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XXXV. *Does an Accelerated Electron* necessarily *Radiate Energy on the Classical Theory?* *By* S. R. MILNER, *D.Sc.*, *Acting Professor of Physics, The University, Sheffield* *.

THIS question is of fundamental importance in modern theory, and it would, I imagine, at the present time receive an affirmative answer from most physicists. It is true that certain adverse experimental results, such as the normal absence of radiation from electronic motions in the atom, call urgently for theoretical explanation; but it seems to be accepted that the necessity in the classical theory for radiation from accelerated charges is so firmly based that it can only be removed by far-reaching and revolutionary changes, such as the quantum theory supplies. Some apology seems necessary for attempting to open the question again at this date; and I should not have ventured to do so, but for the result of the consideration of a certain concrete case of accelerated electronic motion which is amenable to an accurate mathematical treatment. This example shows that it is possible to obtain, even on the classical theory, a solution for a particular case of the accelerated motion of charges, which satisfies completely both Maxwell's equations and the mechanical laws which characterize a conservative system, without any irreversible radiation of energy. A study of it enables us, I think, to prove that a certain step made in the deduction on the classical theory of the general necessity for radiation is not invariably a valid one, and to show that a comparatively minor modification of the boundary conditions of the solution is sufficient to do away with radiation in at any rate one case of accelerated motion.

A remarkable solution of Maxwell's equations for a particular type of accelerated electronic motion has been given by Schott †. It is remarkable in that it is the only case, other than that of uniform motion, which up to the present has been solved in finite terms.

Consider a point-charge moving along the (positive) axis of x, and, ξ being its distance from the origin at time t, let it move in such a way that

$$\xi^2 = k^2 + c^2t^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where k is constant. At $t = -\infty$ the charge is at $\xi = +\infty$ moving towards the origin with the velocity of light c, it

* Communicated by the Author.

† 'Electromagnetic Radiation,' pp. 63–69.

comes to rest at $\xi = +k$ when $t=0$, and moves back towards $\xi = +\infty$, acquiring again at $t = +\infty$ the velocity of light. The motion is that which would be produced, according to the ordinary principles of mechanics, by the action, on a particle of mass

$$m = \frac{m_0}{(1-\beta^2)^{\frac{1}{2}}}, \text{ where } \beta = \frac{1}{c}\frac{d\xi}{dt},$$

of a constant force F, such that

$$\mathrm{F} = \frac{c^2 m_0}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To express the electromagnetic field associated with a point-charge e moving in this way, at any point let χ be the angle included between two lines, lengths r_1 and r_2, respectively drawn from it to the instantaneous position of the point-charge and to that of its image in the plane $x=0$, and let $\psi = \log r_2/r_1$. Then χ, ψ are related to the cylindrical co-ordinates x, y by the equations

$$x = \frac{\xi \sinh \psi}{\cosh \psi - \cos \chi}, \quad y = \frac{\xi \sin \chi}{\cosh \psi - \cos \chi}. \quad . \quad (3)$$

The third co-ordinate ϕ, the angle through which the plane containing the point has rotated about the axis of x from a fixed position, is the same in both systems.

The electric and magnetic forces at any point of the field are given by

$$\mathrm{E} = \frac{e(1-\beta^2)(\cosh \psi - \cos \chi)^2}{\xi^2 (1-\beta^2 \sin^2 \chi)^{\frac{3}{2}}}, \quad \mathrm{H} = \beta \sin \chi \,.\, \mathrm{E}. \quad (4)$$

The nature of the field will be made clear by a reference to figs. 1 and 2. These show meridian sections at the moments $t = \mp \frac{k}{c}$ and $t=0$, when the charge is at the distance $\sqrt{2}k$ and k respectively from the origin. The lines of force in each case form the arcs of circles (χ=const.) passing through the charge and through its image in the median plane $x=0$; in the figures they are drawn so that $\frac{1}{8}$ of the total flux of induction is enclosed by adjacent lines. The changes which the field undergoes can be pictured by supposing that each line of force in fig. 1 at $t = -\frac{k}{c}$ is moving normally inwards with a velocity $\beta c \sin \chi$. The velocity gradually decreases until the line comes to rest momentarily in the position of the corresponding line of fig. 2 at $t=0$; it then moves outwards, passing the position of fig. 1 again at $t=+\frac{k}{c}$. The magnetic force is directed along the parallel

Fig. 1.

