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LXIII. The accelerated motion of an electrified sphere

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The following simple experiment illustrates this effect. Portions of a piece of thin manganin wire are insulated with glass, the rest being left bare. When placed in a current of air and heated electrically the bare pieces of wire glow brilliantly, but the portions covered by the glass are quite dark and are therefore at a much lower temperature.

In very high tension systems for the electric transmission of power the overhead wires are sometimes surrounded with coronæ which appreciably increase the transmission losses. The author has previously suggested that the losses would be diminished by insulating the overhead wires with a suitable material of high electric strength. The above analysis indicates that this procedure instead of diminishing the permissible current in the wires would actually, in many cases, allow an appreciably greater current to be transmitted for the same rise of temperature of the wire.

In conclusion, I have to thank Professor Charles Lees, F.R.S., for his kind help in giving me a long list of references to papers on this subject.

LXIII. *The Accelerated Motion of an Electrified Sphere.*

By J. W. Nicholson, *M.A., D.Sc.**

When a sphere carrying a surface charge is placed in a uniform field of electric force at any instant of time, it is set into motion under the mechanical action on its electrification during the adjustment necessary for the satisfaction of the new conditions at its surface. A direct solution of the appropriate electromagnetic relations, with a determination of the motion of the sphere, has been given by Mr. G. W. Walker † for the general case in which the sphere is assumed to possess a Newtonian mass in addition to its inertia of electrical origin, and in which the applied field of force is small.

The application of the quasi-stationary principle to accelerated motions has never received formal justification, and in addition to certain general considerations tending to throw doubt upon its validity for such problems, Walker has obtained, in a later paper ‡, a formula for the transverse inertia of a moving sphere which is not in accord with that derived by Abraham with the aid of this principle. The

* Communicated by the Author.
† Proc. Roy. Soc. 1906, p. 260.
‡ Phil. Trans. 1910, vol. 210, p. 145.

method employed is to obtain solutions of the primary electromagnetic relations which satisfy definite surface conditions, and neither the relations nor the conditions are dispensed with at any stage. After a calculation of the mechanical reaction on the sphere has been made, the motion of the sphere is worked out by the principles of Newtonian dynamics, and Walker contends that this method, by its direct nature, is the one most fitted to yield correct results. With this view it seems necessary to agree, and as the method does lead to a different formula for the electrical inertia, and, moreover, indicates a redistribution of the charge in certain cases of motion which is again contrary to the results of the quasi-stationary principle, this principle has perhaps been pushed too far. Its use is therefore not to be regarded as definitely justified in cases of accelerated motion, until its exact limits of validity have been examined in a more formal manner, and the more direct method seems preferable in every way for the solution of special problems. But on the other hand, the conditions holding inside a conductor in a state of accelerated motion are at present quite unknown, and there is no certainty that the evanescence of either the tangential electric force or electromagnetic force, conditions hitherto used for a perfect conductor, at all represent the facts. It is difficult to believe that there could be no electrical effect inside a conductor with an acceleration, and all that can be done at present apparently is to work out the consequences of various possible assumptions. Thus Walker's results do not necessarily disprove the quasi-stationary principle for small accelerations, and the results of the present paper will be found to cast some doubt upon the theory that the usual treatment of the perfect conductor is still valid when its motion is accelerated.

The object of the paper is a brief discussion of the initial motion under a small field of electric force, or a small force of a purely mechanical nature, of a sphere whose charge is initially uniform, and whose mass is purely of electric origin. Walker states in his first paper that when the Newtonian inertia is zero, the damped harmonic vibration present at the beginning of the motion becomes evanescent, and it is impossible to satisfy all the initial conditions, so that his solution fails in this case. The formal deduction of this solution as a limiting case from Walker's formulæ is attempted in the present paper.

Prof. A. W. Conway, in a recent paper *, has concluded that when a charged sphere without Newtonian mass is

* Proc. Royal Irish Academy, xxviii. p 1.

placed in a uniform field, it moves in such a way that its charge remains uniform. But his investigation does not take account of the initial conditions of the motion, and it is by no means obvious that the effect of these conditions would vanish in the same way as for a sphere with both electrical and Newtonian inertia.

