will be found—somewhere else. In this way we save time, and we preserve the interest of the student and our reputation for scientific accuracy. Perhaps we shall decide to hear students flounder around over their heads in words a few years more before taking such a radical step. Perhaps the matter will not be settled in quite the way I have suggested. One thing, however, seems clear. From every point of view that I am able to take, the theory of limits in elementary teaching is a failure. If both secondary and college teachers suddenly discover that they agree on this point, the rest is simple.

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## FUNCTIONAL EXPONENTS.

By Arthur Latham Baker, Ph.D.,

*Manual Training High School, Brooklyn.*

"One of the most grotesque types of mathematical symbols is represented by $\sin^{-1}x$ for the angle whose sine is $x$. While $\sin^{\alpha}x$ means the $\alpha$ power of $\sin x$ for every value of $\alpha$ which differs from unity, it assumes an entirely different meaning for the special value of $\alpha$."

I find the quotation above in School Science and Mathematics for May, 1907, p. 409.

The writer seems to have neglected the significance of *exponent,* a number showing how many times the operation of producing the functional expression is successively performed.

Thus if we denote by $f$ the operation of squaring and adding one, we have, 1 being understood in the absence of an exponent,

$$f(x) = x^2+1$$
$$f^2(x) = (x^2+1)^2+1$$
$$f^3(x) = [(x^2+1)^2+1]^2+1$$

If $f$ is a building up operation, then $f^{-1}$ is the tearing down operation, the operation which exactly undoes the thing accomplished by $f$, so that

$$f^{-1}(f(x)) = x\ ,$$

leading us back to exactly the point of starting.

Similarly

$$f^{-2}[f^3(x)] = f^{-2+3=1}(x) = x^2+1$$

The ordinary exponents of algebra are merely special cases of functional exponents; cases in which the functional operation is confined to multiplication and division instead of being allowed to be any combination of the four algebraic operations, addition, subtraction, multiplication and division.

In the transcendental functions, e. g., sin $x$ indicates the operation which builds up sin $x$ from the arc $x$, viz,

$$\text{Sin } x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots\cdots$$

If we desire to indicate the tearing down operation, that which builds the arc from the sine, we must logically indicate it by $\sin^{-1}$

If we should write

$$x + \frac{x^3}{2\cdot3} + \frac{1\cdot3x^5}{2\cdot4\cdot5} + \frac{1\cdot3\cdot5x^7}{2\cdot4\cdot6\,7} + \cdots\cdots$$

it would be difficult to recognize any connection with

$$\text{Sin } x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots\cdots$$

But the moment we write

$$\text{Sin}^{-1} x = x + \frac{x^3}{2\cdot3} + \frac{1\ 3x^5}{2\ 4\ 5} + \frac{1\cdot3\cdot5x^7}{2\cdot4\cdot6\cdot7} + \cdots\cdots$$

the story is told, and the reciprocality of the two functional operations is stentor shouted.

To call this form grotesque because slovenly usage has given currency to a different and illogical interpretation is something like calling Addisonian diction grotesque because a majority of people use slang.

It must be remembered that the exponent can be in three different places, appended to the operand, to the functional operation, or to the result. Thus

$$\text{Sin}^{-1}\frac{\pi}{6} = 31°34'$$

$$\text{Sin}\left(\frac{\pi}{6}\right)^{-1} = \text{Sin } 1.91 = \text{Sin } 109°\ 26' = 0.9430$$

$$\left(\text{Sin}\frac{\pi}{6}\right)^{-1} = 2$$

have entirely different meanings; as also

$$\text{Sin}^2\frac{\pi}{6} = \text{Sin}\left(\text{Sin}\frac{\pi}{6}\right) = \text{Sin } 0.5 = \text{Sin } 28°39' = 0.47942$$

$$\text{Sin}\left(\frac{\pi}{6}\right)^2 = \text{Sin } 0.27315 = \text{Sin } 15°\ 39' = 0.26974$$

$$\left(\text{Sin}\frac{\pi}{6}\right)^2 = 0.25$$

The continental notation, arc sin $x$, has its value, but on the few occasions when we wish to use expressions such as

$$\text{Sin}^{-2}\frac{\pi}{6}$$

it would be exceedingly clumsy. It is an accidental and sporadic notation which has no analogue in the case of

$$\log^{-1} 2 = 100$$
$$\log^{-2} 2 = 10^{100},$$

to say nothing of the higher functional forms.

gd and $\text{gd}^{-1}$
Sinh and $\text{Sinh}^{-1}$
Sn and $\text{Sn}^{-1}$ etc.

Why is $\sin^{-1}$ any more grotesque than $\text{gd}^{-1}$ or $\text{sn}^{-1}$?

To be exact, arc sin $x$ is a misnomer. It should be angle sin $x$.

In the case of the more familiar algebraic exponents, e. g., $x^{2/3}$, the $\frac{2}{3}$ indicates two multiplicative and three divisive operations, the first operation being the production of $x$ from unity, the second the repetition of this operation, in conformity with the rule for multiplication.

The case of algebraic exponents is complicated somewhat by the fact that the symbol for the functional operation and for the result are the same, e. g.,

$$x(1) = x \quad , \quad x^2(1) = x^2,$$

where $x$ indicates the functional operation of producing $x$ from unity, in conformity with the definition of multiplication as the performing upon the operand of the operation which produced the multiplier from unity. The $x$ indicates the functional operation or the result, according to the point of view.

Another complication arises from the fact that we have two ways of indicating the inverse functional operation, viz., by the negative exponent and by the position

$$x^{\ 2} = \frac{1}{x^2}$$

The ambiguity is not lessened by the fact that the inverse operation has received a special name of its own, division.

Instead of calling $\sin^{-1}$ grotesque, because slovenly usage has stolen its birthright, would it not be better to show loyalty to logical consistency and exactitude by insisting that those who mean $(\sin x)^{-1}$ shall use the proper notation therefor and leave $\sin^{-1} x$ in quiet possession of its hereditary and legal rights?