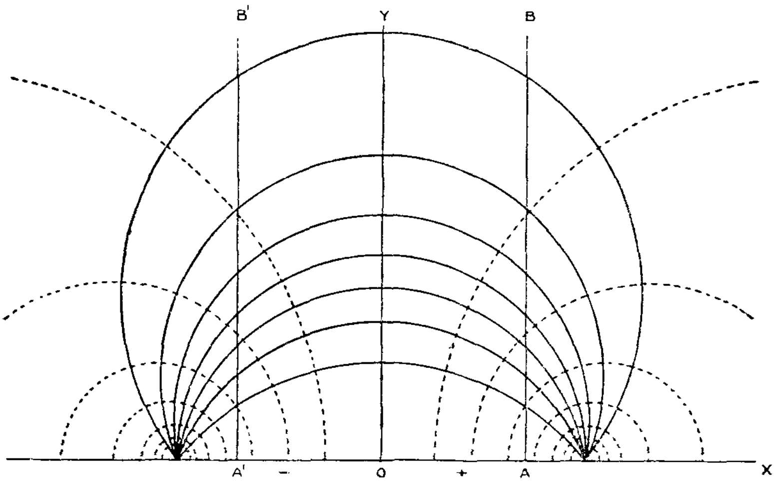
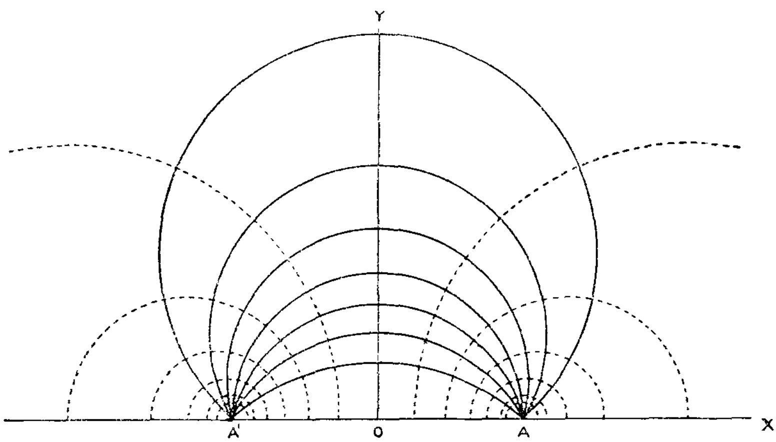


Fig. 2.



of latitude, and, passing through zero value, is reversed in sign at the moment when the electric lines change their direction of motion. The dotted circles, $\psi=$const., orthogonal to the lines of force, are the lines of the Poynting energy-flux.

The field, as given by Schott, is limited by a moving boundary, formed by the plane $x+ct=0$ (AB at $t=-k/c$, A'B' at $t=+k/c$, fig. 1 ; OY, fig. 2). The boundary is a transition layer (of a thickness comparable with the radius of the electron) in which the electric and magnetic forces vary from the values given in (4) on the right of the boundary to zero beyond it. For a point-charge it forms a layer of discontinuity, in which the lines of force emanating from the charge suddenly change their direction, and thenceforth lie in the plane.

It will readily be seen that this field gives an irreversible radiation of energy. The direction of the energy-flux shows that, except at the moment $t=0$ when the moving boundary crosses it, no energy ever passes through the median plane. Thus the field energy which, at positive values of t, exists to the left of the origin, must, along with the boundary, constitute a permanent loss by radiation from the system.