Let ζ denote the displacement, at time t, of the centre of a sphere of radius a initially placed in a uniform field of electric force F of small magnitude, so that F^2 can be neglected. F and ζ are both measured along the axis of z. The uniform charge initially present on the sphere is e, and (x, y, z) denote the coordinates of a point referred to an origin instantaneously coinciding with the centre of the sphere, r being the distance of this point from the origin. Then, within the region defined by $r=ct+a$, Walker shows that the components of the electric and magnetic forces are given by

$$(X, Y, Z)=\frac{e}{r^3}(x, y, z)+(0, 0, 1)\left\{F-\frac{c}{r^3}\left(r^2\chi''+r\chi'+\chi-\frac{e\zeta}{c}\right)\right.$$
$$\left.+\frac{cz}{r^5}(x, y, z)\left\{r^2\chi''+3r\chi'+3\left(\chi-\frac{e\zeta}{c}\right)\right\}\right\},$$
$$(\alpha, \beta, \gamma)=\frac{c}{r^3}(y, -x, 0)(r\chi''+\chi') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where χ denotes $\chi(ct-r)$, and c is the velocity of radiation. χ and $\frac{e\zeta}{c}$ are small after the manner of F.

The surface condition is taken to be the continuity of (X, Y, Z). Whether this or the more probable (X′, Y′, Z′), the electromagnetic force, is to be continuous does not matter in the present case, as they only differ by an order F^2. The surface condition yields, if $\xi=ct-a$, and if the tangential component is zero inside,

$$\frac{d^2\chi}{d\xi^2}+\frac{1}{a}\frac{d\chi}{d\xi}+\frac{1}{a^2}\left(\chi-\frac{e\zeta}{c}\right)=\frac{aF}{c} \quad . \quad . \quad . \quad . \quad (2)$$

and the surface density is found to be given by

$$4\pi\sigma=\frac{e}{a^2}+\frac{cP_1}{a^2}\left(\frac{3a^3F}{c}-2a^2\chi''\right) \quad . \quad . \quad . \quad (3)$$

leading to a mechanical force on the sphere of magnitude

$$eF-\frac{2}{3}\frac{ec}{a}\chi''(ct-a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

along the axis of z, so that if m be the Newtonian mass,

$$m\ddot{\zeta}+\frac{2}{3}\frac{ec}{a}\chi''(ct-a)=e\mathrm{F}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

with initial conditions

$$\zeta=\dot{\zeta}=0 \quad \text{at } t=0 \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the sphere being initially at rest with ζ vanishing.

Other conditions may be deduced from the consideration that the undisturbed portion of the external medium commences where $r=ct+a$, so that $\chi(ct-r)=\chi'(ct-r)=0$ when $r=ct+a$, or

$$\chi(-a)=\chi'(-a)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The solution of these equations and conditions is, so far as ζ is concerned,

$$\zeta=-\frac{2}{3}\frac{e\mathrm{A}}{mac}e^{-ct/2a}\sin\left\{\left(3+\frac{4m'}{m}\right)^{\frac{1}{2}}\frac{ct}{2a}+\epsilon\right\}$$

$$+\frac{1}{2}\frac{e\mathrm{F}}{m+m'}t^2-\frac{e\mathrm{F}m'}{(m+m')^2}\frac{at}{c}-\frac{1}{3}\cdot e\mathrm{F}\frac{(2m^2+4mm'-m'^2)}{(m+m')^2}\cdot\frac{a^2}{c^2},$$

where

$$m'=\frac{2}{3}\cdot\frac{e^2}{ac^2}, \quad \mathrm{A}\sin\epsilon=-\mathrm{D}', \quad \left(3+\frac{4m'}{m}\right)^{\frac{1}{2}}\mathrm{A}\cos\epsilon=-(\mathrm{D}'+2a\mathrm{B}')$$

$$\mathrm{D}'=\frac{m(2m^2+4mm'-m'^2)}{2(m+m')^3}\frac{a^3\mathrm{F}}{c}, \quad \mathrm{B}'=\frac{3}{2}\cdot\frac{mm'}{(m+m')^2}\frac{a^2\mathrm{F}}{c} \quad . \quad . \quad (8)$$

An error of sign has crept into one of the terms as given by Walker, and continues in some of the later analysis, though not interfering with the general conclusions. The value given above has been corrected in this respect.

