ON THE EXPANSIONS OF THE ELLIPTIC AND ZETA FUNCTIONS OF $\frac{2}{3}K$ IN POWERS OF q

By J. W. L. GLAISHER.

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1. In a paper in Vol. xxi. of the *Proceedings*,* I have given the expansions of the twelve elliptic and four zeta functions of $\frac{1}{3}K$ in powers of q, the coefficients in the expansions being expressed by means of certain arithmetical functions. Since the publication of that paper I have reduced the number of these arithmetical functions, which are required for the expansions, to two. The new forms may be deduced from those contained in the previous paper, but it seems preferable to give an independent investigation, deriving them from the general expansions of the elliptic and zeta functions. In the previous paper the results were stated without proof, the methods by which they were obtained being merely indicated.

2. The expansions in powers of q of the general elliptic and zeta functions may be written†

$$\begin{aligned}
k\rho \operatorname{sn} \rho x &= 4\Sigma_1^\infty \Delta(\sin mx)\, q^{\frac{1}{2}m}, \\
kk'\rho \operatorname{sd} \rho x &= 4\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} \Delta(\sin mx)\, q^{\frac{1}{2}m}, \\
k\rho \operatorname{cd} \rho x &= 4\Sigma_1^\infty E(\cos mx)\, q^{\frac{1}{2}m}, \\
k\rho \operatorname{cn} \rho x &= 4\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} E(\cos mx)\, q^{\frac{1}{2}m} ;
\end{aligned}$$

$$\begin{aligned}
\rho \operatorname{zn} \rho x &= 4\Sigma_1^\infty \Delta'(\sin 2nx)\, q^n, \\
\rho \operatorname{zd} \rho x &= 4\Sigma_1^\infty (-1)^n \Delta'(\sin 2nx)\, q^n, \\
\rho \operatorname{dn} \rho x &= 1+4\Sigma_1^\infty E'(\cos 2nx)\, q^n, \\
k'\rho \operatorname{nd} \rho x &= 1+4\Sigma_1^\infty (-1)^n E'(\cos 2nx)\, q^n ;
\end{aligned}$$

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* "On the q-Series derived from the Elliptic and Zeta Functions of $\frac{1}{3}K$ and $\frac{1}{4}K$," *Proceedings*, Vol. xxii., 1890, pp. 143–171.

† *Messenger of Mathematics*, Vol. xviii., 1888, p. 8.

$$\begin{aligned}
\rho \operatorname{ns} \rho x &= \operatorname{cosec} x + 4\Sigma_1^\infty \Delta(\sin nx)\, q^n, \\
\rho \operatorname{ds} \rho x &= \operatorname{cosec} x + 4\Sigma_1^\infty (-1)^n \Delta(\sin nx)\, q^n, \\
\rho \operatorname{dc} \rho x &= \sec x + 4\Sigma_1^\infty E(\cos nx)\, q^n, \\
k'\rho \operatorname{nc} \rho x &= \sec x + 4\Sigma_1^\infty (-1)^n E(\cos nx)\, q^n;
\end{aligned}$$

$$\begin{aligned}
\rho \operatorname{zs} \rho x &= \cot x + 4\Sigma_1^\infty \sigma(\sin 2nx)\, q^{2n}, \\
\rho \operatorname{cs} \rho x &= \cot x - 4\Sigma_1^\infty \zeta'(\sin 2nx)\, q^{2n}, \\
\rho \operatorname{zc} \rho x &= -\tan x - 4\Sigma_1^\infty \zeta(\sin 2nx)\, q^{2n}, \\
k'\rho \operatorname{sc} \rho x &= \tan x - 4\Sigma_1^\infty \xi(\sin 2nx)\, q^{2n};
\end{aligned}$$

where $\rho = 2K/\pi$, n is any number, m any uneven number, and the arithmetical functions Δ, E, ... are defined as follows.

Let $\delta_1, \delta_2, \ldots$ be all the uneven divisors of any number n, and let $\delta_1', \delta_2', \ldots$ be their conjugates (*i.e.*, so that $\delta_1\delta_1' = \delta_2\delta_2' = \ldots = n$). Let $d_1, d_2, \ldots$ be all the divisors of n and $d_1', d_2', \ldots$ their conjugates.

Let
$$\begin{aligned}
\Delta\phi(n) &= \Sigma_n \phi(\delta), & \Delta'\phi(n) &= \Sigma_n \phi(\delta'), \\
E\phi(n) &= \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \phi(\delta), & E'\phi(n) &= \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \phi(\delta'), \\
\sigma\phi(n) &= \Sigma_n \phi(d), & \xi\phi(n) &= \Sigma_n (-1)^{d+d'} \phi(d), \\
\zeta\phi(n) &= \Sigma_n (-1)^{d-1} \phi(d), & \zeta'\phi(n) &= \Sigma_n (-1)^{d'-1} \phi(d),
\end{aligned}$$

where ϕ is any function and n is the number to which the divisors $\delta_1, \delta_2, \ldots, d_1, d_2, \ldots$ relate.* For example,

$$\Delta\phi(n) = \phi(\delta_1) + \phi(\delta_2) + \ldots, \qquad E\phi(n) = \pm\, \phi(\delta_1) \pm \phi(\delta_2) \pm \ldots,$$

where in the second equation the sign of the term involving δ_r is positive or negative according as δ_r is of the form $4k+1$ or $4k+3$.

These definitions are also supposed to hold good when n is replaced by any uneven number m. In this case the d's and δ's are the same.

