Philosophical Magazine Series 3

ISSN: 1941-5966 (Print) 1941-5974 (Online) Journal homepage: http://www.tandfonline.com/loi/tphm14

LXVI. On asymptotic straight lines, planes, cones and cylinders to algebraical surfaces

Thomas Weddle

To cite this article: Thomas Weddle (1847) LXVI. On asymptotic straight lines, planes, cones and cylinders to algebraical surfaces , Philosophical Magazine Series 3, 31:210, 425-434, DOI: 10.1080/14786444708645888

To link to this article: http://dx.doi.org/10.1080/14786444708645888

Published online: 30 Apr 2009.

LXVI. *On Asymptotic Straight Lines, Planes, Cones and Cylinders to Algebraical Surfaces.* By THOMAS WEDDLE*.

IN the Cambridge Mathematical Journal, first series, vol. iv. pp. 42–47, the late D. F. Gregory gave a very excellent method of determining the asymptotes to algebraical curves. I here purpose considering the corresponding subject relative to algebraical surfaces; and as this seems to have as yet engaged but little attention (if any), I trust the discussion will not be unacceptable to the mathematical readers of this Journal.

Definitions.

1. A straight line which passes through a point at a finite distance and touches a surface at an infinite distance, is called an *asymptotic straight line*, or simply an *asymptote* to the surface.

2. If *every* straight line drawn in a plane be an asymptote to a surface, the plane is styled a CONICAL *asymptotic plane* to the surface.

3. If all straight lines drawn in a plane parallel to a straight line in that plane be asymptotes to a surface, the plane is denominated a CYLINDRICAL *asymptotic plane* to the surface.

4. An *asymptotic cone* or *cylinder* to a surface is a cone or cylinder having its generators asymptotes to the surface.

If $\phi_q(xyz)$ denote a homogeneous function of x, y, z of the qth degree, it is plain that a surface of the pth degree may be denoted thus:

$$\phi_p(xyz)+\phi_{p-1}(xyz)\,.\,.\,.\,.\,.+\phi_1(xyz)+\phi_0=0. \quad . \quad (1.)$$

Let

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r\dagger \quad . \quad . \quad . \quad . \quad (2.)$$

be the equations of an asymptote to (1.) passing through the point $(\alpha\beta\gamma)$: hence

$$x=lr+\alpha, \quad y=mr+\beta, \text{ and } z=nr+\gamma;$$

substitute these values of x, y and z in (1.) and develope each term, the result is,

* Communicated by the Author.

† The axes may be either rectangular or oblique; only in the former case we shall have

$$l^2+m^2+n^2=1;$$

but in the latter,

$$l^2+m^2+n^2+2fmn+2gln+2hlm=1,$$

f, g, h denoting the cosines of the angles which the axes make with each other.

$$\left.\begin{array}{l}\phi_p.r^p+(D\phi_p+\phi_{p-1})r^{p-1}+\left(\frac{1}{2}D^2\phi_p+D\phi_{p-1}+\phi_{p-2}\right)r^{p-2}\\ \quad\ldots\ldots+\left(\frac{1}{2.3\ldots s}D^s\phi_p+\frac{1}{2.3\ldots(s-1)}D^{s-1}\phi_{p-1}+\right.\\ \quad\left.\ldots\ldots+D\phi_{p-s+1}+\phi_{p-s}\right)r^{p-s}\ldots\ldots=0^*,\end{array}\right\}\quad(3.)$$

where D denotes the operation

$$\alpha\frac{d}{dl}+\beta\frac{d}{dm}+\gamma\frac{d}{dn}.$$

This equation will determine the values of r at the points in which the straight line (2.) cuts the surface (1.); now for all lines parallel to an asymptote, one of these points is evidently at an infinite distance; hence a root of (3.) being infinite, we must have

$$\phi_p=0;\quad\ldots\ldots\ldots\quad(4.)$$

and this equation determines the directions of the asymptotes. The equation (3.) hence becomes

$$\left.\begin{array}{l}(D\phi_p+\phi_{p-1})r^{p-1}+\left(\frac{1}{2}D^2\phi_p+D\phi_{p-1}+\phi_{p-2}\right)r^{p-2}\ldots\\ \quad+\left(\frac{1}{2\ldots s}D^s\phi_p+\frac{1}{2\ldots(s-1)}D^{s-1}\phi_{p-1}+\ldots\right.\\ \quad\left.+D\phi_{p-s+1}+\phi_{p-s}\right)r^{p-s}+\ldots=0;\end{array}\right\}.\quad(5.)$$

