XXXV.—*On the Hessian.* By Professor CHRYSTAL.

(Read 18th May 1885.)

1. Let

$$U \equiv u_0 z^n + u_1 z^{n-1} + u_2 z^{n-2} + \ldots = 0$$

be the equation to an algebraical curve of the nth degree, the co-ordinates of any point on which in a system of linear co-ordinates are (x, y, z), $u_0, u_1, u_2 \ldots$ being homogeneous functions of x and y of degrees indicated by the attached suffixes; then

$$H \equiv U_{xx}U_{yy}U_{zz} + 2U_{yz}U_{zx}U_{xy} - U_{xx}U_{yz}^2 - U_{yy}U_{zx}^2 - U_{zz}U_{xy}^2 = 0$$

is the equation to its Hessian, which is a curve of the $3(n-2)$th degree.

Every one of the $3n(n-2)$ points of intersection of H and U is a point of inflexion on U if it be not a multiple point on U. In this last case the intersection may or may not be a point of inflexion on some one of the branches of U; but in any case where H passes through a multiple point the total number $3n(n-2)$ of inflexions suffers a reduction. It is therefore a problem of great geometrical interest to calculate the number of the intersections of H and U which are absorbed at a multiple point on the latter. This problem has never been solved directly in any but a few simple cases. It has been shown, for example, that at an ordinary double point on U, H has also a double point the tangents at which are the same as the tangents at the double point on U, and that such a point absorbs $2 \times 2 + 2 = 6$ of the intersections $\widehat{HU}$; also that a multiple point of order k, all of whose tangents are distinct, is a multiple point of order $3k-4$ on H, k of whose tangents are tangents to U, and that such a point absorbs $k(3k-4) + k = 6 \times \frac{1}{2}k(k-1)$ intersections,—in other words, has the same effect as the $\frac{1}{2}k(k-1)$ ordinary double points to which it may be regarded as equivalent. It has also been shown that a point which is a cusp of the ordinary kind on U is a triple point on H, two of whose tangents coincide with the cuspidal tangent of U; this cusp counting for $2 \times 3 + 2 = 8$ among $\widehat{HU}$. Finally, CAYLEY has laid down that every singularity of an algebraical curve can be regarded, for our present purpose, as equivalent to a certain number δ of ordinary double points, and a certain number κ of ordinary cusps. But the proofs which have been given of this theory by NÖTHER, ZEUTHEN, STOLZ, HENRY SMITH, and the methods given for ascertaining the indices δ and κ, are of an indirect nature, and it has been doubted whether any proof of this theory can be given by methods appropriate to co-ordinate geometry.

The direct calculation of the reduction is therefore a general problem, whose interest is quite equal to its difficulty. With a view to clear the way for a general solution (if such be attainable) I have worked out a number of cases, some quite special, others of a more general character, and propose to communicate the solutions to the Society in the following paper.

2. In its ultimate stage the problem reduces to the following:—

To determine the number $\widehat{UV}$ of the intersections of two algebraical curves $U=0$, $V=0$, which coincide with a common point which is multiple on one or both.

Let us suppose that the common point is a multiple point of order k on U and of order κ on V.

So long as no one of the k tangents of U coincides with any one of the κ tangents of V, there is no difficulty; the number of intersections absorbed at the common point is $k\kappa$.

But let us suppose that l of the k tangents and λ of the κ tangents coincide with $x=0$, then we have

$$U \equiv x^l u_{k-l} + u_{k+1} + \dots\dots$$
$$V = x^\lambda u_{\kappa-\lambda} + u_{\kappa+1} + \dots\dots$$

or, what is still worse, that $x=0$ is a multiple inflexional or undulatory tangent, so that

$$U \equiv x^l u_{k-l} + x^m u_{k+1-m} + x^n u_{k+2-n} + \dots\dots$$
$$V \equiv x^\lambda v_{\kappa-\lambda} + x^\mu v_{\kappa+1-\lambda} + x^\nu v_{\kappa+2-\nu} + \dots\dots$$

and the problem becomes one of some difficulty.

In many cases the solution may be obtained by the following process:—

Ex. 1.