For the expression of the coefficients the functions $E(n)$, $H(n)$, $H_1(n)$ will be used, where

$E(n) =$ the number of divisors of n of the form $4k+1$
$-$ „ „ „ $4k+3$,†

$H(n) =$ the number of divisors of n of the form $3k+1$
$-$ „ „ „ $3k+2$,

$H_1(n) =$ the number of divisors of n of the form $6k+1$
$-$ „ „ „ $6k+5$.

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* Thus, when the argument is $2nx$, it is with the divisors of n, not $2n$, that we are concerned. For example, $\sigma(\sin 2nx) = \sin 2d_1x + \sin 2d_2x + \ldots$, where $d_1, d_2, \ldots$ are the divisors of n.

† This definition of $E(n)$ is inconsistent with the general definition of $E\phi(n)$, but, as $\Sigma^n (-1)^{\frac{1}{2}(\delta-1)}\delta$ does not occur, it is convenient to use $E(n)$ to denote $\Sigma_n (-1)^{\frac{1}{2}(\delta-1)}$.

3. Putting $x = \frac{1}{3}\pi$, the first group of expansions becomes

$$
\begin{aligned}
k\rho \operatorname{sn} \tfrac{2}{3}K &= 4\Sigma_1^\infty \,.\, \Sigma_m \sin \tfrac{1}{3}\delta\pi \,.\, q^{\frac{1}{2}m},\\
kk'\rho \operatorname{sd} \tfrac{2}{3}K &= 4\Sigma_1 \; (-1)^{\frac{1}{2}(m-1)}.\, \Sigma_m \sin \tfrac{1}{3}\delta\pi \,.\, q^{\frac{1}{2}m},\\
k\rho \operatorname{cd} \tfrac{2}{3}K &= 4\Sigma_1 \;.\, \Sigma_m (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi \,.\, q^{\frac{1}{2}m},\\
k\rho \operatorname{cn} \tfrac{2}{3}K &= 4\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)}.\, \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi \,.\, q^{\frac{1}{2}m}.
\end{aligned}
$$

Now $\sin \frac{1}{3}\delta\pi = \frac{1}{2}\sqrt{3}$ if δ is of the form $6k+1$, and $= -\frac{1}{2}\sqrt{3}$ if δ is of the form $6k+5$. If δ is divisible by 3, it is zero. Therefore

$$\Sigma_m \sin \tfrac{1}{3}\delta\pi = \tfrac{1}{2}\sqrt{3}\, H_1(m).$$

To calculate the value of $\Sigma_m (-1)^{\frac{1}{2}(\delta-1)} \cos \frac{1}{3}\delta\pi$ we notice that, if δ is not divisible by 3, $\cos \frac{1}{3}\delta\pi = \frac{1}{2}$, and that, if δ is divisible by 3,

$$\cos \tfrac{1}{3}\delta\pi = -1 = \tfrac{1}{2} - \tfrac{3}{2}.$$

Thus $$\Sigma_m (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi = \tfrac{1}{2}\Sigma_m (-1)^{\frac{1}{2}(\delta-1)} - \tfrac{3}{2}\Sigma_m (-1)^{\frac{1}{2}(\epsilon-1)},$$

where the second term occurs only when m is divisible by 3, in which case ϵ is any divisor of m which is divisible by 3.

The first term $= \frac{1}{2}E(m)$. To express the second term, let $m = 3\mu$; then $\epsilon_r = 3\eta_r$, where η_r is any divisor of μ; therefore

$$\Sigma_m (-1)^{\frac{1}{2}(\epsilon-1)} = \Sigma_\mu (-1)^{\frac{1}{2}(3\eta-1)} = -\Sigma_\mu (-1)^{\frac{1}{2}(\eta-1)} = -E(\mu) = -E(\tfrac{1}{3}m).$$

We thus find

$$\Sigma_m (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi = \tfrac{1}{2}E(m) + \tfrac{3}{2}E(\tfrac{1}{3}m).$$

This equation also holds good when m is not divisible by 3, if we define $E(r)$ to be zero when r is fractional.

The group of expansions therefore becomes

$$
\begin{aligned}
k\rho \operatorname{sn} \tfrac{2}{3}K &= 2\sqrt{3}\, \Sigma_1^\infty H_1(m)\, q^{\frac{1}{2}m},\\
kk'\rho \operatorname{sd} \tfrac{2}{3}K &= 2\sqrt{3}\, \Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} H_1(m)\, q^{\frac{1}{2}m},\\
k\rho \operatorname{cd} \tfrac{2}{3}K &= 2\Sigma_1^\infty \left\{ E(m) + E(\tfrac{1}{3}m) \right\} q^{\frac{1}{2}m}.\\
k\rho \operatorname{cn} \tfrac{2}{3}K &= 2\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} \left\{ E(m) + E(\tfrac{1}{3}m) \right\} q^{\frac{1}{2}m}.
\end{aligned}
$$

4. Putting $x = \frac{1}{3}\pi$, the second group is

$$
\begin{aligned}
\rho \operatorname{zn} \tfrac{2}{3}K &= \phantom{1+{}} 4\Sigma_1^\infty \,.\, \Sigma_n \sin \tfrac{2}{3}\delta'\pi \,.\, q^n,\\
\rho \operatorname{zd} \tfrac{2}{3}K &= \phantom{1+{}} 4\Sigma_1^\infty .\, (-1)^n\, \Sigma_n \sin \tfrac{2}{3}\delta'\pi \,.\, q^n,\\
\rho \operatorname{dn} \tfrac{2}{3}K &= 1 + 4\Sigma_1^\infty \,.\, \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{2}{3}\delta'\pi \,.\, q^n,\\
k'\rho \operatorname{nd} \tfrac{2}{3}K &= 1 + 4\Sigma_1^\infty .\, (-1)^n\, \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{2}{3}\delta'\pi \,.\, q^n.
\end{aligned}
$$