A very simple modification of Schott's solution eliminates from it the presence of irreversible radiation. Consider the equations (4) for E and H, and let them now be valid over all space and all time, the moving boundary being dispensed with. The electromagnetic field thus expressed possesses the following properties :—

(1) It satisfies Maxwell's equations

$$\dot{\mathbf{E}}=c\,\mathrm{curl}\,\mathbf{H}, \quad \dot{\mathbf{H}}=-c\,\mathrm{curl}\,\mathbf{E},$$
$$\mathrm{div}\,\mathbf{E}=4\pi\rho, \qquad \mathrm{div}\,\mathbf{H}=0,$$

at every point of space and time from $-\infty$ to $+\infty$.

(2) The third equation is satisfied in the sense that $\mathrm{div}\,\mathbf{E}=0$ everywhere except at the points $x=\pm\xi$, where it becomes ∞ in such a way that $\int \mathbf{E}_n dS$ round any closed surface surrounding the point is equal to $\pm 4\pi e$. The solution thus forms throughout all space and time an electromagnetic field which can be associated with two point-charges, $+e$ and $-e$, moving with the particular type of accelerated motion defined by (1).

The two charges start with the velocity of light at $t=-\infty$ from positive and negative infinity of x respectively; they move symmetrically along the axes inwards towards the origin, come to rest at $t=0$ at the points $x=\pm k$, and then move outwards to infinity, ultimately acquiring again the velocity of light. The lines of force are as in figs. 1 and 2, except that the image is now a real charge and there is no boundary, the lines extending from one charge right up to the other.

(3) There is no irreversible radiation of energy from the system. This is evident, for the field at $t=+t_1$ is identical throughout all space with that at $t=-t_1$ except that the sign of H is reversed. Hence, whatever flux of energy outwards from either charge may have occurred at $t=-t_1$, it will be exactly annulled by an equal inward flux at $t=+t_1$.

(4) It will be useful to express the energy and the momentum of the system. The volume of the orthogonal element comprised between the spheres χ and $\chi+d\chi$, ψ and $\psi+d\psi$, and the planes ϕ and $\phi+d\phi$ is

$$dV=\frac{\xi^3 \sin\chi\, d\chi\, d\psi\, d\phi}{(\cosh\psi-\cos\chi)^3},$$

and the total energy of the field is

$$\int \frac{1}{8\pi}(E^2+H^2)dV.$$

The integration extends from $\phi=0$ to 2π, $\chi=0$ to π, but with regard to that for ψ, the space occupied by the nucleus or charged surface of the electron must be excluded from the integration. The result will depend on the shape assumed for the nucleus. If we integrate from $\psi=0$ to the surface given by

$$\frac{\sinh\psi}{\xi}=\frac{1}{a}\left(\frac{1-\beta^2}{1-\beta^2\sin^2\chi}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (5)$$

we get for the field energy of either charge in the region external to the surface

$$\frac{2}{3}\frac{e^2}{a}(1-\beta^2)^{-\frac{1}{2}}-\frac{1}{6}\frac{e^2}{a}(1-\beta^2)^{\frac{1}{2}}. \quad . \quad . \quad . \quad (6)$$

This is identical with the field energy of a Lorentz electron of the same e and a in uniform motion with the same velocity.

The surface (5) is not precisely the same as the spheroid which forms the surface of the Lorentz electron in a state of uniform motion; but it reduces to it in two cases: when the acceleration is zero, and when the radius of the electron a is indefinitely small. In each of these cases $\xi/\sinh\psi$ (the radius of the ψ sphere passing through a given point) reduces to identity with the radius vector to the point drawn from the centre of the nucleus. There is a difficulty in dealing with the problem for an electron of finite size, partly mathematical if the Lorentz spheroid is assumed, and partly because we cannot say *à priori* what is the exact shape which must theoretically be ascribed to the nucleus of an electron in this type of accelerated motion. The theory of the Lorentz spheroid only applies strictly to uniform motion. There is no need, however, to discuss this difficulty here, as for the present purpose it can be set aside by our assuming, as is now done, that the electrons in the problem are of infinitely small size. The surface (5) then becomes identical with a Lorentz spheroid bounding the charge, and it can be taken to represent the surface of the nucleus without any difficulties being encountered. It is true that the assumption that a is infinitely small makes the energy and the momentum of the system formally infinite; nevertheless, they are definitely evaluated in the limit, and the essential feature of the solution, the absence of radiation from the system, is not affected by the assumption in any way.