in which values of l, m, n satisfying (4.) must be substituted. Now an asymptote being a tangent at an infinite distance, it follows that the asymptote will be distinguished from all lines having the same direction by a root of (5.) being infinite; we must therefore have

$$D\phi_p+\phi_{p-1}=0;$$

that is,

$$\frac{d\phi_p}{dl}\alpha+\frac{d\phi_p}{dm}\beta+\frac{d\phi_p}{dn}\gamma+\phi_{p-1}=0.\quad\ldots\ldots\quad(6.)$$

The equation (4.) shows that every asymptote is parallel to some generator or other of the cone

$$\phi_p(xyz)=0;\quad\ldots\ldots\ldots\quad(7.)$$

* In this paper I restrict θ,ϕ,ψ,χ (either with or without a letter or figure subscribed) to denote homogeneous functions only; and when these symbols stand alone, they are to be understood as functions of l, m, n; in other cases the symbols of quantity must be written; thus $\chi_q(xyz)$ (a homogeneous function of x,y,z of the qth degree) means the same function of x,y,z that χ_q does of l,m,n.

and since $(\alpha\beta\gamma)$ may be any point in each asymptote, (6.) denotes the locus (α, β, γ being the variable coordinates) of the asymptotes parallel to the same generator of (7.); this locus is therefore a cylindrical asymptotic plane, and it is parallel to that tangent plane of the cone (7.) which touches along the generator. Hence, to find the equation of a cylindrical asymptotic plane, we have only to take such values of l, m, n as satisfy (4.) and substitute them in (6.). It thus appears that when the cone (7.) is not imaginary, there is an indefinite number of cylindrical asymptotic planes; one indeed parallel to every tangent plane of the cone (7.), with a few exceptions, which I shall consider presently.

Should (4.), or, which is the same thing, (7.) be resolvable into factors, then (7.) will in reality denote as many conical surfaces; and if any of these factors be of the first degree, the corresponding conical surface will degenerate into a plane.

Let θ_q be any factor of φ_p, and put

$$\psi_{p-q}=\frac{\varphi_p}{\theta_q};$$

hence (6.) becomes

$$\theta_q.\mathrm{D}\psi_{p-q}+\psi_{p-q}\mathrm{D}\theta_q+\varphi_{p-1}=0,$$

when $\theta_q=0$, this reduces to

$$\psi_{p-q}\left\{\frac{d\theta_q}{dl}\alpha+\frac{d\theta_q}{dm}\beta+\frac{d\theta_q}{dn}\gamma\right\}+\varphi_{p-1}=0; \quad . \quad (8.)$$

and this equation, together with $\theta_q=0$, will supply the place of (4.) and (6.) for those cylindrical asymptotic planes that are parallel to the tangent planes of the cone $\theta_q(xyz)=0$. Also similar equations may be found for every factor of φ_p.

If the equations

$$\varphi_p=0,\quad \frac{d\varphi_p}{dl}=0,\quad \frac{d\varphi_p}{dm}=0,\quad \frac{d\varphi_p}{dn}=0,$$

can be satisfied by simultaneous values $(l_1\, m_1\, n_1)$ of l, m, n, (6.) cannot be satisfied unless φ_{p-1} also $=0$; if φ_{p-1} should not $=0$, there will be no cylindrical asymptotic plane corresponding to these values of l, m, n; but if $\varphi_{p-1}=0$, so that we have

$$\varphi_p=0,\quad \frac{d\varphi_p}{dl}=0,\quad \frac{d\varphi_p}{dm}=0,\quad \frac{d\varphi_p}{dn}=0,\quad \varphi_{p-1}=0^*, \quad (9.)$$

then (6.) will be satisfied independently of α, β, γ. We have only to recur however to (5.), and equate to zero the coefficient

* Since ϕ_p is a homogeneous function of l, m, n of the pth degree, we have

$$p\phi_p=\frac{d\phi_p}{dl}l+\frac{d\phi_p}{dm}m+\frac{d\phi_p}{dn}n;$$

hence the equations (9.) amount only to four independent equations—the last four.