Consider the value of $\quad A = \Sigma_m \sin \frac{2}{3}\delta'\pi.$

If n is uneven, the system of numbers δ' consists of $\delta_1, \delta_2, \ldots$, and $A = \frac{1}{2}\sqrt{3}\, H_1(n)$. If $n = 2m$ (m being uneven), the system δ' consists of $2\delta_1, 2\delta_2, \ldots$ and $A = -\frac{1}{2}\sqrt{3}\, H_1(n)$; if $n = 4m$, the system δ' consists of $4\delta_1, 4\delta_2, \ldots$ and $A = \frac{1}{2}\sqrt{3}\, H_1(n)$; and so on. Thus we find, if $n = 2^i m$,

$$\Sigma_n \sin \tfrac{2}{3}\delta'\pi = (-1)^i \tfrac{1}{2}\sqrt{3}\, H_1(n).$$

Consider now the value of

$$A = \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{2}{3}\delta'\pi.$$

We have $\qquad \cos \frac{2}{3}\delta'\pi = -\frac{1}{2}$, except when δ is divisible by 3,

and $\qquad = -\frac{1}{2} + \frac{3}{2}$ when δ is divisible by 3.

Therefore $\qquad A = -\frac{1}{2}\Sigma_n (-1)^{\frac{1}{2}(\delta-1)} + \frac{3}{2}\Sigma_n (-1)^{\frac{1}{2}(\epsilon-1)}$,

where in the second term (which occurs only when n is divisible by 3) ϵ is any uneven divisor of n whose conjugate is divisible by 3.

The first term $= -\frac{1}{2}E(n)$; and to evaluate the second term we notice that, if $n = 2^i . 3\mu$, where μ is uneven, then ϵ is any divisor of μ. Thus the second term $= \frac{3}{2}E(\frac{1}{3}n)$, and we find

$$\Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{2}{3}\delta'\pi = -\tfrac{1}{2}E(n) + \tfrac{3}{2}E(\tfrac{1}{3}n).$$

The second group of expansions therefore becomes

$$\begin{aligned}
\rho \operatorname{zn} \tfrac{2}{3}K &= 2\sqrt{3}\, \Sigma_1^\infty (-1)^i H_1(n) q^n,\\
\rho \operatorname{zd} \tfrac{2}{3}K &= 2\sqrt{3}\, \Sigma_1^\infty (-1)^n (-1)^i H_1(n) q^n,\\
\rho \operatorname{dn} \tfrac{2}{3}K &= 1 - 2\Sigma_1^\infty \{E(n) - 3E(\tfrac{1}{3}n)\} q^n,\\
k'\rho \operatorname{nd} \tfrac{2}{3}K &= 1 - 2\Sigma_1^\infty (-1)^n \{E(n) - 3E(\tfrac{1}{3}n)\} q^n.
\end{aligned}$$

5. The third group is

$$\begin{aligned}
\rho \operatorname{ns} \tfrac{2}{3}K &= \operatorname{cosec} \tfrac{1}{3}\pi + 4\Sigma_1^\infty . \Sigma_n \sin \tfrac{1}{3}\delta\pi . q^n,\\
\rho \operatorname{ds} \tfrac{2}{3}K &= \operatorname{cosec} \tfrac{1}{3}\pi + 4\Sigma_1^\infty . (-1)^n \Sigma_n \sin \tfrac{1}{3}\delta\pi . q^n,\\
\rho \operatorname{dc} \tfrac{2}{3}K &= \quad \sec \tfrac{1}{3}\pi + 4\Sigma_1^\infty . \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi . q^n,\\
k'\rho \operatorname{nc} \tfrac{2}{3}K &= \quad \sec \tfrac{1}{3}\pi + 4\Sigma_1^\infty . (-1)^n \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi . q^n.
\end{aligned}$$

If $n = 2^i m$,

$$\Sigma_n \sin \tfrac{1}{3}\delta\pi = \Sigma_m \sin \tfrac{1}{3}\delta\pi = \tfrac{1}{2}\sqrt{3}\, H_1(m) = \tfrac{1}{2}\sqrt{3}\, H_1(n) \quad (\S\ 3),$$

$$\begin{aligned}
\Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi = \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi &= \tfrac{1}{2}E(m) + \tfrac{3}{2}E(\tfrac{1}{3}m)\\
&= \tfrac{1}{2}E(n) + \tfrac{3}{2}E(\tfrac{1}{3}n) \quad (\S\ 3);
\end{aligned}$$

and therefore

$$\begin{aligned} \rho \operatorname{ns} \tfrac{2}{3}K &= 2/\surd 3 + 2\surd 3\, \Sigma_1^\infty H_1(n)\, q^n, \\ \rho \operatorname{ds} \tfrac{2}{3}K &= 2/\surd 3 + 2\surd 3\, \Sigma_1^\infty (-1)^n H_1(n)\, q^n, \\ \rho \operatorname{dc} \tfrac{2}{3}K &= 2 + 2\Sigma_1^\infty \left\{E(n) + 3E(\tfrac{1}{3}n)\right\} q^n, \\ k'\rho \operatorname{nc} \tfrac{2}{3}K &= 2 + 2\Sigma_1^\infty (-1)^n \left\{E(n) + 3E(\tfrac{1}{3}n)\right\} q^n. \end{aligned}$$

6. In the fourth group the four coefficients are all different in form and depend upon all the divisors, instead of only upon the uneven divisors, of n.