The expression (6) now represents the total electromagnetic energy in the external field of one of a pair of Lorentz electrons moving with the given type of motion, in the limit when they are of infinitely small size. In order to produce agreement between the electromagnetic and the mechanical schemes, the nucleus, precisely like that of the uniformly moving electron, must be supposed to possess a store of internal energy of the amount

$$\frac{1}{6}\frac{e^2}{a}(1-\beta^2)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Then the total energy of each electron in the system becomes

$$w=\frac{2}{3}\frac{e^2}{a}(1-\beta^2)^{-\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (8)$$

(5) The electromagnetic momentum parallel to x is given by

$$\int\frac{\text{EH}\sin\theta}{4\pi c}d\text{V},$$

where θ is the angle made by the line of force, $\chi=$const., with the positive axis of x. On substituting

$$\sin\theta=\frac{\sinh\psi\sin\chi}{\cosh\psi-\cos\chi}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and integrating with respect to ϕ and χ as before, but with respect to ψ up to the surface

$$\frac{\cosh\psi}{\xi}=\frac{1}{a}\left(\frac{1-\beta^2}{1-\beta^2\sin^2\chi}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad (10)$$

we find the x-momentum associated with the positive electron external to this surface to be

$$g=\frac{2}{3}\frac{e^2}{ac}\beta(1-\beta^2)^{-\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (11)$$

This is precisely the same as the momentum of the Lorentz electron of the same e and a in uniform motion with the same velocity.

It must be noted that this result has only been obtained by making the surface (10), to which the integration in ψ was extended, a slightly different one from that (5) used for the determination of the energy. The difference, however, disappears when, as is the case here, the surfaces are of indefinitely small size, as in this case (10) as well as (5) reduces to identity with the Lorentzian spheroid.

There is a distinction between what we may call "fictitious" and "real" cases of electronic motion, which is brought into evidence by this example. Maxwell's equations, as is well known, are of great generality, and can be satisfied by cases of motion which cannot really exist. Theoretically a solution for the field of a charged particle in an arbitrary state of motion can be obtained. But from a physical standpoint the purely arbitrary motion of a charge is a condition impossible to produce. We know of no way in which an electron can actually be set into motion except by the application to it of an electromagnetic field (excluding possible effects of gravitational fields). Not only is this not entirely arbitrary, being subject to the fundamental equations, but also its inclusion alters the problem in an important respect. The problem becomes now, not that of finding the field of an electron whose charge is imagined to be moved about in a given way by external non-electromagnetic means, but that of finding a solution of the fundamental equations which will represent the real time-history of a given electromagnetic field which contains an electron. But for this problem Maxwell's equations are insufficient. Being satisfied by the simple superposition of two electromagnetic fields, they do not reveal the way in which, when an electron is present in a given field, the two fields interact. An additional theory is required for this, such as that of the Lorentz equation.

Leaving this aside for the moment, we can consider in a general way the conditions which a system of two superposed fields must be expected to satisfy for it to represent a real case of electronic motion. There is undoubtedly the law of the conservation of energy. The electromagnetic system must, in fact, be a conservative one, capable of existing without the introduction of energy by imaginary processes from outside. (Radiation, if present, is of course part of the electromagnetic system.) We may, perhaps, further expect that the energy and momentum of each part of the system which is in motion should be related to each other in accordance with the laws of mechanics.

From this point of view the solution of electronic motion which has been discussed is a fictitious one, as it does not obey the conservation of energy. As the electrons are approaching each other the energy of the system gets smaller, and it increases as they move apart. By a simple modification, however, the system can be made conservative.

