PROBLEM DEPARTMENT.

Conducted by J. O. Hassler.

Crane Technical High School and Junior College, Chicago.

This department aims to provide problems of varying degrees of difficulty which will interest anyone engaged in the study of mathematics. Besides those that are interesting per se, *some are practical, some are useful to teachers in class work, and there are occasionally some whose solutions introduce modern mathematical theories and, we hope, encourage further investigation in these directions.*

We desire also to help those who have problems they cannot solve. *Such problems should be so indicated when sent to the Editor, and they will receive immediate attention. Remember that it takes several months for a problem to go through this department to a published solution.*

All readers are invited to propose problems and solve problems here proposed. Problems and solutions will be credited to their authors. Each solution, or proposed problem, sent to the Editor should have the author's name introducing the problem or solution *as on the following pages. In selecting problems for solution we consider* accuracy, completeness, *and* brevity *as essential.*

The Editor of this department desires to serve its readers by making it interesting and helpful to them. If you have any suggestion to make, mail it to the Editor. *Address all communications to J. O. Hassler, 2337 W. 108th Place, Chicago.*

Correction.

In connection with the proofs of No. 554 published in the May number, the Editor's attention has been called to the fact that the proof by I. E. Kline credited to Loomis was really first published in the American Mathematical Monthly (February, 1902) by G. I. Hopkins of Manchester, N. H. It also appears in Hopkins' book, *Inductive Geometry*, and Loomis acknowledges the authorship of the proof.

SOLUTION OF PROBLEMS.

Algebra.

561. *Proposed by the Editor.*

Factor

$x^2y+x^2z+xy^2+xz^2+yz^2+y^2z+2xyz.$

I. *Solution by Garland Martin, First Year Pupil, Warren (R. I.) High School.*

$$\begin{aligned}
&x^2y+x^2z+xy^2+xz^2+yz^2+y^2z+2xyz\\
&=(xy^2+2xyz+xz^2)+(x^2y+x^2z)+(y^2z+yz^2)\\
&=x(y+z)(y+z)+x^2(y+z)+yz(y+z)\\
&=(y+z)(xy+xz+x^2+yz)\\
&=(y+z)[(x^2+xy)+(xz+yz)]\\
&=(y+z)[x(x+y)+z(x+y)]\\
&=(y+z)(x+y)(x+z).
\end{aligned}$$

II. *Solution by Thomas Griffith, Junior, Riverside Polytechnic High School, Cal.*

$$\begin{aligned}
&x^2y+x^2z+xy^2+xz^2+yz^2+y^2z+2xyz\\
&=x(xz+xy+yz)+y(xz+xy+xz)+xz^2+yz^2\\
&=(x+y)(xz+xy+yz)+(x+y)z^2\\
&=(x+y)[x(y+z)+z(y+z)]\\
&=(x+y)(x+z)(y+z).
\end{aligned}$$

III. *Solution by Ruth A. David, Gibson City, Ill.*

By the factor theorem, the expression reduces to zero by the substitution $x = -y$. $\therefore$ $x+y$ is a factor. Dividing the given expression by $x+y$, the other factor is found to be $xy+xz+z^2+yz$. This may readily be factored by grouping so as to show the common binomial factor $y+z$, the other factor being $x+z$.

$\therefore\ x^2y+x^2z+xy^2+xz^2+yz^2+y^2z+2xyz = (x+y)(y+z)(x+z)$

A second solution was also received from Ruth A. David. *Solutions were also received from* Chas. Britton, Louise E. Church, K. C. Fitch, Nellie F. Henderson, Emma V. Hesse, S. J. Keusch, Murray J. Leventhal, Harold M. Lufkin, R. M. Mathews, M. B. Messinger, Ralph Moody, Edward J. O'Leary, Margaret O'Neill, Norris Olson, Howard R. Park, Everett Robinson, Philomathe, Walter S. Rodgers, Nelson L. Roray (2), H. H. Sampson, V. M. Spunar, Donald C. Steele, Sylvia Silverman, Henry S. Titus, Jacob A. Weiss, Josephine Wible *and* Harriet A. Wilcox.

562. *Proposed by Murray J. Leventhal, Stuyvesant High School, New York City.*

If n is a prime number, show that

$$\underline{|n-1}+1 \text{ is a multiple of } n.$$

Solution by Philomathe, Montreal, Can.

I. *Lemma.*—If n is a prime number, the product $2\cdot3\cdot4\cdot\cdot\cdot\cdot\cdot(n-3)(n-2)$ is a multiple of n, plus 1.