The expansions are

$$\begin{aligned} \rho \operatorname{zs} \tfrac{2}{3}K &= \quad \cot \tfrac{1}{3}\pi + 4\Sigma_1^\infty .\, \Sigma_n \sin \tfrac{2}{3}d\pi \,.\, q^{2n}, \\ \rho \operatorname{cs} \tfrac{2}{3}K &= \quad \cot \tfrac{1}{3}\pi + 4\Sigma_1^\infty .\, \Sigma_n (-1)^{d'} \sin \tfrac{2}{3}d\pi \,.\, q^{2n}, \\ \rho \operatorname{zc} \tfrac{2}{3}K &= -\tan \tfrac{1}{3}\pi + 4\Sigma_1^\infty .\, \Sigma_n (-1)^{d} \sin \tfrac{2}{3}d\pi \,.\, q^{2n}. \\ k'\rho \operatorname{sc} \tfrac{2}{3}K &= \quad \tan \tfrac{1}{3}\pi - 4\Sigma_1^\infty .\, \Sigma_n (-1)^{d+d'} \sin \tfrac{2}{3}d\pi \,.\, q^{2n}. \end{aligned}$$

It is evident that

$$\sin \tfrac{2}{3}d\pi = \tfrac{1}{2}\surd 3 \text{ when } d \text{ is of the form } 3k+1,$$

and $$= -\tfrac{1}{2}\surd 3 \quad \text{,,} \quad \text{,,} \quad 3k+2.$$

Therefore $$\Sigma_n \sin \tfrac{2}{3}d\pi = \tfrac{1}{2}\surd 3\, H(n),$$

and $$\rho \operatorname{zs} \tfrac{2}{3}K = 1/\surd 3 + 2\surd 3\, \Sigma_1^\infty H(n)\, q^{2n}.$$

7. A different form of the value of $\Sigma_n \sin \tfrac{2}{3}d\pi$ will now be obtained in connection with the evaluation of

$$\Sigma_n (-1)^{d'} \sin \tfrac{2}{3}d\pi, \qquad \Sigma_n (-1)^{d} \sin \tfrac{2}{3}d\pi, \qquad \Sigma_n (-1)^{d+d'} \sin \tfrac{2}{3}d\pi.$$

Let $n = 2^i m\ (i > 0)$, and let $\delta_1, \delta_2, \ldots$ be the divisors of m (which of course are all uneven). The system of divisors of n is therefore $\delta_1, \delta_2, \ldots,$ $2\delta_1, 2\delta_2, \ldots, \ldots, 2^i\delta_1, 2^i\delta_2, \ldots$. Now

$\sin \tfrac{2}{3}\delta\pi = \tfrac{1}{2}\surd 3$ or $-\tfrac{1}{2}\surd 3$ according as δ is of the form $6k+1$ or $6k+5$,

$\sin \tfrac{4}{3}\delta\pi = -\tfrac{1}{2}\surd 3$ or $\tfrac{1}{2}\surd 3$,, ,, ,, ,,

and, in general,

$\sin \tfrac{2}{3} 2^r \delta\pi = (-1)^r \tfrac{1}{2}\surd 3$ or $-(-1)^r \tfrac{1}{2}\surd 3$

according as δ is of the form $6k+1$ or $6k+5$.

Thus
$$\Sigma_n \sin \tfrac{2}{3}\delta\pi = \tfrac{1}{2}\sqrt{3}\, H_1(n),$$
$$\Sigma_n \sin \tfrac{2}{3}2\delta\pi = -\tfrac{1}{2}\sqrt{3}\, H_1(n),$$
$$\dots \quad \dots \quad \dots \quad \dots$$
$$\Sigma_n \sin \tfrac{2}{3}2^i\delta\pi = (-1)^i \tfrac{1}{2}\sqrt{3}\, H_1(n).$$

The even or uneven characters of d, d', $d+d'$, according to the different forms of d, are shown in the following table, in which $n = 2^i m$ and $i > 0$.

d	d'	$d+d'$
$\delta_1, \delta_2, \dots$ (uneven)	even	uneven
$2\delta_1, 2\delta_2, \dots$ (even)	even	even
$2^2\delta_1, 2^2\delta_2, \dots$ (even)	even	even
... ...	... ...	... ...
$2^i\delta_1, 2^i\delta_2, \dots$ (even)	uneven	uneven

It follows therefore that, if $i > 0$,
$$\Sigma_n \sin \tfrac{2}{3}d\pi = \tfrac{1}{2}\sqrt{3}\,\{1+(-1)+(-1)^2+\dots+(-1)^i\}\, H_1(n)$$
$$= \tfrac{1}{2}\sqrt{3}\,\frac{1+(-1)^i}{2} H_1(n),$$
$$\Sigma_n (-1)^{d'} \sin \tfrac{2}{3}d\pi = \tfrac{1}{2}\sqrt{3}\,\{1+(-1)+(-1)^2+\dots-(-1)^i\}\, H_1(n)$$
$$= \tfrac{1}{2}\sqrt{3}\,\frac{1-3(-1)^i}{2} H_1(n),$$
$$\Sigma_n (-1)^{d} \sin \tfrac{2}{3}d\pi = \tfrac{1}{2}\sqrt{3}\,\{-1+(-1)+(-1)^2+\dots+(-1)^i\}\, H_1(n)$$
$$= \tfrac{1}{2}\sqrt{3}\,\frac{-3+(-1)^i}{2} H_1(n),$$
$$\Sigma_n (-1)^{d+d'} \sin \tfrac{2}{3}d\pi = \tfrac{1}{2}\sqrt{3}\,\{-1+(-1)+(-1)^2+\dots-(-1)^i\}\, H_1(n)$$
$$= -\tfrac{1}{2}\sqrt{3}\,\frac{3+3(-1)^i}{2} H_1(n).$$

When n is uneven, *i.e.* when $i = 0$, the first three formulæ still hold good, but in place of the last we have
$$\Sigma_n (-1)^{d+d} \sin \tfrac{2}{3}d\pi = \tfrac{1}{2}\sqrt{3}\, H_1(n).$$

