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CISOIDAL OSCILLATIONS

BY GEORGE A. CAMPBELL

The oscillations here defined as "cisoidal oscillations" are those of the form

$$C \operatorname{cis} pt = C(\cos pt + i \sin pt) = Ce^{ipt} \qquad (1)$$

where t is the time, e the Napierian base, $i = \sqrt{-1}$ the imaginary symbol,[1] and cis an abbreviation for the complete trigonometric expression. The constants C and p may be any scalar quantities, either real or complex. The oscillations are sustained, logarithmically damped or aperiodic, according as the time coefficient p is real, complex, or pure imaginary. The following discussion will, in general, apply indifferently to all three cases.

The use of the term "cisoidal oscillations" emphasizes the distinctive character of the subject, while tending to keep in mind the close connection between these oscillations and sinusoidal oscillations. The fact that one of the algebraic curves is called a "cissoid" can hardly lead to confusion.

The practical importance of cisoidal oscillations rests upon the following properties:

1. In all cases where the principle of superposition holds, any

1. The use of i (or Greek ι) for the imaginary symbol is nearly universal in mathematical work, which is a very strong reason for retaining it in the applications of mathematics in electrical engineering. Aside, however, from the matter of established conventions and facility of reference to mathematical literature, the substitution of the symbol j is objectionable because of the vector terminology with which it has become associated in engineering literature, and also because of the confusion resulting from the divided practice of engineering writers, some using j for $+i$ and others using j for $-i$.

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oscillation can be regarded as a compound cisoidal oscillation, *i.e.*, the algebraic summation of simple cisoidal oscillations.

2. Cisoidal oscillations are uniquely simple because the ratio of the instantaneous electromotive force to the instantaneous current is not a function of the time.

3. Cisoidal oscillations involve scalar magnitudes only so that all algebraical relations and operations applying to the real physical phenomena may be extended to them.

4. The solution for cisoidal oscillations in any finite network may be written down directly, without solving differential equations or the use of integration or differentiation.

Scalar Character of Cisoidal Oscillations

As complex quantities and exponential functions of complex quantities follow the laws of ordinary algebra, they introduce scalar quantities and not vector quantities. This is a matter of great importance, since ordinary algebra is simpler than vector algebra. The wide-spread use of the term vector in connection with complex quantities in alternating current theory is unfortunate for it is logically incorrect, and so has led to confusion, and it also tends to divert attention from the algebraical theory of complex quantities, which is of great practical assistance in the treatment of cisoidal oscillations.

When the direction of a current is confined to one or the other of two opposite directions by the use of a linear conductor, we can vary its scalar magnitude only; it is no more correct to speak of representing this scalar quantity by a vector when it is complex than when it is real. It is only when the electrical phenomena takes place in two or three dimensions in space that vector variables are involved in the mathematical treatment.

With complex quantities the power continues to be the product of electromotive force and current. A steady imaginary current flowing through a resistance, therefore, dissipates negative real power, that is, energy is absorbed by the electrical phenomena taking place, which tends to cool the conductor. Similarly the magnitudes of the kinetic energy of an inductance and the potential energy of a condenser are real negative quantities in case the instantaneous current and potential are pure imaginary. As the power with complex quantities may be either positive or negative, or in general have any argument, the total power in a portion of a network, such as two or more resistances, may vanish because the several powers in the individual elements mutually cancel when added together.

If the current and electromotive force are each cisoidal the associated power is also cisoidal with a time coefficient equal to the algebraical sum of the time coefficients of the electromotive force and current; when these two coefficients are equal and opposite in sign the power is constant with respect to the time.

We might have defined the cisoidal oscillation using throughout $-i$ in place of i, which would change all quantities, including the impedances, to their conjugates. But we follow, of course, the general practice of taking positive quantities as the norm, in consequence of which the sign for inductive reactances is positive, and the sign for capacity reactances is negative.

Correlated Oscillations

The complete formal solution of a sinusoidal alternating current problem by the aid of complex quantities involves the following steps:

1. Resolution of the periodic data into the sum of cisoidal oscillations having the time factors cis $(+pt)$ and cis $(-pt)$.
2. Solution of the problem for the cis $(+pt)$ component taken alone; the solution for the cis $(-pt)$ component is then obtained directly from this by changing all complex quantities to their conjugates.
3. Superposition of these two cisoidal solutions to obtain the real physical oscillation.

It is however not necessary to carry through the formal proof in individual cases, this being replaced by the following correlation between the real and the complex oscillations.

If throughout any invariable network a cisoidal oscillation and a cosinusoidal oscillation (all of one time coefficient p) have electromotive forces and currents of the same effective values (moduli) and angles (arguments), they will be called correlated oscillations.

The alternating powers involved throughout correlated oscillations are equal to each other as regards amplitudes (moduli) and angles (arguments); the cosinusoidal oscillation having also non-alternating power components which are equal, as regards amplitudes (moduli) and phase angles (arguments) to the powers which would be associated with the correlated cisoidal electromotive forces taken with the conjugates of the correlated cisoidal currents.

Or in other words:

The instantaneous cosinusoidal electromotive forces and currents are the real components of the correlated cisoidal electromotive forces and currents multiplied by the factor $\sqrt{2}$.

The instantaneous powers involved in a cosinusoidal oscillation are equal to the real components of the cisoidal powers in the correlated cisoidal oscillation, augmented by the real components of the powers involved in the correlated cisoidal oscillation after changing the currents (or electromotive forces) to their conjugates.

In the typical notation the correlated oscillations thus defined have, if $p=p_1+p_2 i$

Instantaneous	Cisoidal	Cosinusoidal
e.m.f.	$E e^{ipt}$	$\sqrt{2}\lvert E\rvert e^{-p_2 t}\cos(p_1 t+arg E)$
current	$I e^{ipt}$	$\sqrt{2}\lvert I\rvert e^{-p_2 t}\cos(p_1 t+arg\, I)$ (2)
power	$E I e^{2ipt}$	$\lvert E I\rvert e^{-2p_2 t}\left[\cos(2p_1 t+arg(E I))+\cos arg\frac{E}{I}\right]$
impedance	$\frac{E}{I}$	$\left\lvert\frac{E}{I}\right\rvert\frac{\cos(p_1 t+arg E)}{\cos(p_1 t+arg\, I)}$

In much of the actual algebraical work connected with cisoidal oscillations, we may drop the time factors e^{ipt} and e^{2ipt} and write only E, I and $E\,I$ (or $P=E\,I$) with considerable resulting simplification and no liability of introducing confusion.

It is to be particularly noted that the magnitudes which are equal to the corresponding cisoidal moduli are the *effective* values of the cosinusoidal electromotive forces or currents and the *amplitudes* of the cosinusoidal power components. On the other hand, the cisoidal arguments are uniformly equal to the corresponding real angles, this angle reducing for the non-oscillatory cosinusoidal power component to the constant angle of lag or lead.

The preceding statements supply the working rules for making the change from the real physical cosinusoidal oscillation to the ideal cisoidal oscillation and vice versa. This connection is, as regards electromotive force and current, one of mutual resolvability as is expressed by the following formulæ:

$$\sqrt{2}|C|e^{-p_2 t}\cos(p_1 t+arg\, C)=\frac{1}{\sqrt{2}}\,Ce^{ipt}+\frac{1}{\sqrt{2}}\,C'\,e^{-ip't}$$

$$C\,e^{ipt}=\frac{1}{\sqrt{2}}\,[\sqrt{2}|C|\,e^{-p_2 t}\cos(p_1 t+arg\, C)]$$
$$+\frac{i}{\sqrt{2}}\left[\sqrt{2}|C|e^{-p_2 t}\cos\left(p_1 t+arg\left(C-\frac{\pi}{2}\right)\right)\right] \qquad (3)$$

the first giving the cosinusoid in terms of the correlated cisoid and its conjugate cisoid, the second giving the cisoid in terms of the correlated cosinusoid and the consinusoid with its phase retarded 90 degrees. On account of this mutual resolvability either the cisoidal oscillation or the cosinusoidal oscillation may be regarded as being obtained by summation from the other.

If any particular cisoidal or cosinusoidal oscillation is possible the correlated oscillation is also possible.

It is somewhat arbitrary as to the exact functions which we define as correlated oscillations. The sine might have been taken in place of the cosine and the amplitudes in place of the effective values, but on the whole these alternatives do not seem to afford quite the same convenience, but only because the statements become slightly more involved. We shall however continue to use the term " sinusoid " as the general designation for the sine function having any arbitrary phase angle including thereby the cosine function.

The correlation between the sinusoidal oscillations and cisoidal oscillations is so simple that it is not ordinarily necessary to indicate the step from one to the other in special applications of the method. But this omission has led to the cisoidal solution being in some way regarded as representing the actual sinusoidal oscillation, which is not the case as is very clearly shown by the power relations. It is therefore necessary to lay emphasis upon the fact that the use of complex quantities affords an indirect method, and not a symbolic method of solving real cases of oscillations and that the complete application of the method involves an initial algebraical resolution of the real data and a final algebraical summation of the complex results as an essential and integral part of the method.

General Equations for any Network

In any invariable network the actual distribution of current due to any impressed electromotive forces is such as to make the power dissipated assume the stationary value[2] *which is consistent with the conditions imposed by current continuity and the conservation of*

2. A function assumes a stationary value when it is not altered by any possible infinitesimal change in the system of variables upon which it depends; the first derivatives of the function, with respect to each of a set of independent variables is zero at a stationary value. Stationary is thus a generalization of maximum, minimum and point of inflection, but without any implication beyond the vanishing gradient.

energy. The theorem assumes that each branch or circuit contains resistance, a condition which corresponds to the physical fact and involves no theoretical limitation as the resistances may be as small as desired, or any number of the resistances may be allowed to vanish completely after playing their part in the formation of the general solution.

This theorem may be established directly from the principles of dynamics, but we will here show that it is the equivalent of the generalized Kirchhoff equations.