Superpose on the field (4), which forms one solution of Maxwell's equations, another solution in the form of a uniform electric field X of infinite extent and parallel to the axis of x. Let $Xe=F$, so that X is the field which is required, on the basis of Lorentz's equation, to produce the actual motion which the electrons possess. Then, since for a Lorentz electron we have $m_0=\frac{2}{3}\frac{e^2}{c^2a}$, we have by (2)

$$X=\frac{2}{3}\frac{e}{ak}. \quad \quad (12)$$

The resulting field is now given by

$$E_x=E\cos\theta+X, \quad E_y=E\sin\theta,$$

$$H=\beta\sin\chi\,.\,E,$$

where E and X are given by (4) and (12), and it forms of course a solution valid over all space and time.

These equations represent a system of two electrons of opposite sign and Lorentz mass moving in a prescribed way in a uniform electric field. We shall, however, as before consider only the case in which a is infinitely small, when the electrons become point-charges, still of Lorentz mass, and the field X, though of infinite strength, is the limit of the definite value required to cause in the point-charges a finite acceleration.

Although the total energy of the system is infinite, the question of its conservation can be examined by considering the flux of energy in the field. For mathematical purposes we can regard such a field as a solution of Maxwell's equations which extends over all space and time, except that it is limited by certain moving boundaries, which form the surfaces of the electrons, and inside which **E** and **H** are prescribed to be permanently zero. We know by Poynting's theorem that, if the quantity $\frac{1}{8\pi}(\mathbf{E}^2+\mathbf{H}^2)$ be identified with the energy density, the conservation of energy will hold throughout the whole field except for the space included within the boundaries; for the Poynting flux is equivalent to the transfer of this quantity from one part of the field to another without net loss or gain. The only places, therefore, where energy can enter or leave the electromagnetic system are the boundaries, so that if we calculate the total flux of energy which is passing through them at any time, we shall obtain the energy which is being introduced into or is disappearing from the electromagnetic system per unit of time. It must be noted that this is not necessarily zero for a conservative system. Even in the simplest case, that of the uniformly moving electron, considerations of the continuity of the flux *, as well as those based on dynamics and on relativity †, necessitate the postulation of a definite store of internal energy, of the amount given by (7), within the nuclear boundary. When the internal energies of the electrons are included in the scheme, conservation in the system will be characterized by the net flux of energy outwards across the boundaries being equal to the net rate of decrease of the internal energies of the electrons.

Confining attention to the positive electron and to the positive x half of the field, we shall take the boundary to be the spheroid given by the limit of (5) when a is infinitely small. Let **P** be the Poynting flux outwards through a fixed boundary momentarily coinciding with (5), and let **v** be the velocity outwards of any point of the boundary surface, then the net flux outwards from the whole moving surface is given by

$$\int(\mathbf{P}-\mathrm{W}\mathbf{v})\mathbf{n}\,d\mathrm{S},$$

where **n** is a unit vector outwards normal to the surface, and

$$\mathrm{W}=\frac{1}{8\pi}(\mathrm{E}^2+\mathrm{H}^2+2\mathrm{E}\cos\theta\,.\,\mathrm{X}+\mathrm{X}^2) \quad . \quad . \quad (13)$$

is the energy density of the field at the point. Expressed in

* Milner, Phil. Mag. Oct. 1920, p. 494.
† *Cf.* Lorentz, 'Theory of Electrons,' § 180.

Cartesians, the flux becomes

$$\int\left\{\frac{c}{4\pi}(\mathrm{E}\cos\theta+\mathrm{X})\mathrm{H}+\mathrm{W}\frac{dy}{dt}\right\}2\pi y\,dx$$
$$+\int\left\{\frac{c}{4\pi}\mathrm{E}\sin\theta\,.\,\mathrm{H}-\mathrm{W}\frac{dx}{dt}\right\}2\pi y\,dy. \quad . \quad . \quad . \quad (14)$$