The four expansion formulæ therefore become, if $n = 2^i m$,

$$\begin{aligned}
\rho \,\mathrm{zs}\, \tfrac{2}{3}K &= 1/\sqrt{3} + \sqrt{3}\, \Sigma_1^\infty \{1+(-1)^i\}\, H_1(n) q^{2n}, \\
\rho \,\mathrm{cs}\, \tfrac{2}{3}K &= 1/\sqrt{3} + \sqrt{3}\, \Sigma_1^\infty \{1-3(-1)^i\}\, H_1(n) q^{2n}, \\
\rho \,\mathrm{zc}\, \tfrac{2}{3}K &= -\sqrt{3} - \sqrt{3}\, \Sigma_1 \{3-(-1)^i\}\, H_1(n) q^{2n}, \\
k'\rho \,\mathrm{sc}\, \tfrac{2}{3}K &= \sqrt{3} + 3\sqrt{3}\, \Sigma_1^\infty \{1+(-1)^i\}\, H_1(n) q^{2n} \quad (\text{if } i > 0 \\
&= \sqrt{3} - 2\sqrt{3}\, \Sigma_1^\infty H_1(n) q^{2n} \quad (\text{if } i = 0).
\end{aligned}$$

8. The coefficients in the expansions of the sixteen functions have therefore been expressed by means of two arithmetical functions $E(n)$ and $H_1(n)$, but in connection with the latter the factor $(-1)^i$, depending upon the structure of n, occurs. It will now be shown that this factor may be got rid of, and that all the coefficients can be expressed by means of the two functions $E(n)$ and $H(n)$.

It was shown in § 6 that

$$\Sigma_n \sin \tfrac{2}{3}d\pi = \tfrac{1}{2}\sqrt{3}\, H(n),$$

and in § 7 that, if $n = 2^i m$,

$$\Sigma_n \sin \tfrac{2}{3}d\pi = \tfrac{1}{2}\sqrt{3}\, \frac{1+(-1)^i}{2}\, H_1(n).$$

Comparing these two results, we see that

$$\{1+(-1)^i\}\, H_1(n) = 2H(n).$$

Now, if n is even,

$H(n) =$ the number of divisors of n of the forms $6k+1$ and $6k+4$

$-$ " " " $6k+5$ and $6k+2$

$= H_1(n) - H(\tfrac{1}{2}n)$;

for the divisors of n of the forms $6k+4$, $6k+2$ are the doubles of the divisors of $\tfrac{1}{2}n$ of the forms $3k+2$, $3k+1$ respectively.

We have thus obtained the formulæ

$$H_1(n) = H(n) + H(\tfrac{1}{2}n),$$

$$(-1)^i H_1(n) = 2H(n) - H_1(n) = H(n) - H(\tfrac{1}{2}n),^*$$

which still hold good when n is uneven if we define $H(r)$ to be zero when r is fractional.

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* These equations show that $H_1(n) = H(n)$ or $H(\tfrac{1}{2}n)$ according as i is even or uneven.

9. Expressing in terms of H the coefficients which have been obtained in terms of $H_1(n)$, we have

$$\begin{aligned}
\Sigma_n \sin \tfrac{2}{3}\delta'\pi &= \tfrac{1}{2}\surd 3\,(-1)^i H_1(n) = \tfrac{1}{2}\surd 3\,\{H(n)-H(\tfrac{1}{2}n)\} \quad (\S\ 4),\\
\Sigma_n \sin \tfrac{1}{3}\delta\pi &= \tfrac{1}{2}\surd 3\,H_1(n) = \tfrac{1}{2}\surd 3\,\{H(n)+H(\tfrac{1}{2}n)\} \quad (\S\ 5),\\
\Sigma_n \sin \tfrac{2}{3}d\pi &= \frac{\surd 3}{2}\,\frac{1+(-1)^i}{2}\,H_1(n) = \tfrac{1}{2}\surd 3\,H(n),\\
\Sigma_n (-1)^{d'} \sin \tfrac{2}{3}d\pi &= \frac{\surd 3}{2}\,\frac{1-3(-1)^i}{2}\,H_1(n) = -\frac{\surd 3}{2}\,\{H(n)-2H(\tfrac{1}{2}n)\},\\
\Sigma_n (-1)^{d} \sin \tfrac{2}{3}d\pi &= \frac{\surd 3}{2}\,\frac{-3+(-1)^i}{2}\,H_1(n) = -\frac{\surd 3}{2}\,\{H(n)+2H(\tfrac{1}{2}n)\},\\
\Sigma_n (-1)^{d+d'} \sin \tfrac{2}{3}d\pi &= -\frac{\surd 3}{2}\,\frac{3+3(-1)^i}{2}\,H_1(n) = -\frac{3\surd 3}{2}\,H(n) \quad (\text{if } i>0),\\
\text{and} \qquad &= \tfrac{1}{2}\surd 3\,H(n) \quad (\text{if } i=0).
\end{aligned}$$