Let us suppose

$$U \equiv x^3 u_5 + u_{10}$$
$$V \equiv x^4 v_1 + v_7 .$$

$$\text{Let } K \equiv x v_1 U - u_5 V \equiv x v_1 u_{10} - u_5 v_7 \equiv u_{12} ,$$

say, where u_{12} does not contain x as a factor. Then, since K passes through all the intersections of U and V, taking $\widehat{VU}$ to denote the number of those intersections which coincide with $x=0$ $y=0$ we have, since

$$\left.\begin{matrix} U=0 \\ K=0 \end{matrix}\right\} \text{gives } u_5 V = 0 \text{ and } \therefore \left.\begin{matrix} U=0 \\ u_5 V=0 \end{matrix}\right\} ,$$

$$\widehat{UK} = \widehat{UV} + \widehat{Uu_5}$$

$$\textit{i.e.,}\ \widehat{Uu_{12}} = \widehat{UV} + \widehat{Uu_5} ,$$

whence

$$\widehat{UV} = \widehat{Uu_{12}} - \widehat{Uu_5} .$$

Now, provided none of the linear factors of u_{12} occur in xu_5, and none of those of u_5 in u_{10} we have

$$\widehat{Uu_{12}} = 8 \times 12$$
$$\widehat{Uu_5} = \widehat{u_{10}u_5} = 10 \times 5$$

whence finally $\qquad \widehat{UV} = 96 \quad - 50 = 46.$

We may consider the more general case.

Ex. 2.

$$\begin{aligned} &U \equiv x^m u_{k-m} + u_{k+r} \qquad m > \mu \\ &V \equiv x^\mu v_{\kappa-\mu} + v_{\kappa+\rho} \qquad r < \rho \,; \end{aligned}$$

of which Ex. 1 is a particular case; it may be shown by the above method that

$$\widehat{UV} = k\kappa + r\mu \,,$$

This obviously agrees with the result of Ex. 1, and also with the following.

Ex. 3.

$$\begin{aligned} &U \equiv x^2 - y^3 \\ &V \equiv x \; - y^3 \end{aligned} \qquad\qquad \begin{aligned} \widehat{UV} &= 2 \times 1 + 1 \times 1 \\ &= 3 \,. \end{aligned}$$

The figure corresponding to this case is

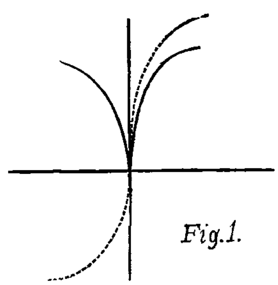

Fig. 1.

which may be looked upon as the limiting case of

Fig 2.

If in Ex. 2 $r > \rho$ the application of the above method is not so simple, and the result is not in all cases the same as will be shown directly.

For the sake of comparison with the results of another process shortly to be indicated, I work out two more examples by the present method.

Ex. 4.

$$\begin{aligned} &U \equiv x^3 u_5 + x^2 u_7 + u_{10} \\ &V \equiv x^4 v_1 + x^3 v_3 + v_7 \,, \end{aligned}$$

we have

$$\begin{aligned} K \equiv xv_1 U - u_5 V &\equiv x^3(v_1 u_7 - u_5 v_3) + xv_1 u_{10} - u_5 v_7 \\ &\equiv x^3 u_8 \quad + u_{12} \quad \text{say} \\ L \equiv u_5 K - u_8 U &\equiv u_{17} \quad + u_{18} \quad \text{say} \\ \therefore\ 8 \times 17 = \widehat{UL} &= \widehat{UK} + \widehat{Uu_5} \\ &= \widehat{UV} + \widehat{Uu_5} + \widehat{Uu_5} \\ &= \widehat{UV} + 2\widehat{(x^2 u_7)u_5} \\ 136 \quad &= \widehat{UV} + 90 \\ \widehat{UV} &= \quad 46 \end{aligned}$$

Ex. 5.

$$\begin{aligned}
&U \equiv x^3u_5 + xu_8 + u_{10}\\
&V \equiv x^4v_1 + x^3v_3 + v_7\\
&K \equiv xv_1U - u_5V \equiv x^2u_9 + u_{12}\\
&L \equiv xu_5K - u_9U \equiv xu_{17} + u_{19}\\
&M \equiv xu_9L - u_{17}K \equiv u_{29}
\end{aligned}$$