of the first power of r that does not vanish independently of any relation among α, β, γ. If this coefficient be that of r^{p-2}, we have

$$\frac{1}{2}D^2\phi_p + D\phi_{p-1} + \phi_{p-2} = 0;$$

that is,

$$\left.\begin{array}{l}\frac{1}{2}\frac{d^2\phi_p}{dl^2}\alpha^2 + \frac{1}{2}\frac{d^2\phi_p}{dm^2}\beta^2 + \frac{1}{2}\frac{d^2\phi_p}{dn^2}\gamma^2 + \frac{d^2\phi_p}{dm.dn}\beta\gamma + \frac{d^2\phi_p}{dl.dn}\alpha\gamma \\ \quad + \frac{d^2\phi_p}{dl.dm}\alpha\beta + \frac{d\phi_{p-1}}{dl}\alpha + \frac{d\phi_{p-1}}{dm}\beta + \frac{d\phi_{p-1}}{dn}\gamma + \phi_{p-2} = 0\end{array}\right\} \quad (10.)$$

This equation denotes a surface which is evidently the locus of the asymptotes which are parallel to that generator of (7.) whose equations are $\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}$. Hence (10.) must denote a cylindrical surface; and as its generators are all asymptotes, it is an asymptotic cylinder of the second degree (which may in certain cases degenerate into one or two cylindrical asymptotic planes). Should the values of l, m, n satisfying (9.) also cause α, β, γ to vanish from (10.), there will be no corresponding asymptotic cylinder, unless $\phi_{p-2}=0$; and in this case we must equate the coefficient of r^{p-3} in (5.) to zero, and we shall have an asymptotic cone of the third degree; and so on.

Hence, to determine the equations of the asymptotic cylinders to the surface (1.), we must find such values (if any) of l, m, n as satisfy (9.), and substitute them in (10.); if all the terms of (10.) also vanish, we must recur to the coefficient of r^{p-3} in (5.); and so on. There will be as many asymptotic cylinders as there are sets of values of l, m, n satisfying (9.), unless, after substituting any set in (10.), &c., the only term that does not vanish is that independent of α, β, γ, in which case there will be no asymptotic cylinder for this set of values.

If ϕ_p contain a factor of the form $\{\theta_q\}^2$, the first four equations of (9.) will be satisfied by $\theta_q=0$; and this, combined with $\phi_{p-1}=0$, will give determinate values for the ratios $l \div m \div n$, and the corresponding asymptotic cylinders will be determined in the way just mentioned. It may happen however that θ_q is also a factor of ϕ_{p-1}; and if so, all the equations (9.) will be satisfied by $\theta_q=0$, and (10.) now admits of simplification as follows. Let

$$\phi_p = \{\theta_q\}^2.\psi_{p-2q}, \text{ and } \phi_{p-1} = \theta_q.\psi'_{p-q-1},$$

then it may easily be shown that when $\theta_q=0$,

$$D^2\phi_p = 2\psi_{p-2q}.\{D\theta_q\}^2 \text{ and } D\phi_{p-1} = \psi'_{p-q-1}.D\theta_q.$$

Hence (10.) becomes

$$\psi_{p-2q}.\{D\theta_q\}^2+\psi'_{p-q-1}.D\theta_q+\phi_{p-2}=0\,;$$

that is,

$$\left.\begin{aligned}&\psi_{p-2q}.\left\{\frac{d\theta_q}{dl}\alpha+\frac{d\theta_q}{dm}\beta+\frac{d\theta_q}{dn}\gamma\right\}^2\\ +\psi'_{p-q-1}&\left\{\frac{d\theta_q}{dl}\alpha+\frac{d\theta_q}{dm}\beta+\frac{d\theta_q}{dn}\gamma\right\}+\phi_{p-2}=0\,;\end{aligned}\right\}\quad . \quad . \quad (11.)$$

which evidently denotes two parallel cylindrical asymptotic planes; also since l, m, n are here only connected by the equation $\theta_q=0$, it appears that there are in general two cylindrical asymptotic planes parallel to every tangent plane of the cone $\theta_q(xyz)=0$.

