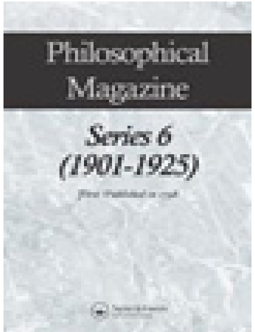

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VIII. The stability of flow of an incompressible viscous fluid

Professor A.H. Gibson D.Sc.

(3) An explanation of this effect is suggested, based on the assumption of a non-uniform distribution of the ionization produced by Röntgen rays, the final values of α corresponding to an effectively uniform distribution.

(4) The value of α is shown to fall off as the pressure of the gas is reduced.

I wish to express my gratitude to Prof. E. M. Wellisch for suggesting these experiments, and for his very helpful interest throughout the work.

Sloane Laboratory,
Yale University.
August, 1912.

VIII. *The Stability of Flow of an Incompressible Viscous Fluid.* *By* Professor A. H. Gibson, *D.Sc., University College, Dundee* *.

As a result of experiment Osborne Reynolds † concluded that the conditions tending to stability of flow, *i. e.* to stream-line as opposed to sinuous flow, of an incompressible viscous fluid are :—

(1) Free (exposed to air) surfaces.

(2) Converging boundaries.

(3) Curvilinear motion with the velocity greatest at the outside of the curve.

(4) An increase in viscosity.

(5) A diminution in density.

From an examination of the equations of motion he also showed ‡ that conditions (4) and (5) might be predicted from theoretical considerations.

As regards conclusion (3), recent experiments by the author § indicate that for increased stability in curvilinear motion the greatest velocity should be at the inside, not at the outside of the curve, and the object of the present paper is to point out that this, as well as the truth of conclusions (1) and (2), might also be inferred from the general equations of motion.

* Communicated by the Author.
† Proc. Royal Institution of Great Britain, 1884.
‡ Phil. Trans. 1883.
§ Memoirs Manchester Lit. and Phil. Soc. vol. lv. (1910-11), ii.

For such a fluid, subject to no external forces, these equations may be written

$$\left.\begin{aligned}\rho\frac{du}{dt}&=-\left\{\frac{d}{dx}(p_{xx}+\rho uu)+\frac{d}{dy}(p_{yx}+\rho uv)+\frac{d}{dz}(p_{zx}+\rho uw)\right\}\\ \rho\frac{dv}{dt}&=-\left\{\frac{d}{dx}(p_{xy}+\rho vu)+\frac{d}{dy}(p_{yy}+\rho vv)+\frac{d}{dz}(p_{zy}+\rho vw)\right\}\\ \rho\frac{dw}{dt}&=-\left\{\frac{d}{dx}(p_{xz}+\rho wu)+\frac{d}{dy}(p_{yz}+\rho wv)+\frac{d}{dz}(p_{zz}+\rho ww)\right\}\end{aligned}\right\} \quad (1$$

and
$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Multiplying respectively by u, v, w, integrating, adding, and writing

$$E=\frac{\rho}{2}(u^2+v^2+w^2)$$

the rate of increase of kinetic energy per unit volume is given by

$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}\right)E=$$

$$-\left\{\begin{aligned}&\frac{d}{dx}(up_{xx})+\frac{d}{dy}(up_{yx})+\frac{d}{dz}(up_{zx})\\ &+\frac{d}{dx}(vp_{xy})+\frac{d}{dy}(vp_{yy})+\frac{d}{dz}(vp_{zy})\\ &+\frac{d}{dx}(wp_{xz})+\frac{d}{dy}(wp_{yz})+\frac{d}{dz}(wp_{zz})\end{aligned}\right\}$$

$$+\left\{\begin{aligned}&p_{xx}\frac{du}{dx}+p_{yx}\frac{du}{dy}+p_{zx}\frac{du}{dz}\\ &+p_{xy}\frac{dv}{dx}+p_{yy}\frac{dv}{dy}+p_{zy}\frac{dv}{dz}\\ &+p_{xz}\frac{dw}{dx}+p_{yz}\frac{dw}{dy}+p_{zz}\frac{dw}{dz}\end{aligned}\right\} \quad . \quad . \quad (3)$$

The first term on the right of this equation represents the rate at which work is being done by the surrounding fluid per unit of volume, so that, for conservation of energy, as pointed out by Stokes * the second term on the right must

* Cambridge Phil. Trans. vol. ix. p. 57.

equal the rate of conversion of energy of motion into heat energy, or *vice versâ*.