$$\begin{aligned}
11 \times 29 = \widehat{MK} &= \widehat{KL} + \widehat{xK} + \widehat{u_9K}\\
&= \widehat{KL} + \widehat{xu_{12}} + \widehat{u_9u_{12}}\\
&= \widehat{KU} + \widehat{Ku_9} + \widehat{xu_{12}} + \widehat{u_9u_{12}}\\
&= \widehat{KU} + \widehat{xu_{12}} + 2\widehat{u_9u_{12}}\\
&= \widehat{UV} + \widehat{u_5U} + \text{ \&c.}\\
&= \widehat{UV} + \widehat{xu_5} + \widehat{u_5u_8} + 2\widehat{u_9u_{12}} + \widehat{xu_{12}}
\end{aligned}$$

$$\widehat{UV} = 319 - 5 - 40 - 216 - 12 = 46\,.$$

3. The process just exemplified is tedious to apply and capricious in its action, and affords, besides, no indication of generality. It clearly contains redundant steps, for in the three examples (1) (4) (5) the same final result, viz., $\widehat{UV} = 46$, is obtained by extremely different developments. Yet it is obvious, *a priori*, that the same final result ought to be arrived at in all these cases, since the additional terms which appear in U and V in examples (4) and (5) are such that they do not affect the forms of U and V at the point $x=0$ $y=0$, which alone can be supposed to affect $\widehat{UV}$.

It is at once suggested, therefore, that the problem will be simplified by substituting for U and V the approximations to their branches at the origin determined by the rule of NEWTON and CRAMER. In this way we can in general reduce the problem to a series of others, of which the following is a type.

To determine the number of intersections of

$$U \equiv x^m - y^n = 0 \ldots . (1) \quad \text{and } V \equiv x^\mu - y^\nu = 0 \ldots . (2)$$

at the point $x=0$ $y=0$.

Since imaginary branches must be considered as well as real branches, it may be well to give a rigorous proof of the solution in this simple case.

If $\alpha_1\alpha_2 \ldots . \alpha_n$ be the n roots of $+1$, the n values of y given by $y^n = x^m$ are

$$\alpha_1 x^{\frac{m}{n}},\ \alpha_2 x^{\frac{m}{n}} \ldots . \alpha_n x^{\frac{m}{n}}\,.$$

The eliminant of the two equations (1) and (2) with respect to y is therefore

$$\left\{x^\mu - \left(\alpha_1 x^{\frac{m}{n}}\right)^\nu\right\}\left\{x^\mu - \left(\alpha_2 x^{\frac{m}{n}}\right)^\nu\right\} \ldots . . \left\{x^\mu - \left(\alpha_n x^{\frac{m}{n}}\right)^\nu\right\} = 0\,,$$

that is $$\left(x^{\frac{m\nu}{n}}\right)^n \left\{x^{\frac{\mu n - m\nu}{n}} - \alpha_1^\nu\right\}\left\{x^{\frac{\mu n - m\nu}{n}} - \alpha_2^\nu\right\} \ldots . . \left\{x^{\frac{\mu n - m\nu}{n}} - \alpha_n^\nu\right\} = 0\,,$$

Now, if g be the G.C.M. of n and ν, and $n = gn'$, $\nu = g\nu'$, then the series $\alpha_1^\nu, \alpha_2^\nu, \ldots . \alpha_n^\nu$, simply consists of the roots of $x^{n'} - 1 = 0$ repeated g times. Hence the equation last written reduces to

$$x^{gm\nu'}(x^{\mu n' - m\nu'} - 1)^g = 0\,, \qquad \text{that is } (x^{\mu n'} - x^{m\nu'})^g = 0 \ldots . (3)\,,$$

where from the nature of the process employed we are sure that there is no redundant factor.

Now $x^{\mu n}$ or $x^{m\nu}$ is a factor in (3) according as $\mu n <$ or $> m\nu$, *i.e.*, the number of zero roots of (3) is the least of the two numbers μn $m\nu$.

Hence denoting for shortness the least of the two μn $m\nu$ by $l(\mu n, m\nu)$, we have the following simple theorem.

The number of intersections of $x^m - y^n = 0$ and $x^\mu - y^\nu = 0$ at the point $x=0$ $y=0$, is

$$l(\mu n, m\nu).$$

4. By means of the NEWTON-CRAMER rule we can, as far as points near $x=0$ $y=0$ are concerned, replace U and V by

$$U' \equiv (x^{m_1} - A_1 y^{n_1})(x - A_2 y^{n_2}) \ldots .$$
$$V' \equiv (x^{\mu_1} - B_1 y^{\nu_1})(x^{\mu_2} - B_2 y^{\nu_2}) \ldots . ,$$

where the factors in U′ will in general be all different, those in V′ all different, and no factor common to U′ and V′.