The corresponding value of χ becomes

$$\chi(ct-r)=\mathrm{A}e^{-(ct-r+a)/2a}\sin\left\{\left(3+\frac{4m'}{m}\right)^{\frac{1}{2}}\frac{ct-r+a}{2a}+\epsilon\right\}$$

$$+\mathrm{A}'(ct-r+a)^2+\mathrm{B}'(ct-r+a)+\mathrm{D}'$$

where

$$\mathrm{A}'=\frac{3}{4}\cdot\frac{m'}{m+m'}\cdot\frac{a\mathrm{F}}{c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In Walker's formula (17), p. 264, for the value of χ after the vibrations have subsided, there is an incorrect sign in the second term.

We proceed to an examination of the case in which m is very small. It may be shown without difficulty that the various constants take the forms

$$A'=\frac{3}{4}\frac{aF}{c},\quad B'=\frac{3}{2}\cdot\frac{m}{m'}\cdot\frac{a^2F}{c},\quad D'=-\frac{m}{2m'}\cdot\frac{a^3F}{c},$$

$$A\sin\epsilon=\frac{m}{2m'}\cdot\frac{a^3F}{c},\quad A\cos\epsilon=-\frac{5}{4}\cdot\left(\frac{m}{m'}\right)^{\frac{3}{2}}\frac{a^3F}{c},$$

so that on reduction,

$$\zeta=-\frac{ea^2F}{c^2m'}e^{-ct/2a}\left\{\cos.\frac{ct}{a}\left(\frac{m'}{m}\right)^{\frac{1}{2}}-\frac{5}{2}\left(\frac{m}{m'}\right)^{\frac{1}{2}}\sin.\frac{ct}{a}\left(\frac{m'}{m}\right)^{\frac{1}{2}}\right\}$$
$$+\frac{1}{2}\frac{eF}{m'}t^2-\frac{eF}{m'}\frac{at}{c}+\frac{1}{3}\frac{eF}{m'}\frac{a^2}{c^2}\quad\ldots\ldots\ldots\quad(10)$$

and, except at $t=0$, ζ tends to involve the sine and cosine of an infinite angle as the Newtonian mass decreases to zero. But even in the immediate neighbourhood of the limit $m=0$, it may be shown that this expression continues, like the corresponding value of χ, to satisfy all the conditions of the problem, and moreover, that no other forms can do so. Whatever the interpretation to be put upon the sine and cosine when m is zero, they cannot exceed unity, so that the vibrational term of ζ will very rapidly disappear on account of the damping. A slight departure from the usual condition of perfect conductivity in the sphere may perhaps remove the indeterminate character of the limit, by preventing the argument of the sine and cosine from becoming infinite, so that when $t=0$, this argument vanishes, and the initial conditions continue to be satisfied with no Newtonian mass present. On this supposition, ζ and $\dot{\zeta}$ vanish with t, and the initial conditions are satisfied, although the initial *acceleration* of the sphere would be practically infinite.

The displacement of the sphere may be regarded as a superposition of a periodic part upon a part corresponding to uniformly accelerated motion, and the damping factor is such that the periodic portion is evanescent after an extremely small time. The displacement thus tends to the form

$$\zeta=\frac{1}{2}\frac{eF}{m'}t^2-\frac{eF}{m'}\frac{at}{c}+\frac{1}{3}\frac{eF}{m'}\frac{a^2}{c^2}\quad\ldots\ldots\quad(11)$$

In the formula as given by Walker (p. 268) the sign of the second term is positive, and the factor $\frac{1}{3}$ has been dropped in the last term.

We proceed to a determination of the surface density on the sphere. The coefficient of the zonal harmonic term in $4\pi\sigma$ becomes from (3)

$$3\mathrm{F}-\frac{2c}{a}\chi'',$$

and, with a little reduction, it may be shown that the part of $\ddot{\zeta}$ not evanescent on account of m is

$$\frac{e\mathrm{F}}{m'}+\frac{1}{3}\frac{e\mathrm{F}}{m}e^{-ct/2a}\cos.\frac{ct}{a}\left(\frac{m'}{m}\right)^{\frac{1}{2}}$$

and thus by (5)