10. Collecting the expansions, the six which depend upon E are

$$\begin{aligned}
k\rho \operatorname{cd} \tfrac{2}{3}K &= 2\Sigma_1^\infty \{E(m)+3E(\tfrac{1}{3}m)\}\,q^{\frac{1}{2}m},\\
k\rho \operatorname{cn} \tfrac{2}{3}K &= 2\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)}\{E(m)+3E(\tfrac{1}{3}m)\}\,q^{\frac{1}{2}m},\\
\rho \operatorname{dn} \tfrac{2}{3}K &= 1-2\Sigma_1^\infty \{E(n)-3E(\tfrac{1}{3}n)\}\,q^n,\\
k'\rho \operatorname{nd} \tfrac{2}{3}K &= 1-2\Sigma_1^\infty (-1)^n\{E(n)-3E(\tfrac{1}{3}n)\}\,q^n,\\
\rho \operatorname{dc} \tfrac{2}{3}K &= 2+2\Sigma_1^\infty \{E(n)+3E(\tfrac{1}{3}n)\}\,q^n,\\
k'\rho \operatorname{nc} \tfrac{2}{3}K &= 2+2\Sigma_1^\infty (-1)^n\{E(n)+3E(\tfrac{1}{3}n)\}\,q^n,
\end{aligned}$$

and the ten which depend upon H are

$$\begin{aligned}
k\rho \operatorname{sn} \tfrac{2}{3}K &= 2\surd 3\,\Sigma_1^\infty H(m)\,q^{\frac{1}{2}m},\\
kk'\rho \operatorname{sd} \tfrac{2}{3}K &= 2\surd 3\,\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)}\,H(m)\,q^{\frac{1}{2}m},\\
\rho \operatorname{zn} \tfrac{2}{3}K &= 2\surd 3\,\Sigma_1^\infty \{H(n)-H(\tfrac{1}{2}n)\}\,q^n,\\
\rho \operatorname{zd} \tfrac{2}{3}K &= 2\surd 3\,\Sigma_1^\infty (-1)^n \{H(n)-H(\tfrac{1}{2}n)\}\,q^n,\\
\rho \operatorname{ns} \tfrac{2}{3}K &= 2/\surd 3+2\surd 3\,\Sigma_1^\infty \{H(n)+H(\tfrac{1}{2}n)\}\,q^n,\\
\rho \operatorname{ds} \tfrac{2}{3}K &= 2/\surd 3+2\surd 3\,\Sigma_1^\infty (-1)^n \{H(n)+H(\tfrac{1}{2}n)\}\,q^n,\\
\rho \operatorname{zs} \tfrac{2}{3}K &= 1/\surd 3+2\surd 3\,\Sigma_1^\infty H(n)\,q^{2n},\\
\rho \operatorname{cs} \tfrac{2}{3}K &= 1/\surd 3-2\surd 3\,\Sigma_1^\infty \{H(n)-2H(\tfrac{1}{2}n)\}\,q^{2n},\\
\rho \operatorname{zc} \tfrac{2}{3}K &= -\surd 3-2\surd 3\,\Sigma_1^\infty \{H(n)+2H(\tfrac{1}{2}n)\}\,q^{2n},\\
k'\rho \operatorname{sc} \tfrac{2}{3}K &= \quad \surd 3+2\surd 3\,\Sigma_1^\infty \{1+2(-1)^n\}\,H(n)\,q^{2n}.
\end{aligned}$$

11. The formulæ in the E-group, which contains the expansions of the six even functions of $\frac{2}{3}K$, are the same as those given on p. 144 of the previous paper, except that by the use of the symbol $E(\frac{1}{3}n)$ two series are combined into one, *e.g.*, in the previous paper the first series was written

$$k\rho \operatorname{cd} \tfrac{2}{3}K = 2\Sigma_1^\infty E(m) q^{\frac{1}{2}m} + 6\Sigma_1^\infty E(m) q^{\frac{3}{2}m}.$$

The coefficients in the ten expansions forming the H-group, which represent the uneven functions of $\frac{2}{3}K$, were originally expressed in the previous paper (pp. 144, 145) by means of six arithmetical functions $H(n)$, $H'(n)$, $H''(n)$, $H_1(n)$,* $I(n)$, $\mathrm{i}(n)$. These six functions were subsequently (p. 148) expressed in terms of $H(m)$ and $(-1)^i H(m)$, where $n = 2^i m$; and it was pointed out (p. 150) that the six functions could also be expressed in terms of H and H_1, so that the expansions of the sixteen functions involved only the three arithmetical functions E, H, H_1.

At that time I failed to notice the very simple formula

$$H_1(n) = H(n) + H(\tfrac{1}{2}n),$$

by means of which H_1 can be expressed in terms of H, so that (as shown in this paper) the ten expansions involve only a single function H, and can be expressed each by a single series if we adopt the convention that $H(r)$ is zero when r is fractional.

12. The following equations express in terms of the function H the arithmetical functions which were defined and used in the previous paper, and which on p. 148 of that paper were expressed in terms of $H(m)$ and $(-1)^i H(m)$,

$$\begin{aligned}
H_1(n) &= H(n) + H(\tfrac{1}{2}n),\\
H'(n) &= H(n) - H(\tfrac{1}{2}n),\\
H''(n) &= H(\tfrac{1}{2}n),\\
\mathrm{h}(n) &= H(n) - 2H(\tfrac{1}{2}n),\\
I(n) &= H(n) + 2H(\tfrac{1}{2}n),\\
I'(n) &= (-1)^{n-1} H(n) + H(\tfrac{1}{2}n),\\
I''(n) &= \{1 + (-1)^n\} H(n) + H(\tfrac{1}{2}n),\\
\mathrm{i}(n) &= -\{1 + 2(-1)^n\} H(n).
\end{aligned}$$

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* In the previous paper $H_1(n)$ was denoted by $J(n)$. I have changed the notation because in subsequent papers I have used $J(n)$ to denote the excess of the number of divisors of n of the forms $8k+1$ and $8k+3$ over the number of those of the forms $8k+5$ and $8k+7$. This function in the previous paper (p. 163) was denoted by $T(n)$.