If u, v, w in these equations denote the component velocities at a given point in the fluid, and if $\bar{u}$, $\bar{v}$, $\bar{w}$ represent the velocities of the instantaneous centre of gravity of the element of fluid surrounding this point, while u', v', w' represent the velocities relative to this centre of gravity, Reynolds has pointed out * that two equations of energy may be obtained, one dealing with the mean motions $\bar{u}$, $\bar{v}$, $\bar{w}$, and the other with the relative motions u', v', w', and from a consideration of these equations has shown that the limit of stability of motion is attained when

$$-\rho\iiint\left\{\begin{array}{l} u'u'\dfrac{d\bar{u}}{dx}+u'v'\dfrac{d\bar{u}}{dy}+u'w'\dfrac{d\bar{u}}{dz} \\ +v'u'\dfrac{d\bar{v}}{dx}+v'v'\dfrac{d\bar{v}}{dy}+v'w'\dfrac{d\bar{v}}{dz} \\ +w'u'\dfrac{d\bar{w}}{dx}+w'v'\dfrac{d\bar{w}}{dy}+w'w'\dfrac{d\bar{w}}{dz} \end{array}\right\}dx\,.\,dy\,.\,dz,$$

$$=\mu\iiint\left\{\begin{array}{l} 2\left\{\left(\dfrac{du'}{dx}\right)^2+\left(\dfrac{dv'}{dy}\right)^2+\left(\dfrac{dw'}{dz}\right)^2\right\} \\ +\left(\dfrac{dw'}{dy}+\dfrac{dv'}{dx}\right)^2+\left(\dfrac{du'}{dz}+\dfrac{dw'}{dx}\right)^2 \\ +\left(\dfrac{dv'}{dx}+\dfrac{du'}{dy}\right)^2 \end{array}\right\}dx\,.\,dy\,.\,dz. \quad (4)$$

The left-hand side of this equation represents the conversion of energy of mean motion into energy of relative motion, while the right-hand side represents the conversion of the energy of relative motion into heat. So long as the first of these terms is less than the second, the motion as a whole is steady, while if the second is the smaller, the motion is essentially unstable, and eddies are formed.

By integrating equation (4) it may be shown † that in the case of flow through a uniform circular tube, the condition for stability is $\dfrac{\mathrm{VL}\rho}{\mu}<\kappa$ where κ is a definite numerical constant.

An examination of the equation, however, enables further general conclusions to be drawn as to the effect of any variation from a state of rectilinear motion, on the stability

* Phil. Trans. 1895.

† Scientific Papers, "Reynolds," vol. ii. p. 561. Also Phil. Trans. *ibid.*

of flow. Evidently any factor which tends to diminish the relative magnitude of the left-hand side of the equation tends to reduce the tendency to eddy formation and to increase the stability of flow and *vice versâ*. Thus, as is well known, an increase in μ or a diminution in ρ tends to increased stability.

Effect of Free Boundaries.—With "free" as opposed to "solid" boundaries, the boundary velocities are increased and, assuming flow to take place in the direction of increasing x, the magnitudes of $\frac{d\bar{u}}{dy}$ and of $\frac{d\bar{u}}{dz}$ are diminished. Since these are negative and since the quantity in brackets on the left-hand side of equation (4) is essentially negative, this tends to diminish this term and hence to increase the stability of flow.

Effect of Converging Boundaries.—The same effect may be seen to follow the introduction of converging as opposed to parallel boundaries, or indeed any convergence of stream lines accompanying a change from pressure to kinetic energy. In such a case $\frac{d\bar{u}}{dx}$, $\frac{d\bar{v}}{dy}$, and $\frac{d\bar{w}}{dz}$ instead of being zero become positive and thus diminish the absolute (negative) value of the term in brackets. Conversely, a retardation in the direction of flow, such as is produced by diverging boundaries, diminishes the stability of flow.

Effect of Curvature of the Path.—If, in curvilinear motion, the direction of x be taken as tangential to the path at a given point and if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are positive, the value of $\frac{d\bar{u}}{dy}$ is greater if the velocity is greatest at the outside of the curve, than it is with rectilinear motion. Since, under these conditions, $\frac{d\bar{u}}{dy}$ is essentially negative, the absolute value of the first term of (4) is greater than with rectilinear motion, and the stability of motion is consequently diminished. Conversely, with the velocity greatest at the inside of the curve the stability is increased.

University College, Dundee.