Generally, let $\{\theta_q\}^s$, $\{\theta_q\}^{s-1}$, $\{\theta_q\}^{s-2}$. . . . . . θ_q be factors of ϕ_p, ϕ_{p-1}, ϕ_{p-2} . . . . . ϕ_{p-s+1}, and put

$$\phi_p=\psi.\{\theta_q\}^s, \phi_{p-1}=\psi'.\{\theta_q\}^{s-1}, \phi_{p-2}=\psi''.\{\theta_q\}_{s-2}\dots\phi_{p-s+1}=\psi^{(s-1)}.\theta_q$$

(here the subscribed letters relative to ψ, ψ', &c. are omitted for simplicity), then it may easily be shown that when $\theta_q=0$, we have

$$D\phi_p=0,\ D^2\phi_p=0\ \dots\ D^{s-1}\phi_p=0,\ D^s\phi_p=2.3\dots s.\psi.\{D\theta_q\}^s, \&c.$$

Moreover, the equation to the asymptotic cylinder parallel to a generator of the cone $\theta_q(xyz)=0$, will, by equating to zero the first coefficient of (5.) that does not vanish independently of α, β, γ, be found to be

$$\frac{1}{2.3\dots s}D^s\phi_p+\frac{1}{2.3\dots(s-1)}D^{s-1}\phi_{p-1}+\dots+D\phi_{p-s+1}+\phi_{p-s}=0;$$

and this, by what precedes, reduces to

$$\left.\begin{aligned}\psi.\{D\theta_q\}^s+\psi'.\{D\theta_q\}^{s-1}+\psi''.\{D\theta_q\}^{s-2}+\dots\\ +\psi^{(s-1)}.D\theta_q+\phi_{p-s}=0.\end{aligned}\right\}\quad . \quad (12.)$$

Since

$$D\theta_q=\frac{d\theta_q}{dl}\alpha+\frac{d\theta_q}{dm}\beta+\frac{d\theta_q}{dn}\gamma,$$

it is evident that the asymptotic cylinder degenerates into s cylindrical asymptotic planes, all parallel to a tangent plane of the cone $\theta_q(xyz)=0$; and there is in general the same number parallel to every tangent plane of this cone.

The asymptotes to the surface (1.) passing through a given point $(\alpha\beta\gamma)$ will be found by determining the ratios $l\div m\div n$ by (4.) and (6.), and substituting, in succession, each set of simultaneous values in (2.); the resulting equations will be those of the asymptotes to the surface that pass through the point $(\alpha\beta\gamma)$.

Since (4.) is of the pth degree and (6.) of the $(p-1)$th, the equation resulting from the elimination of l (suppose) from (4.) and (6.) cannot exceed the $p\,(p-1)$th degree, and consequently there cannot be more than $p\,(p-1)$ values of the ratio $m \div n$. From this we learn, that through any point in space there cannot be drawn more than $p(p-1)$ asymptotes to a surface of the pth degree.

This theorem suffers an exception, however, which I proceed to consider.

It may happen that the point $(\alpha\beta\gamma)$ through which the asymptotes are to be drawn may be so taken as to cause (4.) and (6.) to have a common factor χ_q (which I shall suppose to be their greatest common measure). In this case (4.) and (6.) will be satisfied if $\chi_q=0$; and eliminating l, m, n from this equation by means of (2.), we have

$$\chi_q(x-\alpha,\ y-\beta,\ z-\gamma)=0$$

for the equation to the asymptotic cone, which is the locus of the innumerable asymptotes that pass through the point $(\alpha\beta\gamma)$. (The factor χ_q may sometimes be resolvable into other factors, and then the preceding asymptotic cone of the qth degree will in fact consist of several cones of inferior degrees.)

The division of (4.) and (6.) by χ_q will give two equations, $\chi'_{p-q}=0$, and $\chi''_{p-q-1}=0$, which admit of no common measure. Now (4.) and (6.) will be satisfied by these two equations; but the equations $\chi'_{p-q}=0$, $\chi''_{p-q-1}=0$, will determine not more than $(p-q)(p-q-1)$ sets of values of the ratios $l \div m \div n$, hence (excluding the generators of the cone corresponding to χ_q) not more than $(p-q)(p-q-1)$ asymptotes can pass through the point (α, β, γ).