The condition imposed by the conservation of energy may be expressed in the form of the equation of activity by equating the total power supplied by the impressed forces to the sum of the powers taken separately by the resistances (including conductances), self-inductances, mutual inductances and capacities. That is

$$\sum e_q\, i_q = \sum R_q\, i_q^2 + \frac{d}{dt}\left(\sum \frac{1}{2} L_q\, i_q^2 + \sum M_{qr}\, i_q\, i_r\right)$$
$$+\frac{d}{dt}\sum \frac{(\int i_q\, dt)^2}{2\, C_q} = \sum R_q\, i_q^2 + \sum L_q\, i_q \frac{d\, i_q}{d\, t}$$
$$+\sum M_{qr}\left(i_q \frac{di_r}{dt} + i_r \frac{di_q}{dt}\right) + \sum \frac{i_q \int i_q\, dt}{C_q} \qquad (4)$$

The condition of continuity may be introduced by expressing the currents in terms of any set of independent, circuital currents $c_1, c_2, \ldots c_m$, where m is the number of degrees of freedom of the network. This gives one equation for each of the l branches

$$i_q = a_{q1}\, c_1 + a_{q2}\, c_2 + \ldots a_{qm}\, c_m \qquad (q=1, 2 \ldots l) \qquad (5)$$

where the coefficient $a_{qs} = \pm 1$ or 0, according as branch q is or is not a part of circuit s, the sign in the first case being positive, or negative, according as the positive direction for the branch and for the circuit are or are not concurrent.

The power dissipated $\Sigma R_q\, i_q^2$ is a homogeneous expression of the second order in terms of the m independent circuital currents, while the remainder of equation (**4**) is of the first degree in these currents. The stationary value for the power dissipated under the assumed conditions will therefore be found by first introducing the multiplier $\frac{1}{2}$ as a coefficient for $\Sigma R_q\, i_q^2$ and then differentiating (**4**) with respect to c_s which gives the following set of m equations:

$$\sum a_{qs}\, e_q = \sum a_{qs}\, R_q\, i_q + \sum a_{qs}\, L_q \frac{di_q}{dt} + \sum M_{qr}\left(a_{qs} \frac{di_r}{dt} \right.$$

$$\left. + a_{rs} \frac{di_q}{dt} \right) + \sum a_{qs} \frac{\int i_q\, dt}{C_q} \qquad (s = 1, 2, \ldots m) \qquad \textbf{(6)}$$

The set of equations (**6**) is identical with the generalized Kirchhoff equations of electromotive force for the m circuits taken in the positive direction for the currents c_s, since the coefficients a_{qs} and a_{rs} provide the proper sign for each effective electromotive force occurring in these circuits and exclude all electromotive forces not occurring in the several circuits. The Kirchhoff laws and the above condition of stationary dissipation are therefore mutually equivalent.

In subsequent work it will be more convenient to merge the conditions of continuity in the equation of activity (**4**) than to use separate equations such as (**5**) to cover these conditions. This may be accomplished either by reducing the currents appearing in the equation of activity to a number equal to and so chosen as to correspond with the degrees of freedom of the network, or by adding fictitious currents which correspond to the significant branch points.

The first transformation is accomplished by replacing the branch currents i_q in (**4**) by circuital currents such as c_s by the aid of such equations as (**5**). Rearranging the terms the form of the equation of activity may still be kept the same as in (**4**), but all quantities, e, i, R, L, M, C now refer to complete circuits and not to individual branches.

The second transformation follows from the identity of the condition of continuity, in the form

$$\varphi_f = M_{f1}\, i_1 + M_{f2}\, i_2 + \ldots M_{fr}\, i_r \ldots = 0, \qquad M_{fr} = \pm 1 \text{ or } 0,$$

$$f = (l+1, \ldots l+m), \qquad \textbf{(7)}$$

with the condition that a fictitious circuit of zero impedance can experience no resultant electromotive force whatever be the currents flowing in the branches 1, 2, . . . r . . . with which it has mutual impedances M_{f1}, M_{f2}, . . . M_{fr} . . . This physical consideration shows that the conditions of continuity will be included in (**4**) by extending the summation to cover fictitious circuits of zero self-impedances and with zero mutual impedances between each other and all real branches excepting only $M_{fr} = \pm 1$ when the real branch r terminates in the branch point f, the sign being positive or negative at the positive or negative end of the branch respectively.

To prove the same analytically we multiply each equation of (**7**) by i_f, take their sum, differentiate with respect to t and add this expression, which we may denote by

$$B = \frac{d}{dt} \sum i_f \varphi_f = \frac{d}{dt} \sum \sum M_{fr} \, i_f \, i_r$$

to (**4**), which is permissible since B must be equal to zero. On differentiating (**4**) (with multiplier $\frac{1}{2}$ added to $\Sigma R_q \, i_q^2$) with respect to the real current i_q, B introduces the new terms $\sum \frac{di_f}{dt} \frac{d\varphi_f}{dir}$ to (**6**), and these are precisely the additional terms required by the conditions of continuity, since $\frac{di_f}{dt}$ plays the part of an undetermined multiplier. Again differentiation with respect to the fictitious current i_f gives $\frac{d}{dt} \varphi_f = 0$ or $\varphi_f = 0$, the constant of integration being zero, as infinite energy in the fictitious circuits is to be excluded, and these are the equations of continuity (**7**). Thus after the addition of B, equation (**4**) includes all of the conditions of continuity.

It will be assumed in the subsequent work that the network under discussion has been transformed into a set of simple circuits, thus reducing the conditional equations to the equation of activity. The coefficients occurring in this equation and the number of currents entering it will depend upon the particular choice of simple circuits, but the general discussion of the network will be, to a considerable extent, independent of the choice of the simple circuit system. In concrete applications it will be advantageous, in order to have as few variables as possible, to use the first of the above transformations. In general work, however, the second transformation presents the distinct advantage of including all branches symmetrically.

General Equations for Cisoidal Oscillations

For cisoidal oscillations the preceding theorem may be given the following still simpler form:

The activity of the external sources of power which produce a steady cisoidal oscillation in any invariable network assumes the stationary value which is consistent with the conditions imposed by current continuity and the conservation of energy.

With cisoidal oscillations the differentiations and integrations indicated in the equation of activity (**4**) may be carried out and after dividing by the common factor e^{2ipt} and introducing the self and mutual impedances $Z_{qq}\ (=Z_q)$, $Z_{qr}\ (Z_{rq}=Z_{qr})$, the equation becomes

$$\sum_{q=1}^{q=n} E_q\, I_q = \sum_{q=1}^{q=n} Z_q\, I_q^2 + 2 \sum_{r>q=1}^{q<r=n} Z_{qr}\, I_q\, I_r = \sum_{q=1}^{q=n} \sum_{r=1}^{r=n} Z_{qr}\, I_q\, I_r \qquad (8)$$

The left-hand and the right-hand sides of this equation are homogeneous functions of the first and second orders in terms of the currents. Comparison with the first and second order terms in (**4**) shows that the right-hand side of equation (**8**), which is the total power taken by the network, may be substituted in the general theorem for the power dissipated. Or, since the two sides of equation (**8**) are always equal, the left-hand side, which is the power supplied by the sources, may equally well be taken; whence the above theorem follows.

Stationary activity involves stationary driving point impedance and the theorem might be restated in terms of the impedances.

Differentiating equation (**8**) with respect to each of the n currents (after introducing the multiplier $\frac{1}{2}$ for the right-hand side) we have for the general equations determining the distribution of current:

$$\begin{array}{l} Z_{11}\, I_1 + Z_{12}\, I_2 \ldots\ldots\ldots\ldots\ldots\ldots + Z_{1n}\, I_n = E_1 \\ Z_{21}\, I_1 + Z_{22}\, I_2 \ldots\ldots\ldots\ldots\ldots\ldots + Z_{2n}\, I_n = E_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Z_{n1}\, I_1 + Z_{n2}\, I_2 \ldots\ldots\ldots\ldots\ldots\ldots + Z_{nn}\, I_n = E_n \end{array} \qquad (9)$$

The currents are therefore

$$I_q = \sum_{r=1}^{r=n} \frac{A_{qr}}{A} E_r$$

where A is the determinant of the impedances occurring as coefficients of the currents in (**9**) and A_{qr} is the co-factor of Z_{qr} in this determinant. Substituting these in equation (**8**), we find that the stationary power, that is the power which is actually expended on the network, is

$$P = \sum_{q=1}^{q=n} \sum_{r=1}^{r=n} \frac{A_{qr}}{A} E_q\, E_r = \frac{A_{\prime} - A}{A} \qquad (10)$$

where A_e differs from the determinant A only in having each element Z_{qr} augmented by $E_q E_r$.

Self- and mutual-admittances may be substituted for the self- and mutual impedances in the right-hand side of equation (8), the form of the expression being kept unchanged by simultaneously substituting potential differences for currents. The solution in terms of the admittances will then be obtained from a determinant in which the admittances enter precisely as do the impedances in "A". For certain problems, as will be readily seen, the admittance determinant is much more convenient than the impedance determinant. While the impedance determinant is made the special object of discussion in the remainder of this paper, it is to be understood that corresponding applications may be made of the admittance determinant.

The Discriminant of a Network

The discriminant A of a network is defined as the determinant having the element Z_{qr} in the qth row and rth column; Z_{qr} being the mutual impedance between circuits q and r or the self-impedance of circuit q when $q=r$; the determinant to include the self- and mutual impedances of the system of simple circuits obtained by eliminating the branch points by closing each branch on itself and replacing each branch point, in excess of one in each connected part of the system, by a fictitious circuit of zero self-impedance connected by mutual impedances $+i$ and $-i$ to the several branches which have their positive or negative ends respectively at this branch point.

This will be taken as the normal form of the discriminant, since it is symmetrical in terms of all of the real branches and real closed circuits of the network. That it is also essentially symmetrical in all of the branch points follows from the fact that the value of the determinant is independent of the choice of the particular branch points to be excluded. The discriminant A is of fundamental importance in the discussion of the network because all effective impedances of the network may be determined directly from its array.

The degree of A in terms of the actual impedances of the network is equal to the number of degrees of freedom of the network, which is the same as the number of branches, reduced by the number of branch points, omitting one in each connected part of the system. The determinant A is of the first degree in each self-impedance, and of the second degree in each mutual impedance when physically considered, that is when the order of the subscripts is ignored ($Z_{rq} \equiv Z_{qr}$).