In this case we can at once find the number $\widehat{UV}$. We have, in fact,

$$\begin{aligned}\widehat{UV} = \widehat{U'V'} &= l(m_1 \nu_1, n_1 \mu_1) + l(m_1 \nu_2, n_1 \mu_2) + \ldots . \\ &+ l(m_2 \nu_1, n_2 \mu_1) + l(m_2 \nu_2, n_2 \mu_2) + \ldots . \\ &+ \ldots . .\end{aligned}$$

This result still holds when factors are repeated in U′ or in V′; but when there are factors common to U′ and V′ there is a modification, as will be shown presently.

5. Before proceeding farther, let us apply the above principles to one or two examples.

Ex. 1.
$$U \equiv x^3 u_5 + x^2 u_7 + u_{10} = 0$$
$$V \quad x^4 v_1 + x^3 v_3 + v_7 \quad = 0 .$$

The NEWTON-CRAMER diagrams for U and V show at once that (omitting constant coefficients as irrelevant to the issue) we may write

$$U \equiv u_5(x^3 + y^5) \qquad V \equiv v_1(x^4 + y^6);$$
$$\therefore \widehat{UV} = 5 \times 1 + 5 \times 4 + 3 \times 1 + 3 \times 6 = 46 ,$$

the same result as before.

Ex. 2.
$$U \equiv x^3 u_5 + x u_8 + u_{10}$$
$$V \equiv x^4 v_1 + x^3 v_3 + v_7 .$$

Here it is easily shown that we may write

$$U \equiv u_5(x^2 + y^3)(x + y^2)$$
$$V \equiv v_1(x^4 + y^6);$$
$$\therefore \widehat{UV} = 5 \times 1 + 5 \times 4 + 2 \times 1 + 2 \times 6 + 1 \times 1 + 1 \times 6 = 46.$$

Ex. 3.
$$U \equiv x^m u_{k-m} + u_{k-r}$$
$$V \equiv x^\mu v_{\kappa-\mu} + v_{\kappa+\rho} .$$

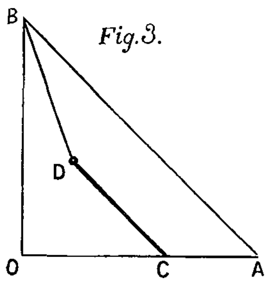

Fig. 3.

The diagrams here are both of the same character; that for U, for example, is figure 3, where AB and CD are parallel and each full of terms; the co-ordinates of D are $(m, k-m)$; and $OA=k+r$, $OC=k$. The two lines CD and BD give approxmations at $x=0$ $y=0$.

We may therefore write

$$U \equiv u_{k-m}(x^m+y^{r+m}), \qquad V \equiv v_{\kappa-\mu}(x^\mu+y^{\rho+\mu}).$$

Hence

$$\begin{aligned}\widehat{UV} &= (k-m)\kappa+m(\kappa-\mu)+l\{(\rho+\mu)m, (r+m)\mu\}\\ &= k\kappa-m\mu+m\mu+l(\rho m, r\mu)\\ &= k\kappa+l(\rho m, r\mu)\\ i.e., &= k\kappa+\rho m \text{ or } =k\kappa+r\mu,\end{aligned}$$

according as $\rho m <$ or $> r\mu$.

This result includes that of Ex. 2 in § 2 as a particular case.

Ex. 4. To illustrate the particular case of Ex. 3, where $m\rho < \mu r$, consider

$$U \equiv x^2-y^4 \qquad V \equiv x^2-y^3$$

Here

$$\begin{aligned}&m=2, \ k=2, \ r=2,\\ &\mu=2, \ \kappa=2, \ \rho=1;\\ &m\rho=2, \ \mu r=4.\\ &\widehat{UV}=2\times2+2=6.\end{aligned}$$

The corresponding figure is

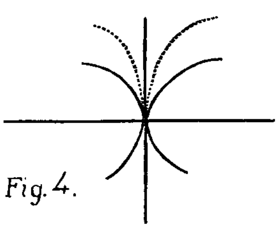
Fig. 4.

which may, in fact, be considered as the limiting case of

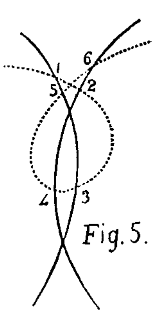

Fig. 5.

(*To be continued in another Communication.*)