67. Circles Are Drawn through Two Fixed Points. Trace the Path of a Point Which Cuts Them All at the Same Angle
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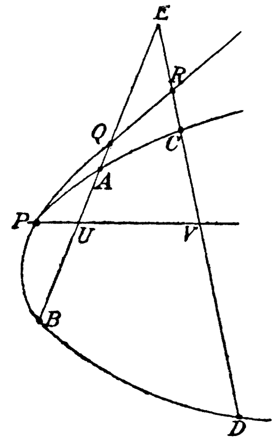
66. *Note on the parabola through four concyclic points.*

Let A, B, C, D be given fixed concyclic points upon a given parabola. Produce AB, CD, which are equally inclined to the axis, to meet externally at E. At *any* point P on the curve draw the tangent PQR and the diameter PUV, meeting AB, CD in Q, R and U, V respectively.



Then, by a known property of the parabola,

$$QU^2 = QA \cdot QB, \text{ and } RV^2 = RC \cdot RD$$

(*Milne & Davis*, p. 26). Hence QR is the radical axis of the circle through A, B, C, D, and a circle touching EAB, ECD in U, V.

Conversely, the envelope of the radical axis of a given circle and a variable circle touching two given fixed straight lines is one of the two parabolas through the four points in which the straight lines intersect the fixed circle. [One parabola corresponds to the series of variable circles inscribed within the acute angle between the straight lines, and another to those within the obtuse angle.]

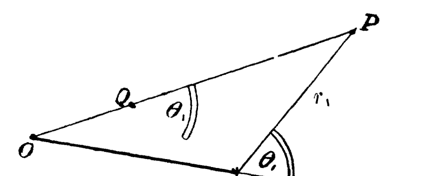
It will be seen that the above property furnishes a method of drawing the tangent *in any proposed direction* to the parabola through four given concyclic points; whence the tangent at the vertex and directrix can be determined.

(Cf. *Educational Times*, May, 1898, Question 13840.) R. F. Davis.

67. *Circles are drawn through two fixed points. Trace the path of a point which cuts them all at the same angle.* (Question 172, p. 89.)

Let O, A be the two fixed points.

Invert the system of circles, taking O as centre and OA $(=a)$ as radius of inversion. The inverted system is a system of straight lines through A.



If A is taken as pole, the spiral curve which cuts these straight lines at a given angle α has the equation

$$r_1 = be^{\mu\theta_1}, \text{ ....(1)}$$

where $\mu = \cot \alpha$, and b is any constant length. The inverse of this spiral with regard to O is the curve required.

Let P be a point on the equiangular spiral, and let Q be its inverse point.

Let
$$OQ = \rho, \quad AQ = \rho',$$
$$QOA = \theta, \quad QAO' = \theta'.$$

Then, in the similar triangles OAP, OQA,

$$\frac{r_1}{a} = \frac{\rho'}{\rho},$$

also
$$\theta' - \theta = \pi - \theta_1;$$

therefore from (1)

$$a\rho' = b\rho \,.\, e^{\mu(\pi+\theta-\theta')} = c \,.\, \rho e^{\mu(\theta-\theta')}, \text{ if } c = be^{\mu\pi}, \text{ i.e. } a\rho' e^{\mu\theta'} = c\rho e^{\mu\theta}.$$

This is the equation of the required curve in bipolar coordinates. Its equation in rectangular or polar coordinates could easily be deduced from this, but would be too complicated for use in drawing the curve, which is probably most easily obtained from the intersections of the equiangular spirals

$$\rho = kae^{-\mu\theta}, \quad \rho' = kce^{-\mu\theta'},$$

drawn for different values of k, which is easily done after one has been drawn.

To save labour the values of k should proceed geometrically, and c should be equal to one of the values of ka. The two systems of curves are then merely differently placed repetitions of each other, one system having O as pole, the pole of the other system being A. Indeed, they are all repetitions of one equiangular spiral, its various positions round O (or A) being obtained by successive rotations round O (or A) through a definite angle each time if k increases geometrically as suggested.

The curves required pass through the intersections of consecutive spirals, and are themselves spiral curves, each coiling an infinite number of times round O in gradually widening circuits, and then coiling round A an infinite number of times in gradually decreasing circuits (or, of course, *vice versâ*). One of the curves (viz. when $c=a$) goes off to infinity after coiling round O, and comes back from the opposite direction of infinity to coil round A, its direction at infinity being parallel to the line $\theta=\alpha$.

If the coaxial circles have real limiting points instead of real points of intersection, O and A will be the limiting points, and μ will then $= \tan\alpha$ instead of $\cot\alpha$.

If A coincides with O, the curves become another similar system of circles through O, the common tangents at O to the two systems cutting each other at the required angle. A. LODGE.

[The general curve of the whole system for all values of the constant angle of section is a spiral with two poles O, A, which does not go to infinity, and which takes roughly the shape of the letter S. The whole system, however, includes three systems which form exceptions to this statement. One is the system which goes to infinity ; and the other two are the system of circles through O, A, and a second system of circles cutting the first or given system orthogonally.

It is clear that any one of the spirals not only cuts the first system of circles at a constant angle, but also the second system. This second system is also a coaxial system, but with imaginary common points, having the poles of the spirals for limiting points. The spirals may be considered as being given by either system of circles.

The whole system of spirals when inverted with respect to any point becomes an exactly similar system, the new poles being the inverses of the old ones. This follows from the fact that angles are unaltered by inversion, and circles invert into circles.

If one pole goes to infinity, the spirals become equiangular spirals, with the remaining pole as pole. Hence, by regarding an equiangular spiral as having in reality two poles, one of which is at infinity, we may regard the system of spirals with two finite poles as being a very simple generalisation of equiangular spirals. From this aspect their properties with respect to inversion and their relation to infinity gain in interest.

A further generalization of an equiangular spiral is the curve cutting at a constant angle the system of circles which touch two given circles ; and this includes, as a special case, the curve which cuts the tangents to a given circle at a constant angle. F. S. MACAULAY.]

68. *In a spherical quadrangle the arcs joining the middle points of the three pairs of opposite sides are concurrent.* (Question 90, p. 16. Taken from Casey's *Spherical Trigonometry.*)

To solve this problem by using the relations between the trigonometrical ratios of the arcs involved would appear to be a very laborious operation.