The algebraical co-factor of the product of the elements located at the intersection of rows $j, q, s, \ldots$ with columns $k, r, t \ldots$ respectively of determinant A will be denoted by $A_{jk.qr.st} \ldots = A_\alpha$, where α stands for the paired list $j\,k\,.\,q\,r\,.\,s\,t \ldots$ The arithmetical value of the co-factor depends thus only on the choice of rows $j, q, s, \ldots$ and columns $k, r, t \ldots$ which occur in the subscript, while its algebraical sign depends upon the sequence of the rows and columns and is changed by each inversion of rows or columns. It follows that if the same row or column occurs twice in the subscript the value of the co-factor is zero. Where we have occasion to restore one or more rows or columns of the original determinant to a co-factor A_α, the elements to be removed from α will be indicated as a divisor of the subscript α. The algebraical value of the expression $A_{\frac{\alpha}{\beta}}$ is uniquely and completely determined by canceling the denominator against a part or the whole of the numerator, making inversions, if necessary, in the numerator or denominator; in case the denominator cannot be entirely eliminated by this process the symbol indicates a determinant with identical rows or columns, and it is therefore equal to zero. For example:

$$A_{\frac{11\cdot 22}{12}} = -A_{\frac{12\cdot 21}{12}} = -A_{21}, \quad A_{12\cdot 13} = 0, \quad A_{\frac{12\cdot 23}{34}} = 0, \quad A_{\frac{11\cdot 22}{12\cdot 21}} = -A$$

and
$$A_{jk.qr\,st\ldots} = \frac{D_{Z_{jk}}\,D_{Z_{qr}}\,D_{Z_{st}}\cdots A}{2^{(\delta_{jk}+\delta_{qr}+\delta_{st}\,\ldots)}}, \tag{11}$$

$$Z_{rq} \equiv Z_{qr} \not\equiv 0, \qquad \delta_{qr} = \begin{cases} 0 \text{ if } q = r \\ 1 \text{ if } \neq qr \end{cases}$$

where the differentiations correspond to actual physical variations in the impedances and therefore treat mutual impedances with interchanged subscripts as identical.

By applying the following rules the expanded expressions for A and its co-factors may be written down directly from the simple circuit system replacing the network, without reference to the determinant. This method of expansion is often more convenient than the use of the ordinary rules for expanding the determinant.

A is the sum of all possible products in which each circuit is represented either by its self-impedance or by its mutual impedance to another circuit, the mutual impedances occurring, however, in closed cycles of two or more constituents only, so that the subscripts may be written $k\,m$, $m\,q$, $q\,u$, . . . $w\,k$, each cycle introducing the sign-factor $+$ or $-$ according as the cycle contains an odd or an even number of terms; each cycle of three or more circuits also introducing the factor 2 to care for the alternative way of associating the mutual impedances and the circuits of the cycle.

A_{qq} is the coefficient of Z_{qq} in A, *i.e.*, A_{qq} is the value taken by A when circuit q is removed from the network.

A_{qr} is the coefficient of Z_{qr} *after* writing A in symmetrical form with respect to Z_{qr} and Z_{rq}, *i.e.*, A_{qr} is the value taken by A if circuits q and r are represented in each product by the mutual impedances 1 and Z_{qr} respectively.

Effective Impedances of Any Network

In the theoretical discussion of networks we are concerned not so much with particular values of the electromotive forces and currents, as with their relative values. For this reason the impedances, which are the ratios of electromotive forces to currents, and the attenuation factors, which are either the ratios of currents to each other, or of electromotive forces to each other, are chosen as the immediate objects of investigation.

Effective impedances may be defined in various ways, for example as:

$$(a)\quad \frac{\text{potential of point } s_j \text{ minus potential of point } s_k}{\text{current at point } s_l},$$

$$(b)\quad \delta\,\frac{\text{power taken by any part } S_p \text{ of network}}{\text{product of currents at points } s_q \text{ and } s_r},$$

($\delta = 1$ or $\frac{1}{2}$ for self and mutual impedances respectively.)

$$(c)\quad \frac{1}{\delta}\;\frac{\text{product of potential differences points } s_t,\ s_u \text{ and } s_v,\ s_w}{\text{power taken by any part } S_x \text{ of network}}$$

($\delta = 1$ or $\frac{1}{2}$ for self- and mutual impedances respectively.)

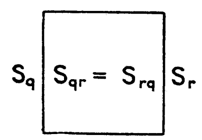
(*d*) The impedances required to make a normal type of network of the requisite number of parameters equivalent to the given network under specified conditions of operation.

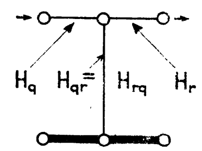
As examples of the above definitions we may instance the following:

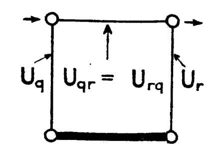
The mutual impedance of a transformer is the ratio, with sign reversed, of the electromotive force induced in either winding

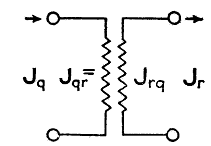
TABLE I. (Campbell.)

EFFECTIVE IMPEDANCES OF EQUIVALENT NETWORKS WITH TWO ACCESSIBLE CIRCUITS, IN TERMS OF EACH OTHER AND OF THE DETERMINANT A.









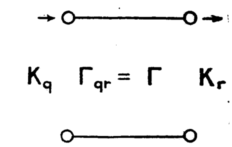


	FIG. 1.	FIG. 2.	FIG. 3.	FIG. 4.	FIG. 5.
$S_q =$	$\frac{A}{A_{qq}}$	$\frac{H_qH_r+H_qH_{qr}+H_rH_{qr}}{H_r+H_{qr}}$	$\frac{U_qU_{qr}}{U_q+U_{qr}}$	$\frac{J_qJ_r-J_{qr}^2}{J_r}$	$\frac{2K_qK_r}{(K_q+K_r)\coth\Gamma+(K_r-K_q)}$
$S_r =$	$\frac{A}{A_{rr}}$	$\frac{H_qH_r+H_qH_{qr}+H_rH_{qr}}{H_q+H_{qr}}$	$\frac{U_rU_{qr}}{U_r+U_{qr}}$	$\frac{J_qJ_r-J_{qr}^2}{J_q}$	$\frac{2K_qK_r}{(K_q+K_r)\coth\Gamma-(K_r-K_q)}$
$S_{qr} =$	$\frac{A}{A_{qr}}$	$\frac{H_qH_r+H_qH_{qr}+H_rH_{qr}}{H_{qr}}$	U_{qr}	$-\frac{J_qJ_r-J_{qr}^2}{J_{qr}}$	$\frac{2K_qK_r}{K_q+K_r}\sinh\Gamma$
$H_q =$	$\frac{S_qS_{qr}(S_{qr}-S_r)}{S^2_{qr}-S_qS_r}$	$\frac{A_{rr}-A_{qr}}{A_{qq\cdot rr}}$	$\frac{U_qU_{qr}}{U_q+U_r+U_{qr}}$	J_q+J_{qr}	$\frac{K_q+K_r}{2}\tanh\frac{\Gamma}{2}+\frac{K_q-K_r}{2}$
$H_r =$	$\frac{S_rS_{qr}(S_{qr}-S_q)}{S_{qr}^2-S_qS_r}$	$\frac{A_{qq}-A_{qr}}{A_{qq\cdot rr}}$	$\frac{U_rU_{qr}}{U_q+U_r+U_{qr}}$	J_r+J_{qr}	$\frac{K_q+K_r}{2}\tanh\frac{\Gamma}{2}-\frac{K_q-K_r}{2}$
$H_{qr} =$	$\frac{S_qS_rS_{qr}}{S_{qr}^2-S_qS_r}$	$\frac{A_{qr}}{A_{qq\cdot rr}}$	$\frac{U_qU_r}{U_q+U_r+U_{qr}}$	$-J_{qr}$	$\frac{K_q+K_r}{2\sinh\Gamma}$
$U_q =$	$-\frac{S_qS_{qr}}{S_q-S_{qr}}$	$\frac{H_qH_r+H_qH_{qr}+H_rH_{qr}}{H_r}$	$\frac{A}{A_{qq}-A_{qr}}$	$\frac{J_qJ_r-J_{qr}^2}{J_r+J_{qr}}$	$\frac{2K_qK_r}{(K_q+K_r)\tanh\frac{\Gamma}{2}+(K_r-K_q)}$
$U_r =$	$-\frac{S_rS_{qr}}{S_r-S_{qr}}$	$\frac{H_qH_r+H_qH_{qr}+H_rH_{qr}}{H_q}$	$\frac{A}{A_{rr}-A_{qr}}$	$\frac{J_qJ_r-J_{qr}^2}{J_q+J_{qr}}$	$\frac{2K_qK_r}{(K_q+K_r)\tanh\frac{\Gamma}{2}-(K_r-K_q)}$
$U_{qr} =$	S_{qr}	$\frac{H_qH_r+H_qH_{qr}+H_rH_{qr}}{H_{qr}}$	$\frac{A}{A_{qr}}$	$-\frac{J_qJ_r-J_{qr}^2}{J_{qr}}$	$\frac{2K_qK_r}{K_q+K_r}\sinh\Gamma$
$J_q =$	$\frac{S_qS_{qr}^2}{S_{qr}^2-S_qS_r}$	H_q+H_{qr}	$\frac{U_q(U_r+U_{qr})}{U_q+U_r+U_{qr}}$	$\frac{A_{rr}}{A_{qq\cdot rr}}$	$\frac{K_q+K_r}{2}\coth\Gamma+\frac{K_q-K_r}{2}$
$J_r =$	$\frac{S_rS_{qr}^2}{S_{qr}^2-S_qS_r}$	H_r+H_{qr}	$\frac{U_r(U_q+U_{qr})}{U_q+U_r+U_{qr}}$	$\frac{A_{qq}}{A_{qq\cdot rr}}$	$\frac{K_q+K_r}{2}\coth\Gamma-\frac{K_q-K_r}{2}$
$J_{qr} =$	$-\frac{S_qS_rS_{qr}}{S_{qr}^2-S_qS_r}$	$-H_{qr}$	$\frac{-U_qU_r}{U_q+U_r+U_{qr}}$	$-\frac{A_{qr}}{A_{qq\cdot rr}}$	$-\frac{K_q+K_r}{2\sinh\Gamma}$
$K_q =$	$\sqrt{\frac{S_qS_rS_{qr}^2}{S_{qr}^2-S_qS_r}+\left[\frac{(S_q-S_{qr})\,S_{qr}^2}{2(S_{qr}^2-S_qS_r)}\right]^2}+\frac{(S_q-S_r)\,S_{qr}^2}{2(S_{qr}^2-S_qS_r)}$	$\sqrt{\frac{(H_q+H_r)(H_q+H_r+4H_{qr})}{4}}+\frac{H_q-H_r}{2}$	$\frac{\sqrt{U_{qr}(U_q+U_r)(U_qU_{qr}+U_rU_{qr}+4U_qU_r)}+U_q-U_r)\,U_{qr}}{2(U_q+U_r+U_{qr})}$	$\sqrt{\left(\frac{J_q+J_r}{2}\right)^2-J_{qr}^2}+\frac{J_q-J_r}{2}$	$\sqrt{\frac{A}{A_{qq\cdot rr}}+\left(\frac{A_{rr}-A_{qq}}{2\ A_{qq\cdot rr}}\right)^2}+\frac{A_{rr}-A_{qq}}{2\ A_{qq\cdot rr}}$
$K_r =$	$\sqrt{\frac{S_qS_rS_{qr}^2}{S_{qr}^2-S_qS_r}+\left[\frac{(S_q-S_r)\,S_{qr}^2}{2(S_{qr}-S_qS_r)}\right]^2}-\frac{(S_q-S_r)\,S_{qr}^2}{2(S_{qr}^2-S_qS_r)}$	$\sqrt{\frac{(H_q+H_r)(H_q+H_r+4H_{qr})}{4}}-\frac{H_q-H_r}{2}$	$\frac{\sqrt{U_{qr}(U_q+U_r)(U_qU_{qr}+U_rU_{qr}+4U_qU_r)}-(U_q-U_r)\,U_{qr}}{2(U_q+U_r+U_{qr})}$	$\sqrt{\left(\frac{J_q+J_r}{2}\right)^2-J_{qr}^2}-\frac{J_q-J_r}{2}$	$\sqrt{\frac{A}{A_{qq\cdot rr}}+\left(\frac{A_{rr}-A_{qq}}{2\ A_{qq\cdot rr}}\right)^2}-\frac{A_{rr}-A_{qq}}{2\ A_{qq\cdot rr}}$
$\Gamma_{qr} =$	$\cosh^{-1}\frac{S_{qr}(S_q+S_r)}{2S_qS_r}$	$\cosh^{-1}\frac{H_q+H_r+2H_{qr}}{2H_{qr}}$	$\cosh^{-1}\left(1+\frac{(U_q+U_r)\,U_{qr}}{2U_qU_r}\right)$	$\cosh^{-1}\frac{-(J_q+J_r)}{2J_{qr}}$	$\cosh^{-1}\frac{A_{qq}+A_{rr}}{2A_{qr}}$

