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FGH; but as the length of these filaments was then considerably reduced, their tendency to twine themselves round A, from any sudden motion of the eyeball, was diminished, and there has been no enlargement of the black spot or *Musca* during the years 1846 and 1847.

St. Leonard's College,
December 11, 1847.

II. *On the Integral* $\int \frac{dx}{x}$, *and on some consequences that have been deduced from it.* *By* J. R. Young, *Professor of Mathematics in Belfast College**.

IT is the object of the present short paper to remove some obscurities connected with the ordinary treatment of the simple integral $\int \frac{dx}{x}$. It will be anticipated therefore that the remarks which I have to offer are of a very elementary character: too much so indeed to entitle them to a place in a Journal of this kind, were it not that such obscurities, in the first elements of science, as experience has often shown, are frequently the source of important errors in its more recondite applications.

The integral just adverted to is a particular case of the more general form $\int x^n dx$, of which the value, disregarding correction, is known to be $\frac{x^{n+1}}{n+1}$. In the particular case noticed, that namely in which $n=-1$, this expression for the value is said to *fail*; though it is admitted to be valid in every other case, whether n be whole or fractional, positive or negative.

It may possibly be remembered by some of the readers of this Journal, that such an isolated failing case, in a general algebraic formula, is an occurrence that I have endeavoured to show can never happen; and that if any such formula hold for all values between a and b, it must equally hold for the extreme limits a and b themselves.

In the instance before us, the particular information, which the general form is supposed to fail in supplying, is obtained from other considerations; and the value of the wanting integral affirmed to be $\log x$. And the reason sometimes assigned for the inefficiency of the general form, in the particular case, is "that the equation $\int \frac{dx}{x} = \log x$, supposes the function of

* Communicated by the Author.

x denoted by $\int \frac{dx}{x}$ to vanish when $x=1$, whilst the equation $\int x^n dx = \frac{x^{n+1}}{n+1}$ supposes $\int x^n dx$ to vanish when $x=0$." It appears to me that this explanation is by no means sufficient to justify the assertion that the general form *fails*. Every student of the calculus knows that, by integrating the same expression by different methods, different functions of the variable will often arise, which can only become identical, in particular applications of the results, when each is connected with its own supplementary constant. One method may lead us to logarithmic functions, another to circular; and though they both arise from one and the same differential, they cannot, in general, be equated till each has received its own peculiar correction. In such instances it appears to me that it would be just as proper to say that one of these methods *fails*, as in the instance before us. The fact is, that in all cases of general integration, where the supplementary constant is suppressed, the process is really performed between limits, one of which is fixed, and the other arbitrary. To be strictly accurate, $\int \frac{dx}{x} = \log x$ should be written $\int_1^x \frac{dx}{x} = \log x$.

Introducing this accuracy of expression, let us now return to the general form, which in the case under consideration is $\int_1^x x^{-1}dx$; or, for convenience, changing x into $1+z$, $\int_1^{1+z} x^{-1}dx$. By the general form, the value of this is $\frac{(1+z)^0 - 1^0}{0}$. Developing by the binomial theorem, we have

$$(1+z)^0 = 1^0 + 0z + \frac{0(-1)}{2}z^2 + \frac{0(-1)(-2)}{2.3}z^3 + \&c.,$$

and consequently

$$\frac{(1+z)^0 - 1^0}{0} = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \&c.,$$

the known development of $\log(1+z)$, or $\log x$; so that the general form really gives us $\int_1^x x^{-1}dx = \log x$, without any failure at all. And we thus get moreover the interesting symbolical result,

$$\frac{(1+z)^0 - 1}{0} = \log(1+z),$$

or rather

$$\infty(1+z)^{\frac{1}{\infty}} - \infty = \log(1+z); \ \therefore \log x = \infty\,(x^{\frac{1}{\infty}} - 1),$$

from which the exponential theorem, and thence the whole theory of logarithms, may be readily derived.

In Liouville's well-known memoir on General Differentiation, in the thirteenth volume of the *Journal de l'Ecole Polytechnique*, the distinguished author has, I think, fallen into error, in consequence of being governed by the prevailing views respecting the failure of the general form here discussed. He is led (page 84) to the formula

$$x^p = \frac{-p}{\Gamma(1-p)}\int_0^\infty (e^{-zx}-1)z^{-p-1}dz,$$

which, in accordance with those views, he affirms (page 85) to be "absolument fausse lorsque $p=0$." In justification of this, he maintains that the definite integral

$$\int_0^\infty (e^{-zx}-1)\frac{dz}{z}$$

is *a finite quantity*.