13. The quantity $H(\frac{1}{2}n)$ may be replaced by $H(2n)$ in all the formulæ, for, if n is uneven, both are zero, and, if n is even,

$$H(\tfrac{1}{2}n) = H(2^2.\tfrac{1}{2}n) = H(2n).$$

The last eight of the H-expansions may therefore be written

$$\begin{aligned}
\rho\,\mathrm{zn}\,\tfrac{2}{3}K &= 2\sqrt{3}\,\Sigma_1^\infty\,\{H(n)-H(2n)\}\,q^n,\\
\rho\,\mathrm{zd}\,\tfrac{2}{3}K &= 2\sqrt{3}\,\Sigma_1^\infty\,(-1)^n\,\{H(n)-H(2n)\}\,q^n,\\
\rho\,\mathrm{ns}\,\tfrac{2}{3}K &= 2/\sqrt{3}+2\sqrt{3}\,\Sigma_1^\infty\,\{H(n)+H(2n)\}\,q^n,\\
\rho\,\mathrm{ds}\,\tfrac{2}{3}K &= 2/\sqrt{3}+2\sqrt{3}\,\Sigma_1^\infty\,(-1)^n\,\{H(n)+H(2n)\}\,q^n,\\
\rho\,\mathrm{zs}\,\tfrac{2}{3}K &= 1/\sqrt{3}+2\sqrt{3}\,\Sigma_1^\infty\,H(n)\,q^{2n},\\
\rho\,\mathrm{cs}\,\tfrac{2}{3}K &= 1/\sqrt{3}-2\sqrt{3}\,\Sigma_1^\infty\,\{H(n)-2H(2n)\}\,q^{2n},\\
\rho\,\mathrm{zc}\,\tfrac{2}{3}K &= -\sqrt{3}-2\sqrt{3}\,\Sigma_1^\infty\,\{H(n)+2H(2n)\}\,q^{2n},\\
k'\rho\,\mathrm{sc}\,\tfrac{2}{3}K &= \sqrt{3}+2\sqrt{3}\,\Sigma_1^\infty\,\{1+2(-1)^n\}\,H(n)\,q^{2n}.
\end{aligned}$$

Whatever the value of n, either $H(n)$ or $H(2n)$ must be zero. Of course both may be zero.

Similarly, $E(\frac{1}{3}n)$ may be replaced by $E(3n)$, for, if n is not divisible by 3, both are zero, and, if n is divisible by 3,

$$E(\tfrac{1}{3}n) = E(3^2.\tfrac{1}{3}n) = E(3n).$$

Thus the E-expansions may be written

$$\begin{aligned}
k\rho\,\mathrm{cd}\,\tfrac{2}{3}K &= 2\Sigma_1^\infty\,\{E(m)+3E(3m)\}\,q^{\frac{1}{2}m},\\
k\rho\,\mathrm{cn}\,\tfrac{2}{3}K &= 2\Sigma_1^\infty(-1)^{\frac{1}{2}(m-1)}\,\{E(m)+3E(3m)\}\,q^{\frac{1}{2}m},\\
\rho\,\mathrm{dn}\,\tfrac{2}{3}K &= 1-2\Sigma_1^\infty\,\{E(n)-3E(3n)\}\,q^n,\\
k'\rho\,\mathrm{nd}\,\tfrac{2}{3}K &= 1-2\Sigma_1^\infty\,(-1)^n\,\{E(n)-3E(3n)\}\,q^n,\\
\rho\,\mathrm{dc}\,\tfrac{2}{3}K &= 2+2\Sigma_1^\infty\,\{E(n)+3E(3n)\}\,q^n,\\
k'\rho\,\mathrm{nc}\,\tfrac{2}{3}K &= 2+2\Sigma_1^\infty\,(-1)^n\,\{E(n)+3E(3n)\}\,q^n.
\end{aligned}$$

Whatever the value of n, either $E(n)$ or $E(3n)$ must be zero. Of course both may be zero.

14. If only $H(n)$ be used, *i.e.* not $H(\frac{1}{2}n)$ or $H(2n)$, the last eight of the H-expansions may be expressed as follows:—

$$\begin{aligned}
\rho\,\mathrm{zn}\,\tfrac{2}{3}K &= 2\sqrt{3}\,\Sigma_1^\infty H(n)\,q^n-2\sqrt{3}\,\Sigma_1^\infty H(n)\,q^{2n},\\
\rho\,\mathrm{zd}\,\tfrac{2}{3}K &= 2\sqrt{3}\,\Sigma_1^\infty\,(-1)^n H(n)\,q^n-2\sqrt{3}\,\Sigma_1^\infty H(n)\,q^{2n},\\
\rho\,\mathrm{ns}\,\tfrac{2}{3}K &= 2/\sqrt{3}+2\sqrt{3}\,\Sigma_1^\infty H(n)\,q^n+2\sqrt{3}\,\Sigma_1^\infty H(n)\,q^{2n},\\
\rho\,\mathrm{ds}\,\tfrac{2}{3}K &= 2/\sqrt{3}+2\sqrt{3}\,\Sigma_1^\infty\,(-1)^n H(n)\,q^n+2\sqrt{3}\,\Sigma_1^\infty H(n)\,q^{2n},
\end{aligned}$$

$$\rho\,\mathrm{zs}\,\tfrac{2}{3}K = 1/\surd 3+2\surd 3\,\Sigma_1^\infty H(n)q^{2n},$$

$$\rho\,\mathrm{cs}\,\tfrac{2}{3}K = 1/\surd 3-2\surd 3\,\Sigma_1^\infty H(n)q^{2n}+4\surd 3\,\Sigma_1^\infty H(n)q^{4n},$$

$$\rho\,\mathrm{zc}\,\tfrac{2}{3}K = -\surd 3-2\surd 3\,\Sigma_1^\infty H(n)q^{2n}-4\surd 3\,\Sigma_1^\infty H(n)q^{4n},$$

$$k'\rho\,\mathrm{sc}\,\tfrac{2}{3}K = \surd 3+6\surd 3\,\Sigma_1^\infty H(n)q^{2n}-8\surd 3\,\Sigma_1^\infty H(m)q^{2m}.$$