In order to find those points (if any) which are the vertices of asymptotic cones, eliminate one of the quantities l, m, n from (4.) and (6.), and find those values of α, β, γ that will render all the coefficients of the resulting equation equal to zero. If no such values be possible, the surface (1.) does not admit of an asymptotic cone; but if values $\alpha_1, \beta_1, \gamma_1$ of α, β, γ can be found, then the point $(\alpha_1\,\beta_1\,\gamma_1)$ will be the vertex of an asymptotic cone. To find the equation of this cone, we must substitute $\alpha_1, \beta_1, \gamma_1$ for α, β, γ in (6.), and ascertain θ_q the common measure of (4.) and (6.) thus modified; then will $\theta_q(x-\alpha_1, y-\beta_1, z-\gamma_1)=0$ be the equation to the asymptotic cone, having its vertex at the point $(\alpha_1\,\beta_1\,\gamma_1)$. If the equation resulting from the elimination of l, m, or n from (4.) and (6.) can be rendered identically zero by other simultaneous values of α, β, γ, there will be as many asymptotic cones as there are sets of values. When the elimination referred to above is effected by the process for the common measure, the factor θ_q will be the last of the remainders that do not vanish when $\alpha_1, \beta_1, \gamma_1$ are substituted for α, β, γ. It will sometimes be found, however, that (4.) and (6.) have a common measure independently of α, β, γ, arising from $\{\theta_q\}^2$ and θ_q being factors of ϕ_p and ϕ_{p-1}; and in this case we must proceed with this common measure in the way to be noticed presently.

When we know that (4.) cannot be resolved into factors, the determination of the asymptotic cone is very easy; for since (4.) admits of no measure but itself, and (6.) is of an *inferior degree*, it is evident that if there be an asymptotic cone, (6.) must be identically zero; hence if such values $\alpha_1, \beta_1, \gamma_1$ can be given to α, β, γ as to cause the coefficients of (6.) to vanish, there will be an asymptotic cone of the pth degree, namely,

$$\phi_p(x-\alpha_1,\ y-\beta_1,\ z-\gamma_1)=0;$$

but if the coefficients cannot be rendered zero simultaneously, there will be no asymptotic cone. Since α, β, γ enter (6.) in the first degree only, there will evidently be at most only one set of values of α, β, γ that will render (6.) identically zero; and hence a surface of the pth degree *may* have one asymptotic cone of the pth degree, but not more, and it is plain that there cannot be an asymptotic cone of a higher degree.

If (4.) admits of being resolved into factors, and these factors can be found, the asymptotic cones may be determined as follows. Let θ_q be one of the factors of ϕ_p, and let θ_q itself be irresolvable into factors. Arrange (6.), or rather (8.), and θ_q according to the powers of either l, m or n (l suppose), and divide the former by the latter until the remainder is of lower dimensions in l than θ_q; then since θ_q is irresolvable into factors, it is clear that this remainder must be identically zero: find therefore $\alpha_1, \beta_1, \gamma_1$, the values of α, β, γ, that make the coefficients of the remainder vanish, then $\theta_q(x-\alpha_1,\ y-\beta_1,\ z-\gamma_1)=0$ will be the asymptotic cone. As α, β, γ enter (8.) in the first degree and do not enter θ_q, there cannot be more than one set of values of α, β, γ, if indeed there be any. The same process being repeated with each of the other prime factors into which (4.) is resolvable, we shall have all the asymptotic cones which the surface admits of.

The preceding process requires modification when the second or any higher power of θ_q is a factor of ϕ_p. As an example, suppose that $\{\theta_q\}^4$ enters as a factor into ϕ_p, and put $\phi_p=\psi\cdot\{\theta_q\}^4$ (θ_q not being a factor of ψ). When $\theta_q=0$, $D\phi_p$