to the inducing current flowing in the other winding, which falls under definition (*a*) if the secondary is open-circuited.

In discussing below the power taken by the actual resistances in a network use is made of definition (*b*) in formula (**26**).

The expression, formula (**10**), for the total power taken by a network in terms of the impressed forces, gives, on breaking up the expression into its individual terms, a set of self-impedances and mutual impedances defined in accordance with definition (*c*).

As an example of definition (*d*) we may take the important case where we are concerned only with two accessible circuits in a network and wish to replace the given network by a normal type having only the required three complex parameters. The normal networks which are ordinarily employed are the " T ", the " Π," the transformer and the artificial line and for these the effective impedances are given in the following table, together with the simple circuit impedances which equal the driving point impedance in either circuit S_q and S_r and the driving-driven point impedance of a single circuit S_{qr} which would give the electromotive force ÷ current ratio actually obtaining when the electromotive force is inserted in q (or r) and the current is measured in r (or q). J_q, J_r, J_{qr} are called the primary, secondary and mutual impedances as they correspond to the primary self-inductance, secondary self-inductance, and mutual inductance following established scientific usage. This terminology is employed throughout this paper, as its extension to three or more circuits is obvious and symmetrical, and it seems to be the only logical system. Many electrical engineers, however, call H_q, H_r, H_{qr}^{-1} (H_{qr} being taken with inductive reactance) the primary impedance, secondary impedance and primary admittance, in case the assumed ratio of turns is 1 to 1.

The table refers to the general case where the two circuits are not symmetrical, but the formulæ are in such form as to facilitate reduction to the special case of symmetrical circuits. In this table different letters are employed for the various effective impedances thus somewhat reducing the multiplication of subscripts.

Elimination of Concealed Circuits

In general we may divide a network into a concealed and an accessible part and it is convenient to eliminate the former from explicit appearance in the impedance determinant A when we are concerned only with the effects which are produced in the accessible part of the network due to causes which are likewise confined to this part of the network.

Elimination of a group of concealed circuits (or of any circuits which contain no impressed forces) from explicit appearance in A is equivalent to the substitution of new effective impedances

$$J_{qr} = \frac{A_{\frac{\alpha}{qr}}}{A_\alpha}$$

between accessible circuits q and r where α stands for the product of the original self-impedances of the accessible circuits.

To prove this we notice that the set of Kirchhoff electromotive force equations for the concealed circuits taken alone give

$$I_c = \frac{1}{A_\alpha} \sum_r I_r A_{\frac{\alpha.xc}{xr}} \qquad \text{where} \begin{cases} c = \text{any concealed circuit} \\ r = \text{any accessible circuit} \\ x = \text{any circuit} \end{cases}$$

which substituted in the electromotive force equation for any accessible circuit q make the new coefficient of I_r in this equation

$$J_{qr} = Z_{qr} + \frac{1}{A_\alpha} \sum_c Z_{qc} A_{\frac{\alpha.qc}{qr}} = \frac{1}{A_\alpha} \left[Z_{qr} A_{\left(\frac{\alpha}{qr}\right)^{qr}} + \sum_c Z_{qc} A_{\left(\frac{\alpha}{qr}\right)^{qc}} \right]$$

after setting $x = q$

$$= \frac{A_{\frac{\alpha}{qr}}}{A_\alpha} = \frac{A_{\frac{\alpha}{rq}}}{A_\alpha} = J_{rq} \text{ as } A \text{ is symmetrical} \tag{12}$$

J_{qr} is thus the new effective mutual impedance (or self-impedance if $q = r$) between accessible circuits q and r.

In the important case where all but two of the circuits are eliminated, we have

$$\begin{aligned} J_{qq} &= \frac{A_{rr}}{A_{qq \cdot rr}} \\ J_{rr} &= \frac{A_{qq}}{A_{qq \cdot rr}} \\ J_{qr} &= \frac{-A_{qr}}{A_{qq \cdot rr}} \end{aligned} \tag{13}$$

And if but one circuit q is regarded as accessible, the driving point impedance of the network to an electromotive force inserted in that circuit, is

$$J_{qq}=\frac{A}{A_{qq}} \tag{14}$$

If we eliminate the circuits corresponding to all of the branch points and to an equal number of the branches which are connected to these branch points but do not form any closed circuit among themselves, it may be shown that: $A_\alpha=1$; the new effective impedances are equal to sums and differences of the original impedances with coefficients which are 0, ± 1, or ± 2; the circuits which are not eliminated are equal in number to the degrees of freedom of the network. The case falls under that directly derived above by the use of circuital currents.

If $A_\alpha=0$ the method of elimination fails, which shows that whenever fictitious branch point circuits are eliminated at least one branch connected to each branch point must be included and that the number of closed circuits formed by the branches must not be greater than the excess of eliminated branches over eliminated branch points.

No change is made in the effective self- or mutual impedance of an accessible circuit q by the elimination of circuits which have no mutual impedance with circuit q. That is $A_{\frac{\alpha}{qr}}=Z_{qr}A_\alpha$ since the added q row has but one term Z_{qr} which differs from zero.

A concealed branch of admittance Y which is free from mutual impedances may be eliminated by adding Y to each of the two self-impedances and subtracting Y from the mutual impedance of the two fictitious circuits which replace the terminal branch points of the concealed branch. Any number of concealed branches may be eliminated in this way; the total self-impedance added to any fictitious circuit will equal the total conductance of the eliminated branches terminating at the corresponding branch point; the total mutual impedance subtracted between any two fictitious circuits will equal the total conductance eliminated between the corresponding branch points.

To prove, let the concealed branch impedance be $Z=1\div Y=A_\alpha$, then, if the self- and mutual-impedances of the fictitious circuits correspond to the terminals of this branch are originally Z_1, Z_2

and Z_{12}, they become after the elimination of the concealed branch

$$J_1 = \frac{A^{\alpha}_{\overline{11}}}{A_\alpha} = Y\begin{vmatrix} Z_1 & \pm i \\ \pm i & Z \end{vmatrix} = Z_1 + Y$$

$$J_2 = \frac{A^{\alpha}_{\overline{22}}}{A_\alpha} = Y\begin{vmatrix} Z_2 & \mp i \\ \mp i & Z \end{vmatrix} = Z_2 + Y$$

$$J_{12} = \frac{A^{\alpha}_{\overline{12}}}{A_\alpha} = Y\begin{vmatrix} Z_{12} & \pm i \\ \mp i & Z \end{vmatrix} = Z_{12} - Y$$

Any concealed part of a network connected to the remainder of the network through a group of terminals (branch points) $q, r, s, \ldots$ only (and having the impedance determinant A_α or A_β according as the concealed part is taken alone or is taken together with the circuits corresponding to the group of accessible terminals) *may be replaced by either of the following*:

(*a*) Self-impedances $A^{\alpha}_{\underline{qq}} \div A_\alpha$ and mutual impedances $A^{\alpha}_{\underline{qr}} \div A_\alpha$ added to the fictitious circuits corresponding to the group of terminals.

(*b*) Branches, devoid of mutual impedance, connecting the group of terminals in pairs and having the admittances $-A^{\alpha}_{\underline{qr}} \div A_\alpha$. These admittances we will call the equivalent direct admittances of the network.

(*c*) Branches radiating from a common concealed point, one to each of the terminals, with self-impedances $A_{\beta.qq} \div A_\beta$ and mutual impedances $A_{\beta\,qr} \div A_\beta$.

(*d*) Branches radiating from a common concealed point, one to each of the group of terminals, these branches being devoid of self-impedance and having mutual impedances $-(A_{\beta.qq} + A_{\beta\,rr} - 2\,A_{\beta.qr}) \div 2\,A_\beta$.