The last formula may also be written

$$k'\rho\,\mathrm{sc}\,\tfrac{2}{3}K = \surd 3+6\surd 3\,\Sigma_1^\infty H(n)q^{8n}-2\surd 3\,\Sigma_1^\infty H(m)q^{2m}.$$

The following mode of expressing the first group may be noticed, as the even and uneven powers of q are separated :

$$\rho\,\mathrm{zn}\,\tfrac{2}{3}K = 2\surd 3\,\Sigma_1^\infty \{H(2n)-H(n)\}\,q^{2n}+2\surd 3\,\Sigma_1^\infty H(m)q^{m},$$

$$\rho\,\mathrm{zd}\,\tfrac{2}{3}K = 2\surd 3\,\Sigma_1^\infty \{H(2n)-H(n)\}\,q^{2n}-2\surd 3\,\Sigma_1^\infty H(m)\,q^{m},$$

$$\rho\,\mathrm{ns}\,\tfrac{2}{3}K = 2/\surd 3+2\surd 3\,\Sigma_1^\infty \{H(2n)+H(n)\}\,q^{2n}+2\surd 3\,\Sigma_1^\infty H(m)q^{m},$$

$$\rho\,\mathrm{ds}\,\tfrac{2}{3}K = 2/\surd 3+2\surd 3\,\Sigma_1^\infty \{H(2n)+H(n)\}\,q^{2n}-2\surd 3\,\Sigma_1^\infty H(m)\,q^{m}.$$

15. The values of the elliptic and zeta functions for the argument $\frac{1}{3}K$ are deducible at once from those for the argument $\frac{2}{3}K$ by the formula

$$\mathrm{cd}\,\tfrac{2}{3}K = \mathrm{sn}\,\tfrac{1}{3}K,\quad \mathrm{cn}\,\tfrac{2}{3}K = k'\,\mathrm{sd}\,\tfrac{1}{3}K,\quad \mathrm{zc}\,\tfrac{2}{3}K = -\,\mathrm{zs}\,\tfrac{1}{3}K,\quad \ldots,$$

but in this paper I have preferred to express the results by means of the argument $\frac{2}{3}K$, instead of $\frac{1}{3}K$ as in the previous paper, because with the former argument the groups of formulæ are more regular, *e.g.*, when so expressed the six E-formulæ represent the even functions and the ten H-formulæ the uneven functions.

Many of the formulæ in the previous paper are improved by the change from $\frac{1}{3}K$ to $\frac{2}{3}K$, *e.g.*, the last three relations in § 22 (p. 152) become

$$\mathrm{cd}\,\tfrac{2}{3}K+\mathrm{cn}\,\tfrac{2}{3}K = 1,$$

$$\mathrm{nc}\,\tfrac{2}{3}K-\mathrm{nd}\,\tfrac{2}{3}K = 1,$$

$$\mathrm{dc}\,\tfrac{2}{3}K-\mathrm{dn}\,\tfrac{2}{3}K = 1,$$

and the six formulæ in § 23 (pp. 152, 153) represent the six even functions of $\frac{2}{3}K$. Also the three formulæ at the top of p. 151 represent $\mathrm{sn}^2\,\frac{2}{3}K$, $k'^2\,\mathrm{sd}^2\,\frac{2}{3}K$, $k'^2\,\mathrm{sc}^2\,\frac{2}{3}K$.

16. By extending the convention that the function is zero when the argument is fractional from E and H to the arithmetical functions Δ', ζ, σ, ... we may combine into one the two series which occur in the expansions of the squared elliptic and zeta functions on p. 158 of the previous paper.

Selecting from each of the first three groups the expansions in which the constant term can be combined with the first of the two series, we have

$$k^2\rho^2 \operatorname{sn}^2 \tfrac{2}{3}K = 12\Sigma_1^\infty \{\Delta'(n)-3\Delta'(\tfrac{1}{3}n)\}\, q^n,$$

$$k^2k'^2\rho^2 \operatorname{sd}^2 \tfrac{2}{3}K = 12\Sigma_1^\infty (-1)^{n-1}\{\Delta'(n)-3\Delta'(\tfrac{1}{3}n)\}\, q^n,$$

$$k'^2\rho^2 \operatorname{sc}^2 \tfrac{2}{3}K = 3-12\Sigma_1^\infty \{\zeta(n)-3\zeta(\tfrac{1}{3}n)\}\, q^n.$$

In the fourth group a term in ρ^2 occurs in each of the expansions, *e.g.*,

$$\rho^2 \operatorname{ds}^2 \tfrac{2}{3}K+\tfrac{1}{3}(k^2-k'^2)\rho^2 = 1+12\Sigma_1^\infty \{\sigma(n)-3\sigma(\tfrac{1}{3}n)\}\, q^{2n}.$$

17. A table of the values of $E(n)$ up to $n = 1000$ was given in the *Proceedings* of this Society, Vol. xv., 1884, p. 106,* and tables of the same extent of $H(n)$, and of $J(n)$, *i.e.*, of the $T(n)$ of the previous paper, have been given in the *Messenger*, Vol. XXXI., 1901, pp. 64–72 and 82–91. The introductions prefixed to the latter two tables contain references to other papers in which the functions $H(n)$ and $J(n)$ are considered.

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* Two errors in this table are pointed out in the *Messenger*, Vol. XXXI., p. 66, viz., the arguments 802 and 922 should not be omitted, for the values of $E(802)$ and $E(922)$ are each 2.