Here x and y are the co-ordinates of a point of constant χ on the surface (5), which is bodily moving through space and at the same time altering its shape since ξ and β are functions of t. To effect the integration we substitute

$$dx=\left(\frac{\partial x}{\partial\chi}+\frac{\partial x}{\partial\psi}\frac{\partial\psi}{\partial\chi}\right)d\chi, \quad \frac{dx}{dt}=\left(\frac{\partial x}{\partial\xi}+\frac{\partial x}{\partial\psi}\frac{\partial\psi}{\partial\xi}\right)\frac{d\xi}{dt},$$

etc., and use the transformation equations (3). On writing for E and H, X, W, and θ their values (4), (12), (13), and (9), and substituting for ψ, $\frac{\partial\psi}{\partial\chi}$, $\frac{\partial\psi}{\partial\xi}$ everywhere their values in terms of χ and ξ obtained from (5), and then expressing by (1) ξ in terms of k and β, (14) becomes a determinate function of χ, which, integrated from 0 to π, gives the net flux outwards from the boundary. On finally taking the limiting value when a is indefinitely small, it reduces to

$$\frac{\beta ce^2(1-\beta^2)^2}{4ak}\int_0^\pi\frac{(1+2\cos^2\chi+\beta^2\sin^2\chi)\sin\chi\,d\chi}{(1-\beta^2\sin^2\chi)^{\frac{5}{2}}}$$
$$-\frac{\beta ce^2(1-\beta^2)}{3ak}\int_0^\pi\frac{\sin\chi\,d\chi}{(1-\beta^2\sin^2\chi)^{\frac{3}{2}}}.$$

The first term represents the flux corresponding to the electronic field E, H alone, the second the additional flux due to the superposition of the field X. On integration the net flux outwards becomes

$$\frac{1}{6}\frac{\beta ce^2}{ak}(1-\beta^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The internal energy is given by (7), and its rate of decrease with the time, in the existing state of motion (1), is identical with the expression (15) for the net outward flux.

It follows that the total energy of the system is conserved. As the electrons are being brought to rest, a continual process of conversion of magnetic energy into electric is going on in the field, and at the same time field energy is being transferred into the nucleus, where it shows itself as the increase in the internal energy of the electron. When the electrons, having been brought to rest, begin to move away from each other, the stored internal energy comes out into the field, and at the same time the excess of electric energy which has accumulated in the field is reconverted into the magnetic form.

If we distinguish that part of W given by $\frac{E^2+H^2}{8\pi}$ as the "electronic" from the "mutual," $\frac{E \cos\theta \,.\, X}{4\pi}$, and the "external field," $\frac{X^2}{8\pi}$, energy densities, the equations (8) and (11) show that the total electronic energy w and momentum g fit into a mechanical scheme. In fact, since

$$\frac{dg}{dt}=Xe \text{ and } \frac{dw}{dt}=Xe\,.\,\beta c,$$

their rates of increase are identical with those which would be created by a mechanical force Xe acting on the moving nucleus. This shows that the system formally satisfies the Lorentz equation

$$\mathbf{F}=\rho\left(\mathbf{E}+\frac{1}{c}[\mathbf{vH}]\right),$$

as applied to a point-charge e in a uniform electric field X *.

There seem to be only two respects in which the solution fails to represent a real case of electronic motion. (1) It refers to a limiting case in which the electrons are concentrated to a point and are moving in a correspondingly infinite field. (2) Extending over all space and time, it does not represent the initiation of the motion, and does not correspond substantially with a practical case, such as is given when an electron, passing through a hole in a charged plate, comes into an ultimately uniform field. Neither of these points, I imagine, has any significance with regard to the question at issue. (1) clearly does not affect the radiation from the system; and with regard to (2), electromagnetic solutions, like those of other differential equations, in general only represent steady states under idealized conditions, and without reference to the manner of their initiation (*cf.* the solution for the uniformly moving electron).

* It may be noted that the example does not reveal whether a term of the order $e^2/4\xi^2$, expressing a mutual action of the charges, should be included, as, if present, it would be negligible compared with the infinite Xe.

In the light of this solution it seems desirable to examine with some care the proof of the presence of radiation as a necessary accompaniment to the accelerated motion of charges. The proof, as given by Larmor in 'Æther and Matter' (Chapter XIV.), and by Lorentz in 'The Theory of Electrons' (§ 39), is based on the solution in terms of retarded potentials for the field of a point-charge in a prescribed state of motion. This solution shows that the alterations in the field at any point can be considered as due to disturbances which emanate from the charge at each point of its path, and are propagated outwards with the velocity of light. Outside a moving boundary, marking the farthest points to which the disturbances have travelled, the previously existing field is unaffected by the motion of the charge. It is now found that in the resulting field, at a sufficiently great distance from the charge, there is an outward flux of energy proportional to the square of the acceleration, which clearly seems to represent an irreversible loss by radiation.