(*e*) Branches connecting any one of the terminals q to each of the remaining accessible terminals $r, s, \ldots$, the branch connected to terminal r having the self-impedance $(A_{\beta.qq} + A_{\beta.rr} - 2\,A_{\beta\,qr}) \div A_\beta$ and the mutual impedance $(A_{\beta\,qq} + A_{\beta.rs} - A_{\beta.qr} - A_{\beta.qs}) \div A_\beta$ to the branch connected to terminal s.

Substitution (*a*) is a restatement of the results previously established for the case of concealed and accessible parts which are not connected the one to the other by mutual impedances.

To show that (b) is equivalent to (a) apply the theorem for eliminating concealed branches which are devoid of mutual impedances to (b); the fictitious circuits corresponding to the group of terminals will thereby have their mutual impedances increased by $A_{\frac{\alpha}{qr}} \div A_\alpha$ and their self-impedances increased by

$$-\frac{1}{A_\alpha}\sum_{q \pm r} A_{\frac{\alpha}{qr}} = \frac{1}{A_\alpha}\left(A_{\frac{\alpha}{qq}} - \sum A_{\frac{\alpha}{qr}}\right) = A_{\frac{\alpha}{qq}} \div A_\alpha$$

since the complete summation with respect to r of the bordered determinants $A_{\frac{\alpha}{qr}}$ equals the determinant A_α bordered by the row q and a column equal to the sum of all of the fictitious circuit columns r, and vanishes since terms $+i$ and $-i$ occur in pairs and cancel, making the column identically equal to zero. Substitution (b) having been transformed into substitution (a) the two are mutually equivalent.

Substitutions (c), (d) and (e) are readily shown to be mutually equivalent to each other and to the original network by showing that the impedance between any two terminals u and v with all others insulated is $(A_{\beta.uu} + A_{\beta.vv} - 2A_{\beta.uv}) \div A_\beta$.

The direct conductance between two terminals of any network, as defined under (b), is equal to one-half of the excess of the grounded conductance of the two terminals taken separately over their grounded conductance when taken together as a single terminal. By the grounded conductance of a terminal is understood the conductance between that terminal and ground with all of the other terminals grounded. As grounded conductances can be readily measured with simple apparatus, this always affords one method of experimentally determining the direct conductances in any network.

Complete Elimination of Either Mutual Impedances or Self-Impedances

It is shown by what has preceded that if we retain only a group of terminals as the accessible part, any network may be replaced either by a set of direct impedances connecting the terminals in pairs, or by a set of mutual impedances between branches radiating from a common point and terminating one at each of the terminals. In the first case all mutual impedances are avoided; in the second case all self-impedances are avoided. Applications to the simple transformer are of interest as showing that in these substitutions an open circuit is taken care of either by parallel self-impedances which are equal but of opposite signs or by infinite mutual impedances differing by finite amounts. The substitutions show that a transformer J_1, J_2, J_{12}, is equivalent to either

(*a*) The six-branch network directly connecting the four terminals, the impedances of which are

$$\frac{J_1 J_2 - J_{12}^2}{J_2}, \quad \frac{J_1 J_2 - J_{12}^2}{J_1}, \quad \frac{J_1 J_2 - J_{12}^2}{J_{12}} \quad - \frac{J_1 J_2 - J_{12}^2}{J_{12}} \tag{15}$$

between the primary terminals, the secondary terminals, each of the two pairs of correspondingly poled terminals of primary and secondary and each of the two pairs of non-corresponding terminals of primary and secondary, respectively. (In Figs. 6 and 7 terminals 1-2, 3-4, 1-3, and 2-4, 1-4 and 2-3 respectively.)

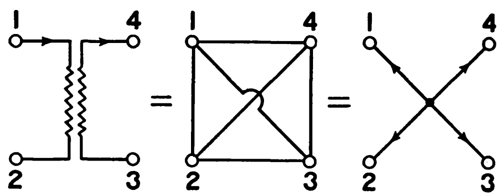


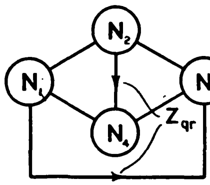
FIG. 6 FIG. 7 FIG. 8
Transformer and equivalent networks having four accessible terminals

Or (*b*) the four-branch network connecting the four terminals to a concealed common point, the mutual impedances being

$$\frac{J_1}{2}, \quad \frac{J_2}{2}, \quad \infty - \frac{J_{12}}{4}, \quad \infty + \frac{J_{12}}{4} \tag{16}$$

between the branches (taken with their positive directions diverging from the common point) which terminate at the primary terminals, the secondary terminals, each of the two pairs of corresponding terminals of primary and secondary and each of the two pairs of non-corresponding terminals of primary and secondary, respectively. See Figs. 6 and 8.

In certain cases a mutual impedance may be eliminated by properly augmenting the impedances of not more than four branches, without altering the arrangement of branches in any way or imposing any restriction as to whether they are concealed or accessible. These cases are all included under that of mutual impedance between diagonally opposite branches of a generalized bridge, by which we will understand a network differing from the ordinary bridge only in having the four bridge corners replaced by four arbitrary networks; these corner networks may have mutual impedances between each other, but the only branches connecting them are to be the six branches corresponding to the simple bridge. *Mutual impedance between diagonally opposite branches in the generalized bridge is replaceable by an equal amount*



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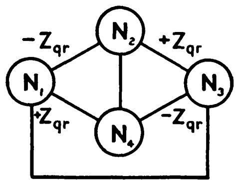
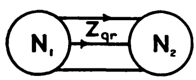


FIG. 9 FIG. 10
Generalized bridge with equivalent mutual and self impedances

of self-impedance in each of the four bridge-arms, added to or subtracted from the original self-impedance of the arm, according as the arm connects the branches having the mutual impedance with their positive directions concurrent or opposed. (Figs. 9 and 10.) An important special case is that in which one arm of the bridge is open-circuited and the network reduces to three branches connecting two arbitrary networks otherwise unconnected except possibly by mutual impedances. (Figs. 11 and 12.)

The correctness of the substitution is shown by the fact that



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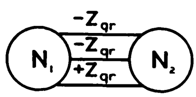

FIG. 11 FIG. 12
Three-branch connection with equivalent mutual and self impedances

the impedance of every closed circuit is the same before and after the substitution, and that this is the most general case is proven by noticing: (1), that the generalized bridge becomes an unrestricted network by admitting any number of branches connecting the four corner networks in pairs; and (2), that with a single branch added to Figs. 9 and 10, it is impossible to keep the self-impedance of every closed circuit the same in the two cases for the added branch requires different increments according to the circuit through which it is closed.

In the simple bridge circuit there are 15 possible mutual impedances which may be eliminated by taking as the effective branch impedances the six permutations of

$$Z_{12}'=Z_{12}+Z_{12,23}+Z_{12,24}+Z_{12,31}+Z_{12,41}+Z_{13,14}+Z_{23,24}+Z_{13,42}+Z_{14,32} \tag{17}$$

where 1, 2, 3, 4 stand for the bridge corners. The condition for a balance of the bridge arms 12, 23, 34, 41 is therefore always

$$Z_{12}'\,Z_{34}'=Z_{23}'\,Z_{41}' \tag{18}$$

IMPEDANCE LOCI

It is often of importance to know how the impedances of a network will vary if the self-impedances or mutual impedances of one or more of the branches of the network are varied over lines or areas in any physically possible manner. On account of the magnitude of this subject we shall touch on the simplest case only, namely that of the driving point impedance with a variable impedance added to one branch of the network.

As the discriminant A and its minors are of the first degree in terms of each self-impedance which they contain, it follows that the effective impedances of the network, being equal to the quotient of two of these determinants, are bilinear functions of the individual impedances; thus the driving point impedance of a network at circuit q is connected with a self-impedance Z inserted in any circuit r by a relation of the form

$$S_q=\frac{a\ Z+b}{c\ Z+d} \tag{20}$$

where a, b, c and d are constants.

The property of the bilinear transformation which is of special importance to us is that it transforms circles into circles, that is, if Z be regarded as a variable and be made to traverse any circle whatsoever, the driving point impedance S will also describe a circle. In making this statement the straight line is included as the limit of a circle so that the loci of S and Z may be straight lines as well as circles. This property of the bilinear transformation is discussed at length in the theory of analytic functions and need not be entered into here.

We are especially concerned with the cases where the locus of Z is a straight line such as the axis of reals or the axis of imaginaries, because the first is a variation which it is convenient to make use of in practical measurements and the second forms the extreme boundary realizable with physically possible values of the inserted impedance. We shall find it better to replace the constants a, b, c and d by others, such as the effective transformer impedances or the effective line constants, which have a physical significance.

A network having effective transformer constants J_1, J_2, J_{12} *effects the transformation of the half of the* $Z-$*plane on the positive side of the reactance axis into the area bounded by a circle with center at* Z_i *and radius* R_i:

$$Z_i = J_1 - \frac{J_{12}^2}{J_2 + J_2'}, \qquad R_i = \frac{|J_{12}^2|}{J_2 + J_2'} \tag{21}$$

the axis of reals going over into the circumference of a circle having its center at Z_r *and radius* R_r:

$$Z_r = J_1 - \frac{J_{12}^2}{J_2 - J_2'}, \qquad R_r = \frac{|J_{12}^2|}{|J_2 - J_2'|} \tag{22}$$

where J_2' *is the conjugate of* J_2 *and* $|J_{12}^2|$ *the modulus of* J_{12}^2. *The two circuits cut each other orthogonally at* J_1 *and* $(J_1 - J_{12}^2 \div J_2)$ *which correspond to open and short-circuited secondary. The double points (effective line impedances—far end with sign reversed) are*

$$\left.\begin{matrix} K_1 \\ -K_2 \end{matrix}\right\} = \frac{J_1 - J_2}{2} \pm \tfrac{1}{2}\sqrt{(J_1 + J_2)^2 - 4J_{12}^2} \tag{23}$$