$$\chi''=\frac{3a}{2ec}(e\mathrm{F}-m\ddot{\zeta})$$

$$=\frac{3\mathrm{F}a}{2c}-\frac{a\mathrm{F}}{2c}e^{-ct/2a}\cos.\frac{ct}{a}\left(\frac{m'}{m}\right)^{\frac{1}{2}}; \quad . \quad . \quad . \quad (12)$$

so that the surface density is finally given by

$$4\pi\sigma=\frac{e}{a^2}+\mathrm{F}\cos\theta\,.\,e^{-ct/2a}\cos.\frac{ct}{a}\left(\frac{m'}{m}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (13)$$

and tends very rapidly to the uniform value belonging to a sphere at rest with no applied field, whatever the meaning given to the cosine. This conclusion is in accord with that of Conway. Thus a sphere with no Newtonian mass must move, when placed in an electric field of small intensity, without a change in its electrical distribution, if the usual conditions for a perfect conductor can continue to be valid. The value of σ at $t=0$, before the field has influenced the distribution by setting up vibrations, is of course

$$\sigma=\frac{1}{4\pi}\left(\frac{e}{a^2}+\mathrm{F}\cos\theta\right). \quad . \quad . \quad . \quad . \quad (14)$$

When Newtonian mass is present, the surface density soon settles down to the steady value

$$\sigma=\frac{1}{4\pi}\left(\frac{e}{a^2}+\frac{3m}{m+m'}\mathrm{F}\cos\theta\right) \quad . \quad . \quad (15)$$

(Walker's first result for this case, given in (19) p. 265, is corrected in a footnote in the second paper), and for a large value of m, gives the ordinary electrostatic formula, as it should.

Effect of a small Mechanical Force.

The corresponding solution for a small applied force of purely mechanical nature, which we may call G, has been given in Walker's second paper. With the previous notation, the primary equations, of which the first expresses the vanishing of the tangential electric or electromagnetic force (these only differ to the second order) at the surface, become

$$a^2\chi''(ct-a)+a\chi'(ct-a)+\chi(ct-a)-\frac{e\zeta}{c}=0$$

$$m\ddot{\zeta}+\frac{2}{3}\frac{ec}{a}\chi''(ct-a)=\mathrm{G} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

with the conditions

(1) $\chi=\chi'=0$ when the functions have argument $-a$,

(2) $\zeta=\dot{\zeta}=0$ at $t=0$; . . . . (17)

and the solutions are

$$\chi(ct-r)=\mathrm{A}e^{-(ct-r+a)/2a}\sin\left\{\left(3+\frac{4m'}{m}\right)^{\frac{1}{2}}\frac{ct-r+a}{2a}+\epsilon\right\}$$
$$+\frac{1}{2}\cdot\frac{e\mathrm{F}}{c^3(m+m')}\left\{(ct-r+a)^2-\frac{2am'}{m+m'}(ct-r+a)-\frac{2a^2mm'}{(m+m')^2}\right\}$$

$$\zeta=-\frac{2}{3}\frac{e\mathrm{A}}{mac}e^{-ct/2a}\sin\left\{\left(3+\frac{4m'}{m}\right)^{\frac{1}{2}}\frac{ct}{2a}+\epsilon\right\}$$
$$+\frac{1}{2}\frac{\mathrm{F}}{m+m'}\left\{t^2+\frac{2m'}{m+m'}\cdot\frac{at}{c}+\frac{2m'^2}{(m+m')^2}\frac{a^2}{c^2}\right\}, \quad . \quad . \quad . \quad . \quad (18)$$

where

$$\mathrm{A}\sin\epsilon=\frac{e\mathrm{F}a^2mm'}{c^3(m+m')^3},\quad \left(3+\frac{4m'}{m}\right)^{\frac{1}{2}}\mathrm{A}\cos\epsilon=\frac{e\mathrm{F}a^2m(2m+3m')}{c^3(m+m')^3}.$$