Let us first test this general conclusion by applying the method to Schott's solution, as originally limited by the moving boundary. The point law of retarded potentials gives for the field of a point-charge moving arbitrarily *

$$\mathbf{E}=e\left[-\frac{\dot{\mathbf{v}}}{c^2\mathrm{K}^2\mathrm{R}}+\frac{\left(\mathbf{R}_1-\frac{\mathbf{v}}{c}\right)\{(\dot{\mathbf{v}}\mathbf{R})+c^2-v^2\}}{c^2\mathrm{K}^3\mathrm{R}^2}\right],$$

$$\mathbf{H}=[\mathbf{R}_1\mathbf{E}], \quad \text{where } \mathrm{K}=\left(1-\frac{\mathbf{v}\mathbf{R}_1}{c}\right).$$

E, **H** are here the electric and magnetic forces which exist at the time $t=\tau+\frac{\mathrm{R}}{c}$ at any point which is at a distance R from the point where the charge was at the time τ. For simplicity take $\tau=0$, so that the disturbances considered are those emitted at the turning-point A, $(k, 0)$, of the charge, whence in the formula we shall have $\mathbf{v}=0$ and $\dot{\mathbf{v}}=c^2/k$ parallel to x. Then at the time $t=\mathrm{R}/c$, and on the spherical surface of radius R described about A as centre, **E** and **H** will be given by

$$\mathbf{E}=e\left(-\frac{\sin\alpha}{k\mathrm{R}}\mathbf{P}_1+\frac{1}{\mathrm{R}^2}\mathbf{R}_1\right), \quad \mathbf{H}=[\mathbf{R}_1\mathbf{E}],$$

where α is the angle between the axis of x and **R**, and $\mathbf{P}_1$, $\mathbf{R}_1$

* Schott, 'Electromagnetic Radiation,' p. 23.

are unit vectors perpendicular to and parallel to **R**. This is consistent with (4) as it should be. When R is so large that the second term is negligible compared with the first, **E** is perpendicular to **R**, and there is a flux of energy outwards through the surface of amount

$$\frac{c}{4\pi}\int_0^\pi \mathbf{EH}\,.\,2\pi \mathrm{R}^2 \sin\alpha\, d\alpha = \frac{2}{3}\frac{e^2 c}{k^2} = \frac{2}{3}\frac{e^2}{c^3}\dot{\mathbf{v}}^2. \quad . \quad (16)$$

This agrees with the usual formula, and apparently verifies the presence of radiation in the solution. Nevertheless, an examination of the lines of energy-flow in figs. 1 and 2 shows that, as regards more than half of this flux—namely, that through the part of the surface on the positive side of the origin,—the energy concerned is not being radiated to infinity irreversibly at all, but is travelling along curved paths which in the course of time inevitably bring it back again to the nucleus of the electron ; and, in fact, it forms a permanent part of the electronic field. The reason for this is that the proof of the existence of the flux (16) does not apply to *any* large sphere surrounding the point A, but only to a specified sphere at a given time. If we apply the calculation to a sphere of different radius R′ at the same time t, this new sphere is not centred on A, but on the position of the charge at the time $t-\mathrm{R}'/c$. The instantaneous lines of energy-flow, which are normal to both spheres, are therefore necessarily curved lines, however far out they may be traced, and in this case their curvature is such as to keep the energy permanently in the electronic system.