Proof: Close the secondary through the added impedance $Z\,x$, where x is a real variable, and the effective driving point impedance at the primary is

$$S_1 = \frac{J_1(J_2 + Zx) - J_{12}^2}{J_2 + Zx} = J_1 - \frac{J_{12}^2}{J_2 + Zx}\;\frac{Z'(J_2 + Zx) - Z(J_2' + Z'x)}{J_2\,Z' - J_2'\,Z} \tag{24}$$

$$= \left(J_1 - \frac{J_{12}^2\,Z'}{J_2\,Z' - J_2'\,Z}\right) + \left(\frac{J_{12}^2\,Z}{J_2\,Z' - J_2'\,Z}\right)\left(\frac{J_2' + Z'\,x}{J_2 + Zx}\right) \tag{25}$$

and in this form the expression obviously represents a circle, the center of which is the first term, the radius being the modulus of the last term, since the variable x occurs only in the last factor of the last term and variations in x change the angle but not the modulus of this factor as its numerator is the conjugate of its denominator. If $Z = - Z' = i$, the inserted impedance is pure imaginary and we obtain from (**25**) the constants for the boundary circle as given in (**21**). If $Z = Z' = 1$, the added impedance is real and the effective driving point impedance falls on a circle with the constants as given in (**22**). To determine the double points substitute $Zx = S = K$ in the first part of

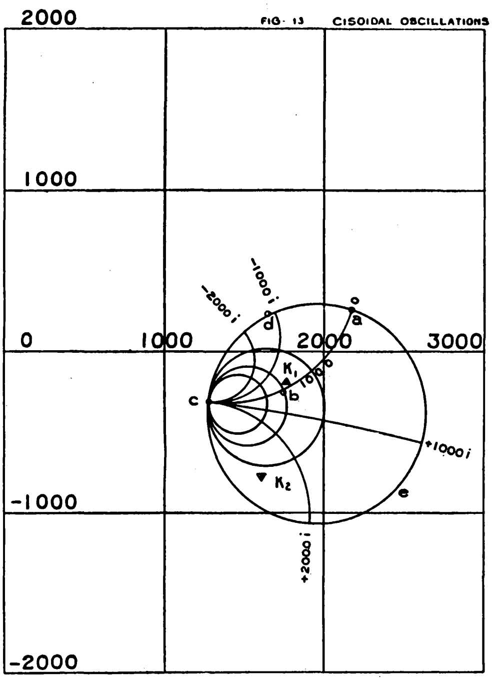

FIG. 13.—Bilinear transformation for a line containing 6.201 wave lengths and having the attenuation constant 0.9123 and the line impedances $K_1 = 1762 - 191\ i$ and $K_2 = 1614 - 781i$ which maps the half plane on the positive side of the imaginary axis into the circle *a d c e* with the rectangular ruling mapping into the orthogonal system of circles.

(**24**) and solve the resulting quadratic in K which gives the values (**23**).

As a practical example of impedance loci, consider Fig. 13, which shows the driving point impedance of a transmission line for a frequency of 1,300 cycles per second, the line containing 6.201 wave lengths, presenting an attenuation constant of 0.9123 and having line impedances $K_1 = 1762 - 191i$ and $K_2 = 1614$

$-781i$ for transmission from the driving point to the receiving end and vice versa. The driving point impedances actually measured are the points marked by circles near a, b, c and d for which the far end of the line was closed through a short circuit, through 2,000 ohms, through an open circuit and through a capacity of 0.107 mf. respectively. As but three measurements are necessary in order to completely determine the three bilinear constants, it was necessary to adjust the four observations to the most probable bilinear transformation. It will be seen that the corrections which it was necessary to apply to the observations were small, being in fact well within the errors of observation. The circle $a\ d\ c\ e$ corresponds to the entire imaginary axis of Z; the arc $a\ b\ c$ corresponds to the entire positive axis of Z. Circles are also shown corresponding to values of Z having constant real components of 1,000, 2,000 and 3,000 ohms and constant imaginary components of $\pm 1{,}000$ and $\pm 2{,}000$ ohms. The line impedances K_1 and K_2 are also shown. As the particular line under measurement effects the transformation of the rectangular network shown in Fig. 13 into the orthogonal system of circles, the diagram shows that the driving point impedance has the resistance limits 1,270 to 2,640 ohms and the reactance limits $-1{,}070$ to $+300$ ohms. The diagram as it stands is sufficiently complete to permit of reading off approximately the value of the driving point impedance for any value of the impedance Z bridged at the receiving end of the line.

The following construction will be required below and may be proven here.

The effective joint impedance S of two impedances Z_1, Z_2 in parallel coincides with the intersection of the circles which are tangent to these impedances at the origin and have the individual impedances as chords. This construction follows at once from the circular locus of S for variable modulus of either Z_1 or Z_2 and the fact that if one of the parallel impedances Z_1 vanishes or the other impedance Z_2 becomes infinite the joint impedance is equal to Z_1. This construction is employed in Fig. 15 for obtaining S from Z_1 and Z_2 or vice versa.

Division of Power between the Resistances and Reactances of a Network

The total power taken by a network is the sum of the powers taken by the individual self-impedances and mutual impedances, and to determine the division of this power between parts of the network it is merely necessary to find the summations for each part separately. As the total power and all of its components are directly proportional to the square of the current entering the network at the driving point, it is more convenient to consider, as the immediate object of discussion, the effective impedances which are defined as the ratios of the powers to the driving current squared. Accordingly we shall discuss the effective impedances S, U, $V\,i$ which correspond respectively to the total powers taken by the entire network, by the true resistances alone, and by the reactances alone. From this definition of these impedances it follows that

$$S = U + Vi = \sum_{j=1}^{n}\sum_{k=1}^{n} Z_{jk}\frac{I_j I_k}{I_d^2} = \sum_{j=1}^{n} Z_j \left(\frac{I_j}{I_d}\right)^2 + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} Z_{jk}\frac{I_j I_k}{I_d^2}$$

$$U = \sum_{j=1}^{n}\sum_{k=1}^{n} R_{jk}\frac{I_j I_k}{I_d^2} \qquad \text{where } Z_{jk} = R_{jk} + i\,X_{jk}$$

$$Vi = i\sum_{j=1}^{n}\sum_{k=1}^{n} X_{jk}\frac{I_j I_k}{I_d^2}$$

The impedance U corresponding to the power taken by the resistances in the general passive network may have any argument, and any modulus which is not greater than the effective resistance of the network. To prove:

Consider an ideal line of zero attenuation containing s wave lengths, closed at the far end through a resistance equal in value to the line impedance K, with an impedance $(R - K) + Bi$ in series at the sending end so as to make the total impedance at the sending end equal to $R + Bi$. A current I flowing at the sending end gives rise to a current I cis $(-2\pi s)$ at the receiving end so that the total power taken by the resistances is

$$P = (R - K)\, I^2 + K\, I^2 \operatorname{cis}(-4\pi s)$$

Therefore

$$U = (R - K) + K \operatorname{cis}(-4\pi s)$$

an impedance which may obviously assume any argument and any modulus not exceeding R with positive real values of K, $(R - K)$, and s.

The modulus of U can under no circumstances be greater than the effective resistance of the network for if this were the case the correlated sinusoidal oscillation would have, at some part of each oscillation, a negative total consumption of power by the resistances which is obviously impossible when the network contains neither sources of power nor so-called negative resistances, which are excluded throughout this discussion.

In Fig. 14 S and R represent the effective driving point impedance and the effective driving point resistance of the network, while U and Vi show a possible resolution of the impedance S into components corresponding to the powers taken by the true resistances and the reactances respectively. The circle $R\ b\ c\ d$ drawn about the origin with $O\ R$ as a radius is the maximum possible locus of U. If U falls at point b or d the power taken by the reactances has its maximum or its minimum value. If U falls at R the power taken by the reactances is 90 degrees ahead of the power taken by the resistances and this case corresponds to the series arrangement of a resistance and a reactance. (This is for positive reactance; with negative reactance the lead becomes a 90 degree lag.) At point c the relative phases are reversed, the resistances taking power 90 degrees in advance of that taken by the reactances; this case, as follows from the formulæ deduced below for parallel circuits, may be realized theoretically by the association of a pure resistance and a pure positive reactance in parallel. The point of special interest is the origin O; if U vanishes the cisoidal powers taken by the various resistances cancel each other in the summation for the resultant; in the correlated sinusoidal oscillation the power taken by the resistances is constant, that is the total generation of heat in the network does not fluctuate during an oscillation. This would seem to be a property which might have practical application.

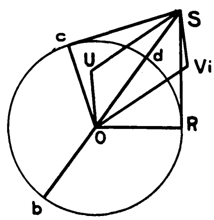

FIG. 14.—Resolution of the driving point impedance S into the impedances corresponding to the power taken by the resistances (U) and by the reactances ($V\,i$); $R\ b\ c$ is the extreme boundary for U.

If λ_1, λ_2, are the maximum and minimum driving point impedance arguments obtainable from the elements employed in a network of driving point impedance $S = R + Xi = |S| \text{cis}\, \sigma$ the impedance U must lie in the lenticular area common to the two circles which intersect at the effective resistance of the network (R) and are centered at the projections of S on lines drawn through the origin at the angles $(\sigma - \lambda_1)$ and $(\sigma - \lambda_2)$.

Multiply each impedance Z_{jk} in the network by cis $\left(\frac{\pi}{2}-\lambda\right)$. As this leaves the current ratios unchanged the new value of U will be

$$\underline{U}=\sum\sum|Z_{jk}|\cos\left(\frac{\pi}{2}-\lambda+\sigma_{jk}\right)\frac{I_j I_k}{I_d^2}$$

$$=i\cos\lambda\sum\sum|Z_{jk}|\ \text{cis}\ \sigma_{jk}\frac{I_j I_k}{I_d^2}$$

$$-\,i\ \text{cis}\ \lambda\sum\sum|Z_{jk}|\cos\sigma_{jk}\frac{I_j I_k}{I_d^2}$$

$$=i\,S\cos\lambda-i\,U\ \text{cis}\ \lambda$$

Therefore $\quad U=i\,\underline{U}\ \text{cis}\ (-\lambda)+S\cos\lambda\ \text{cis}\ (-\lambda)$

which expresses the actual value of U in terms of the modified value $\underline{U}$. We may make $\lambda=\lambda_1$ without introducing any resultant negative resistance in the network, for the multiplication of each impedance by cis $\left(\frac{\pi}{2}-\lambda\right)$ has increased the argument of each simple or combination impedance by $\left(\frac{\pi}{2}-\lambda\right)$, which raises the maximum arguments from λ_1 to $\left(\frac{\pi}{2}-\lambda+\lambda_1\right)$. The extreme possible boundary limit for $\underline{U}$ will then be the circle of radius equal to the new effective resistance or

Extreme limit for $\underline{U}=|S|\cos\left(\sigma+\frac{\pi}{2}-\lambda_1\right)$ cis μ where μ is any real angle

$$=S\sin(\lambda_1-\sigma)\ \text{cis}\ (\mu-\sigma)$$

Substituting this in the above equation, we obtain as a necessary condition

$$\text{Limit for } U=|S|\cos\lambda_1\ \text{cis}\ (\sigma-\lambda_1)+i|S|\sin(\lambda_1-\sigma)\ \text{cis}\ (\mu-\lambda_1)$$

Since the only variable is the unrestricted real quantity μ this locus for U is the circle of which the center is the first term and the radius the modulus of the second term. The first term is the point at the foot of the perpendicular let fall from the extremity of S on the line cis $(\sigma-\lambda_1)$ and the distance from this point to the extremity of R is $||S|\cos\sigma-|S|\cos\lambda_1\ \text{cis}\ (\sigma-\lambda_1)|$ $=|S||\text{cis}-\lambda_1||\cos\sigma\ \text{cis}\ \lambda_1-\cos\lambda_1\ \text{cis}\ \sigma|=|S\sin(\lambda_1-\sigma)|$, the modulus of the second term.