$+\phi_{p-1}=0$, reduces to $\phi_{p-1}=0$, and consequently there will be no asymptotic cone unless θ_q be a factor of ϕ_{p-1}; if so, let $\phi_{p-1}=\psi'.\theta_q$, then

$$\frac{1}{2}D^2\phi_p+D\phi_{p-1}+\phi_{p-2}=0$$

becomes $\psi'.D\theta_q+\phi_{p-2}=0$, which is of the first degree in α,β,γ, and this (instead of (8.)) being combined with $\theta_q=0$, may give an asymptotic cone. If $\{\theta_q\}^2$ however be a factor of ϕ_{p-1}, then

$$\frac{1}{2}D^2\phi_p+D\phi_{p-1}+\phi_{p-2}=0$$

becomes $\phi_{p-2}=0$, and there will be no asymptotic cone unless θ_q be a factor of ϕ_{p-2}. If this be the case, assume $\phi_{p-1}=\psi'.\{\theta_q\}^2$, and $\phi_{p-2}=\psi''.\theta_q$, then

$$\frac{1}{2.3}D^3\phi_p+\frac{1}{2}D^2\phi_{p-1}+D\phi_{p-2}+\phi_{p-3}=0$$

reduces to

$$\psi'.\{D\theta_q\}^2+\psi''.D\theta_q+\phi_{p-3}=0;$$

and this equation, which replaces (8.), combined with $\theta_q=0$, may give one or two asymptotic cones (but not more, as will be shown below), unless θ_q should enter both ϕ_{p-1} and ϕ_{p-2} in a higher power than has been supposed; we shall then have $\phi_{p-3}=0$; and hence θ_q must (if there be an asymptotic cone) be a factor of ϕ_{p-3}. Suppose therefore

$$\phi_{p-1}=\psi'.\{\theta_q\}^3,\ \phi_{p-2}=\psi''.\{\theta_q\}^2,\ \text{and}\ \phi_{p-3}=\psi'''.\theta_q,$$

then

$$\frac{1}{2.3.4}D^4\phi_p+\ldots\ldots+\phi_{p-4}=0$$

becomes

$$\psi.\{D\theta_q\}^4+\psi'.\{D\theta_q\}^3+\psi''.\{D\theta_q\}^2+\psi'''.D\theta_q+\phi_{p-4}=0;$$

and this equation (which cannot be satisfied independently of α,β,γ, for θ_q is not a factor of ψ), combined with $\theta_q=0$, may give four asymptotic cones.

Similarly, if $\{\theta_q\}^s$ be the highest power of θ_q that is a factor of ϕ_p, it may be shown that α,β,γ enter the equation to be combined with $\theta_q=0$, only through $D\theta_q$, and that this equation may rise to any degree in $D\theta_q$ (except the $(s-1)$th) not exceeding s.

Hence when a power (s) of θ_q is a factor of ϕ_p, we must ascertain the highest powers of θ_q that are factors of ϕ_{p-1}, ϕ_{p-2}, ...ϕ_{p-t+1}, also ϕ_{p-t} the first term of (1.) that has not θ_q for a factor; we must then equate to zero the coefficient (reduced as above) of the highest power of r in (5.) that does not vanish independently of α,β,γ. If α,β,γ disappear from this equation so that it becomes $\phi_{p-t}=0$, there will be no asymptotic cone; but if this be not the case, then the reduced equation must be combined with $\theta_q=0$, in the same way as directed for (8.) and $\theta_q=0$, and we may get asymptotic cones though not more than s of them. I proceed to establish the last assertion.

It has been shown above that if $\{\theta_q\}^s$ be the highest power of θ_q that is a factor of ϕ_p, then the equation to be combined with $\theta_q=0$ will be of the form

$$\psi.\{D\theta_q\}^t+\psi'.\{D\theta_q\}^{t-1}+\ldots\ldots=0, \quad . \quad . \quad (13.)$$

where ψ, ψ' ... do not involve α, β, γ, and t may be equal to, but cannot be greater than s. Now if there be a corresponding asymptotic cone, let $(\alpha_1\beta_1\gamma_1)$ denote its vertex; then if

$$\frac{d\theta_q}{dl}\alpha_1+\frac{d\theta_q}{dm}\beta_1+\frac{d\theta_q}{dn}\gamma_1$$

(which I shall denote by $D_1\theta_q$) be substituted for $D\theta_q$ in (13.), the resulting equation will be satisfied by aid (if necessary) of $\theta_q=0$; hence (13.) must be divisible by $D\theta_q-D_1\theta_q$, so that it may be written

$$(D\theta_q-D_1\theta_q)(\psi.\{D\theta_q\}^{t-1}+\ldots\ldots)=0. \quad . \quad . \quad (14.)$$