Let A be any factor of the product $2\cdot3\cdot4\cdot\cdot\cdot\cdot\cdot(n-3)(n-2)$, then there is another factor A′ of the same product, and only one, such that $A\times A' =$ mult. $n, +1$. For, if we divide $A\times1, A\times2, A\times3, \cdot\cdot\cdot\cdot\cdot, A(n-1)$ by n we obtain for remainders, in any order, $1\cdot2\cdot3\cdot4\cdot\cdot\cdot\cdot\cdot, n-2, n-1$. The multiple of A giving 1 as remainder is neither $A\times1$, nor $A\times A$, nor $A(n-1)$, as could be easily made evident. Therefore, there exists a number A′, and only one, in the product $2\cdot3\cdot4\cdot\cdot\cdot\cdot\cdot(n-2)$ such that $A\times A' =$ mult. $n, +1$.

Now, the product $2\cdot3\cdot4\cdot\cdot\cdot\cdot\cdot(n-2)$ has an even number of factors, and it may be decomposed in groups of two factors, each group being a multiple of n, plus 1. Therefore, the product $2\cdot3\cdot4\cdot\cdot\cdot\cdot\cdot(n-2)$ is also a multiple of $n, +1$.

Theorem: $\underline{|n-1}+1$ is a multiple of n

Proof. From the lemma,

$$1\cdot2\cdot3\cdot4\cdot\cdot\cdot\cdot\cdot(n-3)(n-2) = \text{mult. } n, \ +1$$

Multiply both numbers by $n-1$.

$$\underline{|n-1} = (\text{mult. } n, +1)(n-1)$$

or $\underline{|n-1} =$ mult. $n, -1$

$\therefore\ \underline{|n-1}+1 =$ mult. n.

Note. This is Wilson's theorem.

Geometry.

563. *Proposed by Nelson L. Roray, Metuchen, N. J.*

Using the figure for the Theorem of Pythagoras as given on page 194 of Wentworth's *Plane Geometry* (Edition of 1899), prove

(1) BK, AL and FC are concurrent.

(2) Let X be the intersection of FC and AB, then GX, AD and BC are concurrent.

Prove by means of theorems that are usually given in Book I of plane geometry.

Solution by Philomathe, Montreal, Canada.

I. Let FG and KH meet at O. Then it can easily be proved that FC is $\perp$ to BO, BK is $\perp$ to CO, OA produced is $\perp$ to BC, $\therefore$ coincident with AL. Hence, FC, BK, AL are concurrent.

II. Next, produce AB in M, making BM = AC, and join MF, MG, MC; FC is $\perp$ to MG, MA is $\perp$ to GC, $\therefore$ GX produced is $\perp$ to MC. Now, let Y be the intersection of AD and MC; join XY, which is $\parallel$ to MF or BD. In triangle XYC, DA is $\perp$ to XC, BC is $\perp$ to XY, GX (produced) is $\perp$ to YC; $\therefore$ GX, AD and BC are concurrent.

564. *Proposed by N. P. Pandya, Sojitra, B. B. and C. I. Ry., India.*

ABC is a triangle, right-angled at A. AP, AQ are squares on AC, AB, respectively. PB cuts AC at D. QC cuts AB at E. Show that DE is parallel to PQ.

I. *Solution by Emlyn Roberts, Pupil Dunmore (Penna.) High School.*

Let M be the vertex of the square AQ opposite B and R opposite C. Produce lines QM and RP and letter their intersection T. Triangle BAD is similar to triangle BRP. Line AD is parallel to line RP. Then BA : BR = AD : RP. Triangle QPC is similar to triangle QAE, line AE is parallel to line PC. Then QA : QP = AE : CP. Substituting CP's equal, AR, we get QA : QP = AE : AR. Triangle QBA is similar to triangle QTP, angle QTP equals angle QBA and angle TQA equals angle BAQ (alternate interior angles.) Then BA : QT = QA : QP; substituting equals, BA : BR = QA : QP.

From the three equations,

BA : BR = AD : RP,
QA : QP = AE : AR,
BA : BR = QA : QP,

AD : RP = AE : AR. Angle EAD equals the angle ARP (right angles.)

Therefore triangle EAD is similar to triangle ARP. Angle RAP equals angle AED, then ED is parallel to QP (if two lines are cut by a transversal and the corresponding angles are equal the lines are parallel).

II. *Solution by Emma V. Hesse, Petaluma High School, Petaluma, Calif.*

Produce DE to meet PC at X and BQ at Y. Then the rt. $\triangle$s XCD, DAE and EBY are similar.

Hence DC/DA = XD/DE, (1) and AE/EB = DE/EY. (2)
Multiply (1)$\times$(2): DC/DA $\cdot$ AE/EB = XD/DE $\cdot$ DE/EY.
Cancel DE, and DC/DA $\cdot$ AE/EB = XD/EY.
Since DC/EB = XD/EY, AE/DA must = 1, that is, AE = DA; $\triangle$ADE is isosceles, and so must $\triangle$XDC.
$\therefore \angle CXD = 45^\circ$
But $\angle CPA = 45^\circ$, $\therefore \angle CXD = \angle CPA$.
$\therefore$ DE (produced) is parallel PQ.