When the Newtonian mass becomes small,

$$\mathrm{A}\sin\epsilon=\frac{e\mathrm{F}a^2m}{c^3m'^2},\quad \mathrm{A}\cos\epsilon=\frac{3}{2}\cdot\frac{e\mathrm{F}a^2}{c^3m'}\left(\frac{m}{m'}\right)^{\frac{3}{2}},$$

and on reduction,

$$\zeta=-\frac{\mathrm{G}a^2}{c^2m'}e^{-ct/2a}\left\{\cos.\frac{ct}{a}\left(\frac{m'}{m}\right)^{\frac{1}{2}}+\frac{3}{2}\left(\frac{m}{m'}\right)^{\frac{1}{2}}\sin\frac{ct}{a}\left(\frac{m'}{m}\right)^{\frac{1}{2}}\right\}$$
$$+\frac{\mathrm{G}}{2m'}\left(t^2+\frac{2at}{c}+\frac{2a^2}{c^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

satisfying all necessary conditions for values of m tending to zero. The formula for the surface density of electrification at any time is

$$4\pi\sigma = \frac{e}{a^2} - \frac{2c}{a}\chi''\cos\theta. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

But

$$\chi''(ct-a) = \frac{3a}{2ec}(G - m\ddot{\zeta}),$$

and finally, when m is nearly zero,

$$4\pi\sigma = \frac{e}{a^2} - \frac{3G}{e}\cos\theta\left(1 - e^{-ct/2a}\cos\frac{ct}{a}\left(\frac{m'}{m}\right)^{\frac{1}{2}}\right). \quad . \quad (21)$$

A constant surface density (as regards time) is therefore speedily established, with a term involving the first zonal harmonic. Initially, the value is

$$\sigma = e/4\pi a^2,$$

as it should be.

But the infinite acceleration with $m=0$ again appears, although it may be formally shown that these are the only expressions capable of satisfying all the hypothetical conditions. The motion does not seem, therefore, to be physically likely to occur, and the results serve to indicate that an assumption of perfect conductivity with the ordinary condition cannot readily be justified in an accelerated system, and is of a very artificial character. That the electrical motions of the conductor should be confined to the surface in this case is very unlikely, and in the case of a single electron, it is difficult to find a physical meaning for the assumption.

In the more difficult case in which the sphere has a steady motion on which a longitudinal or transverse acceleration is superposed, a calculation of the electrical inertia on the basis of the two usually adopted surface conditions only leads to two values which must be regarded as somewhat arbitrary, and although one formula may be more supported by, for example, the experiments of Kaufmann, than the other, it still remains as but one of many perhaps equally likely results. The agreement with experiment may indicate that the proper vector has been made continuous, but not that it is zero inside the conductor. Yet in the present state of the theory, it seems necessary to emphasise Walker's contention that the Newtonian type of analysis affords the safest mode of attack on the problems of accelerated motion.

The contracted electron is rejected by Walker as having no apparent dynamical foundation, but this may be only apparent, and certainly it does not seem possible to dispense with the Principle of Relativity and its consequences. Moreover, Bucherer's contracted electron gives a very good agreement with Kaufmann's experiments, and it is desirable that a direct mode of analytical treatment of an electron which changes its shape, not associated with the quasi-stationary principle, should be found, but none has been suggested as yet.

There is one combination of a small mechanical force with a weak electric field which would give a finite initial acceleration to a sphere whose inertia is electrical only, no electrical effect being maintained inside. This combination satisfies the condition

$$G = -\tfrac{1}{3}eF, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and the corresponding value of ζ is the limit of

$$\zeta = \frac{4}{3}\frac{ea^2F}{c^2m'}\left(\frac{m}{m'}\right)^{\frac{1}{2}} e^{-ct/2a}\sin.\frac{ct}{a}\left(\frac{m'}{m}\right)^{\frac{1}{2}} + \frac{1}{3}\frac{eF}{m'}t^2 - \frac{4}{3}\frac{eF}{m'}\frac{at}{c}. \quad (23)$$

so that the acceleration at $t=0$ is $\frac{2}{3}\frac{eF}{m'}$. But it becomes infinite afterwards. The surface density remains permanently equal to

$$\frac{1}{4\pi}\left(\frac{e}{a^2} + F\cos\theta\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

so long at least as ζ is small.

In connexion with the question of electrical inertia, the investigations of Conway and Walker, starting from the same differential equations and surface conditions, lead to different values of the transverse inertia, that of Conway being identical with Abraham's expression. A comparison of the two methods will be made in a later note, for it seems that the formula given by Walker in this case is the only possible result of a rigorous analysis applied with the vanishing of the tangential electromagnetic force as its surface condition.

Trinity College, Cambridge,
1910, May 28th.