent thought is quite insufficient to form any compensation. It rests with the mass of teachers to make up their minds what mathematics really is, and why and how it should be taught; and the need is urgent. G. St. L. Carson.

DISCUSSION.

Mr. Daniell: I should like to ask Mr. Carson if, in using the term "functional dependence," he had in his mind any idea of causality? It seems to me that the graph expresses correlation only.

In general, I would support what Mr. Carson has said, particularly about the study of the aspect of the graph. It often falls to my lot to read examination answers dealing with the subject of graphs. It does not much matter whether candidates have been told to give the form and name of the graph that they have drawn; they volunteer the information, and very cheering it is sometimes. On the last occasion on which I asked some candidates to plot a graph from measurements, they informed me that the result was a straight line, a circle, an ellipse, and a hyperbola, the curve being actually a roulette of sorts. This description was not a function of the data provided by the question, but of the particular lessons they had recently had. It is well known that quite eminent discoverers have made the mistake of smoothing their results too much in order to get a curve, a mistake which was so powerfully illustrated in the example given by Mr. Carson.

Mr. A. Lodge: In continuation of Mr. Daniell's remarks, I should like to ask Mr. Carson whether he would not feel that the idea of function is the idea of causality, but when you get to a formula you only have correlation. In thinking of the idea of the function you are thinking of causality. With regard to Mr. Carson's first illustration, in which he first of all gave two things which apparently had no connection at all—the number of hairs on a man's head and the number of hairs in his wife's muff—certainly he suggested it by means of a formula—but the causality was the first thing which came into our minds when he suggested that the numbers were in inverse ratio. I have often felt that when you use the word function in mathematics you are really thinking of some formula or another. When you say there are certain functions, you usually mean certain mathematical expressions or formulæ, but Mr. Carson's statement that the function is so different from the formula, rather suggests that with the idea of function you are thinking of causality, and in the case of a formula you are only thinking of correlation.

A Member: May I ask quite a different question on a point of practical detail? I did not quite understand how far you could explain the meaning and shape of a curve until you had explained the nature of plotting it—the different methods by which the graph is drawn, so as to give any indication of the increase in the function. It seems to me that we must start off with our axes and explanation of the method properly before we can explain at all why the alteration in the curve indicates the increase.

Mr. Andrade: I should like to suggest one other curve that could be given to suggest very clearly the shape that is wanted rather than anything else. I very often give my students to draw a hunger-time curve—the feeling of hunger at different times—and I find that rather interesting, because it leads sometimes to a discussion as to whether during your meal your hunger diminishes at a greater rate at the beginning of the meal or at the end.

Mr. Carson has warned us that it is not safe to join up a few plotted points by a smooth curve unless we know from other sources that the graph should be a smooth one. What would he do if a few points were given and no other information regarding the function could be obtained? Would he join up the points by a number of straight lines, as doctors do in temperature charts?

Mr. Carson: Mr. Daniell and Mr. Lodge both raised the same point, that is, whether I regard functional dependence as based on causality or correlation. The answer depends on the individual who is thinking of functional dependence. Causality is a concept which becomes familiar to human beings quite early in life. It can be appreciated by a boy or girl, in simple cases at any rate, and therefore the original idea of functionality should be developed mainly from notions of causality. All concrete illustrations, such as I have mentioned, involve causality rather than correlation. I am not an expert in these matters, but I should say if a formula is regarded merely as a correlation, nevertheless the proof of the formula may come very near to involving causality. I do not think the two can be separated in elementary teaching. I am very suspicious of any attempt to say in teaching "Here we are going to deal with causality and there with correlation," for the one thing to do is to build in every possible way on the pupil's own ideas, and these involve each factor almost indistinguishably. I believe, however, that in early years we shall in practice rely on impressions of causality rather than on correlation. Mr. Lodge, I noticed, said that we ourselves instead of thinking of "function" usually think of "formula." I agree that very often we mathematicians, in attempting to think of functions, do think of formulæ, but I do not think we are wise, and certainly it has been shown in recent investigations that the idea of functionality and its development is most simply evolved without reference to formulæ of any kind. All the modern courses of analysis proceed thus, and what I suggest is that the same procedure should be followed in elementary teaching, concrete illustrations being used throughout.