The corresponding proof for the minimum limit is made by substituting $-\left(\frac{\pi}{2}-\lambda_2\right)$ for $\left(\frac{\pi}{2}-\lambda\right)$ That the lenticular area thus defined is a sufficient as well as necessary restriction is proven by the properties of parallel circuits discussed below.

We may note that U cannot vanish unless there is a range of at least 90 degrees in the impedances of the elements entering the network.

All possible distributions of power between the resistances and the reactances, with any given total driving point impedance may be obtained from two reactive resistances in parallel, and it will now be of interest to examine this case in detail. We will assume that the impedances Z_1, Z_2, when connected in parallel are to have a given total effective impedance S and a given impedance U corresponding to the total power taken by the resistances. These conditions give

$$S=\frac{Z_1 Z_2}{Z_1+Z_2}$$

$$U=\frac{Z_1+Z_1'}{2}\left(\frac{Z_2}{Z_1+Z_2}\right)^2+\frac{Z_2+Z_2'}{2}\left(\frac{Z_1}{Z_1+Z_2}\right)^2$$

$$=\frac{(Z_1 S'-Z_1' S)^2}{Z_1^2(Z_1'-S')}+\frac{S+S'}{2} \tag{27}$$

where the first expression for U is in terms of the resistances and current ratios and the second expression is found by substituting for Z_2 its value in terms of Z_1 and S.

$$\left.\begin{aligned} \text{Let } F&=|F|\text{cis }\varphi=U-\frac{S+S'}{2}\\ S&=|S|\text{cis }\sigma\\ Z&=|Z|\text{cis }\theta \end{aligned}\right\} \tag{28}$$

and put the last expression for U in the form

$$2\,F Z^2(Z'-S')=(Z S'-Z' S)^2=-4|Z^2 S^2|\sin^2(\theta-\sigma)$$

where subscripts are omitted as the equation applies equally to Z_1 and Z_2. Taking the imaginary part of this equation before and after multiplying by cis $(-\theta-\varphi)$ we have, after dropping the common factors $2|FZ^2|i$ and $2|Z^2S|\sin(\theta-\sigma)\,i$,

$$|Z|\sin(\theta+\varphi)-|S|\sin(2\theta-\sigma+\varphi)=0$$

$$-|F|=2|S|\sin(\theta-\sigma)\sin(\theta+\varphi)=|S|[(\cos(\sigma+\varphi)-\cos(2\theta-\sigma+\varphi)]$$

therefore $Z=|S|\,\dfrac{\sin(2\theta-\sigma+\varphi)}{\sin(\theta+\varphi)}\operatorname{cis}\theta$ **(29)**

with values of θ given by

$$\cos(2\theta-\sigma+\varphi)=\cos(\sigma+\varphi)+|F\div S|$$

which is the required solution for Z_1 and Z_2.

The graphical construction for determining S and U when Z_1 and Z_2 are given, or vice versa, are sufficiently simple to be of assistance. The construction rules which are readily deducible from the preceding work are as follows:

Given Z_1 and Z_2 to find S and U, Fig. 15. Find the impedance S of Z_1 and Z_2 in parallel and draw the circle having S as a diameter, on this circle locate points d_1 and d_2 so that arc $S\,d_1=$arc $c_1\,R$, and arc $S\,d_2=$arc $c_2\,R$ where c_1, c_2 and R are the intersections of the circle with Z_1, Z_2, and the resistance axis, using d_1 and d_2 as centers strike circles passing through point R. The other intersection of the circles is the effective impedance U.

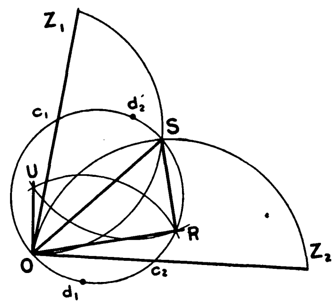

FIG. 15.—Graphical construction for determining the effective driving point impedance (S) and the effective impedance corresponding to the power taken by the resistances (U) for two impedances Z_1 and Z_2 in parallel.

Given S and U to find Z_1 and Z_2. Find the intersections d_1 and d_2 of the circle having S as a diameter and the normal right line bisecting $U\,R$, lay off arc $c_1\,R=$arc $S\,d_1$, and arc $c_2\,R=$arc $S\,d_2$. Then $O\,c_1$ and $O\,c_2$ are the direction lines for Z_1, Z_2 the magnitude of which are found by the intersection therewith of the circles tangent to $O\,c_1$ and $O\,c_2$ which have $O\,S$ as a chord.

The vanishing of U requires a difference of 90 degrees in the two impedances Z_1 and Z_2; if the driving point impedance S is to be pure resistance $(=R)$ we have the important case where the parallel impedances are $(R\pm R\,i)$.

Free Oscillations

The characteristic feature of free oscillations is that, throughout the part of the network over which the oscillation extends, the driving point impedance is equal to zero. This follows from the fact that as the driving point impedance is equal to the impressed electromotive force divided by the current, it vanishes when the electromotive force vanishes, provided the current does not vanish. The criterion for free oscillations is therefore

$$A=O \tag{30}$$

The solution of this equation contains all of the possible values of the time coefficient p. Each possible oscillation is aperiodic or not according as p is pure imaginary or not; p cannot be real for any actual system, since energy must be dissipated in any oscillation which may occur in such a system.

In present day practical applications, complex or imaginary values of p occur, as a rule, only for free vibrations; but there is no inherent reason why such vibrations should not arise as forced vibrations, for that requires only that an alternator be used which gives an electromotive force of constant period and logarithmically decreasing amplitude. This condition is approximately realized by a freely vibrating system which is loosely coupled to the network under consideration.

As an illustration of the application of the method to free oscillations, determine the time coefficients (*i.e.*, the free periods and associated damping constants) for two coupled circuits of impedances Z_1, Z_2, Z_{12}. For this case

$$A=\begin{vmatrix} Z_1 & Z_{12} \\ Z_{12} & Z_2 \end{vmatrix} = Z_1 Z_2 - Z_{12}^2 = 0$$

$$=[(2\,\delta_1+p\,i)\,(2\,\delta_1'+p\,i)+p_1^2]\,[(2\,\delta_2+p\,i)\,(2\,\delta_2'+p\,i)+p_2^2]$$

$$+k^2\,p^2\,(2\,\delta_1'+p\,i)\,(2\,\delta_2'+p\,i) \tag{31}$$

where $\delta=\dfrac{R}{2\,L}$, $\delta'=\dfrac{G}{2\,C}$, $p=\dfrac{1}{\sqrt{L\,C}}$, $k=\dfrac{M}{\sqrt{L_1\,L_2}}$,

taken with subscripts 1 and 2 to correspond with the circuits. For small damping constants δ, δ' (**31**) may be developed into a series of which the first terms are

$$p\,i = p_0\,i - \frac{\delta_1(p_0^2-p_2^2)+\delta_1'(p_0^2(1-k^2)-p_2^2)+\delta_2(p_0^2-p_1^2)+\delta_2'(p_0^2(1-k^2)-p_1^2)}{2p_0^2\,(1-k^2)-p_1^2-p_2^2}+\ldots \quad (32)$$

$$\text{where } p_0 = \sqrt{\frac{p_1^2+p_2^2 \pm \sqrt{(p_1^2-p_2^2)^2+4\,k^2\,p_1^2\,p_2^2}}{2\,(1-k^2)}} \quad (33)$$

are the time coefficients which would obtain if the circuits were free from all dissipative losses.

In the special case of two identical circuits ($Z_1 = Z_2 = Z$) the determinantal equation becomes $A = (Z+Z_{12})\ (Z - Z_{12}) = 0$

$$\text{or } R+(L \pm M)\,p\,i+\frac{1}{G+C\,p\,i}=0$$

$$\text{whence } i\,p = -\left(\frac{R}{2\,(L \pm M)}+\frac{G}{2\,C}\right)$$

$$\pm i\sqrt{\frac{1}{(L \pm M)\,C}-\left(\frac{R}{2\,(L \pm M)}-\frac{G}{2\,C}\right)^2} \quad (34)$$

without any restrictions as to the values R, L, M, G and C.

Infinite Number of Circuits—Eddy Currents.

When the number of circuits is increased indefinitely the determinant A becomes of infinite order. The particular application which at once suggests itself is that of eddy currents in a cylindrical core. Consider the core of radius a as being made up of a large number n of concentric hollow tubes of thickness $a \div n$ and radii $q\,a \div n$, ($q = 1, 2, \ldots n$) and take as the driving winding another tube of radius $(n+1)\ a \div n$ which has infinite conductivity. Then the impedance for tubes q and r per unit of length is

$$Z_{qr} = Z_{rq} = 2\,\pi\,\rho\left(\delta_{qr}\,q+2\,z\,i\,\frac{q^2}{n^2}\right) \qquad \begin{array}{l} 1 \leqq q \leqq r \leqq n+1 \\ \delta_{qr} = \begin{cases} 1 \text{ if } q = r < n+1 \\ 0 \text{ in other cases.} \end{cases} \end{array} \quad (35)$$

with $z = \dfrac{\pi\,a^2\,\mu\,p}{\rho} = \dfrac{\mu\,p}{R} = \dfrac{Lp}{4\,\pi\,\rho}$, R and L being the resistance ($\rho \div \pi\,a^2$) and inductance $4\,\mu\,\pi^2\,a^2$ of the core per unit length.