In the negative half of the field, the energy, although here also it is not travelling outwards indefinitely, but in curves related in a similar way to the image, is apparently true radiation, since, as is shown by the direction of the flux-lines, no energy ever comes back to the electron through the median plane. Nevertheless, a curious point may be noticed in this connexion. If we investigate the flux of energy from the field into the moving plane, $x+ct=0$, we find that it is invariably negative; consequently the boundary, considered as the limit of a thin transverse field separating the electronic field from zero, is always engaged in laying down field energy behind it as it advances, and never, even at $t=-\infty$, in receiving any energy from the electron. The solution therefore premises an initial intrinsic energy in the boundary, apart from that of the electronic field. Now the field energy on the negative side of the median plane is clearly laid down by the boundary, since no energy ever passes through the plane except at the moment when the boundary crosses it. We consequently see that the only radiation which the solution gives is, strictly speaking, not from the electron at all, but is to be attributed ultimately to the moving boundary which is postulated to be the limit of the field.

The limitation of the solution by a moving boundary seems to be included, although perhaps tacitly, in the proof of radiation; but a comparison with the unlimited solution discussed in this paper raises the question whether the boundary is essential in every case for the representation of real motions. The arbitrary motion discussed in the proof makes the case considered there fictitious in the sense previously explained; but this point is only material in that it makes it clear that an additional conception is involved in the complete proof, which must necessarily concern itself with a charge set in motion by an electromagnetic field. Lorentz's equation is applied, and interpreted by assuming that the electron will move in the same way as it would do if it were a particle of given mass acted on by determinate mechanical force. The retarded potential solution corresponding to the resulting motion of the charge, superposed on the external field which gives the motion, then forms the complete solution of the problem, and the boundary is present in it as before.

The question whether the boundary is necessary or not seems to be largely a question of the physical interpretation made of the point law and of Lorentz's equation. The conclusion that it is necessary is based on the conception that the charges or nuclei of the electrons are first set into motion by the operation of the field in their immediate neighbourhood, and that the resulting changes in the field are *actually* propagated outwards from them. But it does not follow, from the mathematical fact that the changes in the field at a point are the same as if the disturbances were propagated from the charge, that the propagation is a physical fact. The field variations at a point can also be described in terms of the differential coefficients of the field at a point in a way which does not bring in the charge at all. Moreover, if we regard the nucleus of the electron from a mathematical standpoint as a small closed surface limiting the field and characterized by the constancy of the flux of force over it, once the field is known at all points, not only the field variations but also the motion of the electrons is uniquely determined. It seems from first principles as logical to consider the motion of the charges and the changes in the field to be two connected aspects of a single solution regarded as a whole, as to suppose that one is antecedent to the other. The adoption of this point of view would not affect the character of the field of being expressible in terms of retarded potentials, but it would permit other boundary conditions than those of a simple moving boundary to be applicable for the representation of real motions. The example which has been discussed in this paper seems to show, in one case at least, the legitimacy of a solution with less prescribed boundary conditions.

October 10th, 1920.

XXXVI. *On a Graphical Method for determining the Frequencies of Lateral Vibration, or Whirling Speeds, for a Rod of Non-Uniform Cross-Section.* By R. V. SOUTHWELL, *M.A., Fellow and Lecturer of Trinity College, Cambridge* *.

[Plate VII.]

THE determination of the normal modes and frequencies of lateral vibration for a rod of varying cross-section is a problem which has attracted the attention of many elasticians, and several papers on the subject have appeared in this Magazine †. It would seem that a complete solution,

* Communicated by the Author.

† References to the investigations of Bernouilli, Euler, Kirchhoff, Sturm, Liouville, and others will be found in Lord Rayleigh's 'Theory of Sound,' or in the 'Dynamical Theory of Sound' of H. Lamb. The following is believed to be a fairly complete list of recent papers bearing on the problem :—

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S. Dunkerley, Phil. Trans. Roy. Soc. (A), vol. clxxxv. (1894), p. 279.
C. Chree, Phil. Mag., vol. vii. (1904), p. 504, and vol. ix. (1905), p. 132.
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A. Morley, 'Engineering,' July 30th and August 13th, 1909.
P. F. Ward, Phil. Mag., vol. xxv. (1913), p. 85.
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W. L. Cowley and H. Levy, Advisory Committee for Aeronautics, R. & M. 485 (1918).
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