I was also asked how the meaning of the shape of a graph can be explained before any points have been plotted with axes and scales. What is in my mind is that the graph idea should be evolved from consideration of continuous automatic recorders of all kinds; barometers, an automatic recorder of rainfall, where a jar is exposed to the rainfall, and its weight depresses a pen which marks a revolving drum; an automatic recorder of the growth of a plant, made by fixing a pen to the stem; and so on. By such means the idea of a graph can be developed, starting from a complete curve instead of a series of plotted points. I find from experience that this can be done with pupils younger than I should have thought possible.

The final question was with regard to the matter of joining up points. What I intended to say is, that if we do not know anything about the general shape of the curve, and if the points plotted do suggest some curve, then as a counsel of despair we may sketch that curve, and at once try to ascertain, by obtaining more data, whether the curve is right or wrong. But if the points suggest no curve, or perhaps even if they do suggest one rather vaguely, the right course is to leave them as plotted, a series of discrete points. No one lie is much better than another, but what I will call the straight-line lie is perhaps the most despicable of all, because it is the most easily told and is farthest from the truth.

Sir G. Greenhill: Mr. Carson spoke of the difficulty of constructing formulæ to suit the operations of every-day life, which reminds me of a formula. I am afraid you will think it trivial and of no utility for teaching in a school. The formula is $Y=\frac{X}{2}-7$. X is the man's age, and Y the age of the girl of his choice. If you substitute in that formula, and Mr. Carson draws the graph, in the moderate range of human life, it works out in a good manner. But my experience is that when I have given the formula in the inverse manner of interchange of the variables, by saying to her, "The girl of my choice ought to be 40; how old am I?" she will not reply. She suspects something behind it.

We thank Mr. Carson for his interesting paper, and now Dr. Shaw, who is present, will read us his paper on "Principia Atmospherica." *

ON THE SETTING OUT OF CERTAIN EASY CUBICS.

If we wish to develop a race of geometers, I think we should begin by encouraging the habit of speculation. This we do nowadays by using squared paper for plotting graphs. Here, however, we place ourselves very much in the hands of the printer: and although ideas of geometrical shape are enlarged and systematised, the right angle plays too great a part in whatever knowledge of curves is acquired; indeed a knowledge of curves is not the professed object in plotting graphs so much as a visualised representation of corresponding changes in related physical quantities.

In teaching elementary geometry there lingers an aversion from the use of dividers in pricking out equal distances: for the transference of distances is not recognised in the postulates of Euclid, and Euclid's influence remains.

Later on a study of the conics, carried on for a prolonged period, and the application of the theorems of Pascal and Brianchon, must exercise their fascination. Ideas about infinity become clarified: the use of parallels is seen to be necessary, to exercise the right accorded by Euclid's first postulate to join any given point to any other, even if one be at infinity. But the mind inevitably tends to become obsessed with the second-degree idea. Mere straight-edge methods are found to be so potent in the construction of conics that metrical properties are neglected. Primary notions about equal distances interfere with the whole scheme of a projective geometry confined to the conics; and the bisection of a given finite straight line becomes a search for the harmonic conjugate of the point at infinity in its direction, with regard to its two ends.

By being encouraged in the use of parallels and dividers the young mind may be led to a quantity of information. Experimental geometry leads to speculation, speculation to information, information to knowledge.

Among the earliest notions that can be developed in this way is that of symmetry. We recognise the value of encouraging an intuitive sense of symmetry, nowadays, one is glad to say, in quite elementary courses of geometry. But I do not know that boys and girls are ever required to pick out the points that are symmetric correspondents, with regard to a straight line, of other points. If they are it is probably always rectangular symmetry; *i.e.* the parallels are always slid perpendicular to the axis of symmetry.

I think we might usefully recognise a geometrical property of figures, which I will call oblique symmetry: got by sliding the parallels at a constant angle to the axis of symmetry. In this way a scalene triangle can be seen to be obliquely symmetrical about each of its medians. A parallelogram has four axes of oblique symmetry. The "regular" figures have none.

With the use of dividers only, the conic hyperbola can, I think, be introduced at a much earlier stage of elementary geometry, if the speculative idea is to be encouraged. I would begin in this way:

* The paper is not printed in this number of the *Gazette*.