Also, if $\alpha_2, \beta_2, \gamma_2$ be another set of values of $\alpha, \beta\gamma$, satisfying (13.), they must reduce the second factor of (14.) to zero, for the first is of a lower degree than θ_q. Hence $\psi.\{D\theta_q\}^{t-1}+\ldots\ldots$ must be divisible by $D\theta_q-D_2\theta_q$; and so on. In this way we shall, after a certain number (v) of divisions, get an equation,

$$\psi.\{D\theta_q\}^{t-v}+\ldots\ldots=0,$$

which either does not contain $D\theta_q$ (and hence α, β, γ) at all, or which cannot be satisfied by any values of α, β, γ. Rejecting this factor then as affording no solution, (13.) is equivalent to

$$(D\theta_q-D_1\theta_q)(D\theta_q-D_2\theta_q)\ldots\ldots(D\theta_q-D_v\theta_q)=0,$$

and each of these factors will give but one set of values of α, β, γ; hence there will be but v asymptotic cones,

$$\theta_q(x-\alpha_1,\ y-\beta_1,\ z-\gamma_1)=0\ \ldots\ldots\ \theta_q(x-\alpha_v,\ y-\beta_v,\ z-\gamma_v)=0;$$

and since v cannot exceed t, nor t exceed s, it follows that there cannot be more than s asymptotic cones resulting from a factor of ϕ_p of the form $\{\theta_q\}^s$.

When θ_q is of the first degree, it is clear that instead of an asymptotic cone we shall have a plane; and since any point in it may be regarded as the vertex, every straight line drawn in it will be an asymptote; hence the asymptotic cone will in this case become a conical asymptotic plane: also since θ_q is here of the form $Al+Bm+Cn$,

$$D\theta_q=A\alpha+B\beta+C\gamma,$$

which does not involve l, m or n. Hence to determine the conical asymptotic planes (if any) to the surface (1.), we must take those factors of φ_p that are of the first degree, and proceed as directed above for asymptotic cones; with this modification, however, that $D\theta_q$ not containing l, m or n must be regarded as a single constant, and consequently the process will be much simplified. If V_1, $V_2 \ldots V_t$ (t not $\succ s$) be the values of $D\theta_q$ corresponding to the factor

$$\{\theta_q\}^s = \{Al + Bm + Cn\}^s,$$

we shall have

$$Ax + By + Cz = V_1,\ Ax + By + Cz = V_2 \ldots Ax + By + Cz = V_t$$

as the equations to the conical asymptotic planes relative to this factor.

It appears from the preceding reasoning, that if the equation (4.), or, which is the same thing, the highest homogeneous function in the equation to the surface (1.) can be resolved into a factors of the first degree, b factors of the second degree, c factors of the third degree, &c. (here a factor of the form $\{\theta_q\}^s$ is to be accounted s factors), then the surface may admit of, but cannot have more than a asymptotic cones of the first degree, that is, a conical asymptotic planes, b asymptotic cones of the second degree, c asymptotic cones of the third degree, &c. Some of these cones may have the same vertex; and since $a + 2b + 3c \ldots = p$, the degree of the aggregate of all the asymptotic cones to a surface can never exceed that of the surface itself.

It will be seen that unless equal factors enter the highest homogeneous function, the asymptotic cones to a surface depend only on the two highest homogeneous functions in its equation; and hence (the above case excepted) all surfaces having the two highest homogeneous functions in their equations identical, will have the same asymptotic cones. Also conversely, it is plain that those surfaces that have the same asymptotic cones must have the two highest homogeneous functions in their equations identical, providing the degree of the equations to the surfaces be exactly equal to that of the aggregate of the cones. Now this aggregate may be considered one of these surfaces; hence if

$$u_1 = 0,\ u_2 = 0, \ldots u_t = 0$$

be the equations to cones, the aggregate of which is of the pth degree, the equation to all the surfaces of the pth degree having these for asymptotic cones may be denoted by

$$u_1 u_2 \ldots u_t + \chi_{p-2}(xyz) + \chi_{p-3}(xyz) \ldots\ldots + \chi_1(xyz) + \chi_0 = 0. \quad (15.)$$

Wimbledon, Surrey, Nov. 10, 1847.