Now since

$$\int\frac{e^{-zx}dz}{z} = \int\frac{dz}{z} - xz + \frac{x^2z^2}{1.2^2} - \frac{x^3z^3}{1.2.3^2} + \text{\&c.},$$

$$\therefore \int_0^\infty \frac{e^{-zx}dz}{z} = \int_0^\infty \frac{dz}{z} - x\infty + \frac{x^2\infty^2}{1.2^2} - \frac{x^3\infty^3}{1.2.3^2} + \text{\&c};$$

and consequently the proposed integral, namely,

$$\int_0^\infty \frac{e^{-zx}dz}{z} - \int_0^\infty \frac{dz}{z} = -x\infty + \frac{x^2\infty^2}{1.2^2} - \frac{x^3\infty^3}{1.2.3^2} + \text{\&c.}$$

How this can be pronounced to be *zero*, I am at a loss to conceive. That it is infinite, instead of zero, necessarily follows from its interpretation in the left-hand member of the original equation, even if there were no internal evidence of the fact. In the particular or extreme case considered, that left-hand member becomes $x^0=1$; and the right-hand member is the series here exhibited multiplied by 0, the limiting value of p: the form therefore is merely a particular instance of $0\times\infty$; interpretable, as all the cases which this terminates are interpretable, by the unambiguous form on the left. Several errors of like kind occur in Liouville's memoir; all traceable to the same oversight respecting fundamental principles.

It may not be superfluous to remark, in reference to the foregoing series for $\log(1+x)$, that whenever that series is not convergent, a supplementary correction is considered to be comprehended under the "&c." In the various transformations which this series is made to undergo, in order that it may serve the purpose of the actual construction of logarithmic tables, it will be found on examination that they are always such as to preserve throughout the convergency of the series, so that the correction adverted to disappears. If however we replace a by $a-1$, a being the base of the system, we then render the series necessarily divergent; and it is common in writings on this subject (see for instance Miller's Diff. Calc., p. 10) to apply to it in this state certain transformations, by which it is said to be converted into a converging series. But no diverging series can admit of such conversion; and whenever this appears to be accomplished, it will always be found that the original series is taken, not by itself, but in conjunction with its correction; and thus the change apparently brought about independently of this correction, is, in reality, a new development of the function generating the original series.

There is a well-known theorem of Lagrange, which, in the case of Taylor's series, enables us to assign the limits within which must lie the error we commit by taking any finite number of terms of the series as an equivalent for the undeveloped function. In the form in which Lagrange delivered it, the theorem is

$$f(z+x)=fz+xf'z+\frac{x^2}{2}f''z+\frac{x^3}{2.3}f'''z+\ldots.\frac{x^n}{2.3\ldots n}f^{(n)}(z+u),$$

in which he says "u désigne une quantité inconnue, mais renfermée entre les limites 0 et x" (*Théorie des Fonctions*, p. 68); and the same conditions are always said to be necessary whenever the theorem is announced. It is certainly of little or no practical moment to correct this statement; yet in order that extreme cases even may not be improperly excluded, it is necessary to widen the limits so as actually to include 0 and x. For it is plain that if the series be finite, and we stop at the last term, which it is conceivable we might sometimes do without knowing that the final term was reached, u would be actually 0; and if we stop at the first term, then in every case u would be equal to x; so that, leaving the term at which we stop entirely unrestricted, the generality of the theorem requires that the limits 0 and x be included. From an examination of this theorem, I am inclined to think that it is capable of greater definiteness and precision than is at present given to it, the range between the limits depending in general upon the place of the term at which we stop: but the discussion of this point must be reserved for a future occasion.

I may perhaps be permitted to add in conclusion, for the information of those interested in such elementary matters, that the statement in the former part of this paper, in reference to the ease with which the exponential theorem may be derived from the above expression involving ∞, is confirmed in a communication to the Mechanics' Magazine; in which also will be found a short algebraical investigation of the development of the important function $\frac{x}{l^x - 1}$. The Part of the Magazine containing the communication here alluded to, will appear simultaneously with the present Number of this Journal. It is a publication which has of late devoted considerable space to mathematical speculations, and is enriched with interesting papers by Professor Davies, Mr. Cockle, and other distinguished contributors to the Philosophical Magazine.

Belfast, Nov. 19, 1847.

III. *On the Action of Nitric Acid on Cymol. First Part.* *By* H. M. Noad, *Esq.**

Formation of Toluylic and Nitrotoluylic Acids.

WE possess in benzoic acid and its derivatives a well-defined group of substances connected in a variety of ways with a large number of organic families. These interesting bodies have been made subjects of investigation by several chemists. The study has been a fascinating one, and has resulted in a thorough development of their history, and of the products of their decomposition.

This group may be considered the prototype, as it were, of several parallel groups, presenting a very close relation with the composition of the benzoyle family. The careful study of the former has gradually made these known to us, in the same manner as the study of alcohol and its derivatives made us acquainted with several corresponding classes of bodies.

The methyle compounds, with which we have become familiar through the experiments of Dumas and Peligot† on pyroxylic spirit, and the amyle series, the origin of which we owe to the investigations of Cahours‡ on fusel oil, form two groups, the analogy of which with the alcohol series can be traced in every direction; they differ in composition from the former only by a multiple of $C_2 H_2$, thus—

$HO, C_2 H_3 O$ = hydrated oxide of methyle.
$HO, C_4 H_5 O$ = hydrated oxide of ethyle.
$HO, C_{10} H_{11} O$ = hydrated oxide of amyle.

* Communicated by the Chemical Society; having been read June 7, 1847.

† Liebig's *Annalen*, xv. 1.   ‡ Ibid. xxx. 288.