II. *Solution by Nellie F. Henderson, Martins Ferry, Ohio, and R. M. Mathews, Riverside (Cal.) Polytechnic High School and Junior College.*

If PB and QC intersect at O, then $\triangle$s QBO and DCO are similar and QO/OC = BO/OD.
Also $\triangle$s BEO and POC are similar and EO/OC = BO/OP.
Divide first equation by the second and QO/EO = PO/DO.
$\therefore$ DE $\parallel$ QP.

Solutions were also received from Ruth A. David, Paul W. Harnley, Murray J. Leventhal, Philomathe (2), Oscar J. Johnson, Nelson L. Roray, Jason A. Zurflich, *and one with no name attached.*

565. *Proposed by L. E. Lunn, Heron Lake, Minn.*

Prove that the lines joining the vertices of a tetrahedron to the centers of the opposite faces are concurrent at the centeroid of the tetrahedron and are quadrisected at this point. Use only elementary geometry.

Solution by R. M. Mathews, Riverside, California.

In tetrahedron ABCD let M be the center of edge CD, G_1 the centroid of face BCD and G_2 that of face ACD.

The three planes determined by one vertex and the median lines of the opposite face are coaxial for they contain said vertex and the centroid of that face. Each pair of these axes lies in one plane and they intersect. Thus AG_1 and BG_2 lie in plane ABM. When each of four non-coplanar lines cuts each of the others they are concurrent. So the axes meet at a point O.

Consider $\triangle ABM$. A parallel to AG_1 through G_2 cuts MG_1 in X such that $XG_1 = 2/3\ MG_1$. But MG_1 is 1/3 MB, whence $XG_1 = 2/9$ MB $= 2/8$ XB $= 1/4$ XB. $\therefore G_2O = 1/4\ G_2B$.

A solution was received from MURRAY J. LEVENTHAL *and one incorrect solution.*

Late Solutions Received.

554. Two solutions from A. E. Breece.
555. R. M. Mathews.
557. Gerald W. Willard, Heron Lake (Minn.) High School, N. P. Pandya.
560. N. P. Pandya.

PROBLEMS FOR SOLUTION.

Geometry.

576. *Proposed by G. Ross Robertson, Polytechnic High School and Junior College, Riverside, Cal.*

Construct an equilateral triangle with one vertex on each of three given unequally spaced parallel lines.

577. *Proposed by M. Costello, Brentwood, Cal.*

Construct a circle passing through two given points and bisecting a given circle.

578. *Proposed by Nelson L. Roray, Metuchen, N. J.*

Equilateral triangles are constructed outward upon the sides of any triangle. Prove by elementary geometry only that the given triangle and the equilateral triangle whose vertices are the centroids of the equilateral triangles have the same median point.

579. *Proposed by N. P. Pandya, Amreli, Kathiawad, India.*

Two circles intersect at A and B. The sides DE, FD of a triangle touch both circles. EF intersects AB in C. If AB : AC = EK : EC, where K is the mid point of EF, find the possible positions of EF.

Trigonometry.

580. *Proposed by R. N. Mathews, Riverside, Cal.*

If P is the Brocard point of a triangle such that the angles PAB, PBC, PCA are equal, and p_1, p_2, p_3 its distances to a, b, c, respectively, then $p_1 : p_2 : p_3 = (a+c) : (a+b) : (b+c)$.

S. O. S.

The editor needs a fresh supply of good algebra problems.

P. S. Do not neglect the other kinds of problems.

P. P. S. Many problems proposed have been published before, either in books or in this journal. New practical problems are especially welcome.

PERSONALS.

Menton Rowand, recently principal of the high school at Bellefontaine, Ohio, has been appointed superintendent of schools at Zanesville, Ohio.

Professor Florian Cajori of Colorado College has been made professor of the history of mathematics at the University of California, Berkeley.

Associate Professor B. Clifford Hendricks, who has for the past ten years served the Peru, Neb., State Normal School in its department of physical science and nature study, has accepted a call to a position as assistant professor of general and physical chemistry in the University of Nebraska.

Mr. R. M. Mathews, for many years in connection with the high school at Riverside, Cal., has been made head of the department of mathematics in the Central High School, Duluth, Minn.

Professor Philip B. Woodworth, dean of electrical engineering of Lewis Institute, Chicago, has entered the Government service as a major in the aviation section of the Signal Corps.

Professor M. E. Graber, fellow in mathematical physics at the Univer-