The driving point impedance of the winding if $x=2zi\div n^2$ is

$$S_{n+1}=\frac{A}{A_{n+1,\,n+1}}=2\pi\rho\,\frac{\begin{vmatrix} 1+x & x & x & \cdots & x & x \\ x & 2+2^2x & 2^2x & \cdots & 2^2x & 2^2x \\ x & 2^2x & 3+3^2x & \cdots & 3^2x & 3^2x \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ x & 2^2x & 3^2x & \cdots & n+n^2x & n^2x \\ x & 2^2x & 3^2x & \cdots & n^2x & (n+1)^2x \end{vmatrix}}{\begin{vmatrix} 1+x & x & x & \cdots & x \\ x & 2+2^2x & 2^2x & \cdots & 2^2x \\ x & 2^2x & 3+3^2x & \cdots & 3^2x \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x & 2^2x & 3^2x & \cdots & n+n^2x \end{vmatrix}},$$

$$=2\pi\rho\,\frac{2(n+1)!\displaystyle\sum_{k=1}^{n+1}\left(\frac{zi}{n^2}\right)^k\frac{(n+k)!}{(k-1)!\,k!\,(n+1-k)!}}{n!\displaystyle\sum_{k=0}^{n}\left(\frac{zi}{n^2}\right)^k\frac{(n+k)!}{(k!)^2\,(n-k)!}}$$

the transformation of the determinants into series of powers of $x\div 2=zi\div n^2$ is proven to be correct by its being correct for $n=1$ and 2 and satisfying the difference equations for the numerator (N_n) and the consecutive values of the denominator (D_{n-2}, D_{n-1}, D_n and D_{n+1})

$$N_n=D_{n+1}-(n+1)\,D_n$$

$$D_n=(2n-1)\,(1+x)D_{n-1}-(n-1)^2D_{n-2}$$

which are obtained by subtracting the next to the last rows and columns from the last rows and columns and expanding according to the last rows and columns. For n = infinite,

$$
\left.\begin{aligned}
S &= 2\pi\rho \frac{2\sum\limits_{k=1}^{\infty} \frac{(z\,i)^k}{(k-1)!\,k!}}{\sum\limits_{k=0}^{\infty} \frac{(z\,i)^k}{(k!)^2}} \\
&= 2\pi\rho \frac{2\,z\,i + (z\,i)^2 + \frac{(z\,i)^3}{6} + \ldots}{1 + z\,i + \frac{(z\,i)^2}{4} + \frac{(z\,i)^3}{36} + \ldots} \\
&= 2\pi\rho \frac{-\sqrt{-4\,i\,z}\,J_1\sqrt{-4\,i\,z}}{J_0\sqrt{-4\,i\,z}} \\
&= 4\pi\rho\,z \frac{d}{d\,z} \log J_0\sqrt{-4\,i\,z} \\
&= 4\pi\rho\,z\,i \left(1 + \frac{J_2\sqrt{-4\,i\,z}}{J_0\sqrt{-4\,i\,z}}\right)
\end{aligned}\right\} \qquad (36)
$$

which are the well known results expressed in Bessel's functions, To make the formula perfectly general for any driving winding it is necessary only to multiply by the length of the core and the square of the total number of turns in the winding and to add the impedance of the winding which arises externally to the core.

This example shows that certain infinite systems of circuits which are ordinarily solved by partial differential equations may be handled by the general determinantal solution, but of course when transcendental functions are involved, as in the case of eddy currents, the algebraical reduction may introduce some complexity.

As a further example of infinite systems of circuits take the eddy currents in transformer plates gives the following results:

For a plate of thickness $2a$, width w and axial length l divided into a large number of $2n$ of sheets of equal thickness and surrounded by a close fitting driving winding of a single turn and zero resistance:

$Z_{qr}=Z_{rq}=\frac{2\,\rho\, w\, n}{a\, l}\left(\delta_{qr}+4\,z\,i\,\frac{q}{n^2}\right)$, $z=\frac{\pi\, a^2\,\mu\, p}{\rho}$ q, r and δ_{qr} as above and the driving point impedance at limit $n=$infinite is

$$S=\frac{2\,\rho\, w}{a\, l}\sqrt{4\,z\,i}\tanh\sqrt{4\,z\,i} \tag{37}$$

Skin Effect

For a cylindrical conductor of radius a, length l and steady current resistance R with close fitting return shell of zero resistance, the conductor being divided into n concentric tubes of equal cross section with circuit q comprising adjacent tubes q and $q+1$:

$$Z_{qr}=Z_{rq}=\frac{\mu\, l\, p\, n}{z}\left(\delta_{qr}+\frac{z\, i}{n\, q}\,\delta'_{qr}\right) \qquad \delta_{qr}=\begin{cases} 2, \text{ if } q=r<n \\ 1, \text{ if } q=r=n \\ -1, \text{ if } q=r\pm 1 \\ 0, \text{ in other cases}\end{cases}$$

$$\text{with } z=\frac{\pi\, a^2\,\mu\, p}{\rho}=\frac{\mu\, p\, l}{R} \qquad \delta'_{qr}=\begin{cases} 1. \text{ if } q=r<n \\ 1\div 2, \text{ if } q=r=n \\ 0, \text{ in other cases}\end{cases}$$

and the driving point impedance at limit $n=$infinite is

$$S=-\frac{2\,\mu\, l\, p\, i}{\sqrt{-4\,i\,z}}\,\frac{J_0\sqrt{-4\,i\,z}}{J_1\sqrt{-4\,i\,z}} \tag{38}$$

Details of Proof. Regard each hollow cylindrical tube as concentrated on its mean diameter, while retaining its actual resistance, $\rho\, n\, l\div\pi\, a^2$. This resistance with sign changed will then be the mutual impedance between any two adjacent circuits, as each tube carries the difference between the currents in the two adjacent circuits of which it forms a common part; no other mutual impedances occur, as no other current products enter the expression for the total energy. The self-impedance of the qth circuit is made up of twice this resistance, together with the inductance $l\div q$; the inductance being found by the single turn solenoid formula 4π cross-section$\div$length, the cross section being $a\, l\div 2\sqrt{q\, n}$ and the length (*i.e.*, mean circumference) being $2\,\pi\, a\sqrt{q\div n}$. For the outermost circuit ($q=n$) this impedance is to be divided by two, since its return circuit is of zero resistance and zero thickness. The impedances Z_{qr} are thus, as stated above. After removing the factor $\mu\, p\, l\, n\div z$ from each element of determinants A and A_{nn} and placing $x = zi \div n$, we have

$$S = \frac{A}{A_{nn}} = \lim_{n=\infty} \frac{\mu p l n}{z} \frac{\begin{vmatrix} 2+\frac{x}{1} & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2+\frac{x}{2} & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2+\frac{x}{3} & \cdots & 0 & 0 \\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdots & \cdot\ \cdot & \cdot\ \cdot \\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdots & \cdot\ \cdot & \cdot\ \cdot \\ 0 & 0 & 0 & \cdots & 2+\frac{x}{n-1} & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1+\frac{x}{2n} \end{vmatrix}}{\begin{vmatrix} 2+\frac{x}{1} & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2+\frac{x}{2} & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2+\frac{x}{3} & \cdots & 0 & 0 \\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdots & \cdot\ \cdot & \cdot\ \cdot \\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdots & \cdot\ \cdot & \cdot\ \cdot \\ 0 & 0 & 0 & \cdots & 2+\frac{x}{n-2} & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2+\frac{x}{n-1} \end{vmatrix}}$$

$$= \frac{\mu p l}{z} \frac{\displaystyle \lim_{x=\infty} \sum_{k=0}^{n} \frac{(2n-k)\ (n-1)!}{2\,(k!)^2\,(n-k)!} x^k}{\displaystyle \lim_{x=\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{n!}{k!\,(k+1)!\,(n-k-1)!} x^k}$$

the correctness of the series expansions for the determinants is readily proven for $n=1$ and 2 directly from the determinants and then extended step by step to any value of n by expanding the determinants according to the terms in the last row and column so as to obtain expressions for the denominator (D_n) and the numerator (N_n) in terms of the denominator with different values of n, *viz*:

$$D_n = \left(2+\frac{x}{n-1}\right) D_{n-1} - D_{n-2}$$

$$N_n = \left(1+\frac{x}{2\,n}\right) D_n - D_{n-1} = D_{n+1} - \left(1+\frac{x}{2\,n}\right) D_n = \tfrac{1}{2}\,(D_{n+1} - D_{n-1})$$

which are readily shown, by substituting the above series expressions, to be identically satisfied for all values of n.

Finally replacing x by its value $z\,i \div n$ and passing to the limit, $n=\infty$

$$S = \frac{\mu\,p\,l}{z}\;\frac{\displaystyle\sum_{k=0}^{\infty}\frac{(z\,i)^k}{(k!)^2}}{\displaystyle\sum_{k=0}^{\infty}\frac{(z\,i)^k}{k!\,(k+1)!}}$$

which is identically formula (**38**), as the numerator is the series for $J_0\sqrt{-4\,i\,z}$ and the denominator is the series for $2\,J_1\sqrt{-4\,i\,z} \div \sqrt{-4\,i\,z}$.

Summary

1. The complex exponential function is shown to be, not a symbolic vector representation of the sinusoidal function, but a scalar function of fundamental importance in its own right, and enjoying algebraical power and energy relations as important as those of real functions. In order to emphasize the basic and distinctive character of the complex exponential function, it is given the name "cisoidal oscillation."

2. The correlation between sinusoidal oscillations and cisoidal oscillations is reduced to a few simple rules which cover power as well as currents and electromotive forces.

3. The general law of distribution of currents in any invariable network is shown to be that of stationary dissipation of power.

4. The law of distribution of cisoidal currents in any invariable network is reduced to that of stationary total power, or to the equivalent condition of stationary driving point impedance or admittance.

5. The cisoidal power is employed as the most convenient means for investigating the division of the instantaneous power between the resistances and reactances of a network.

6. The general solution for cisoidal oscillations in any invariable network is given in determinantal form and it is shown how the various impedances of any particular network may be written down at once and how the elimination of concealed circuits, mutual impedances or self-impedances may be accomplished. Applications to impedance loci, free oscillations and infinite systems of circuits are also given.