A METHOD FOR DETERMINING THE BEHAVIOUR OF CERTAIN
CLASSES OF POWER SERIES NEAR A SINGULAR POINT
ON THE CIRCLE OF CONVERGENCE

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1. Suppose that the coefficients $a_v$ of a power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots,$$

whose radius of convergence is unity, are analytic functions of certain
parameters $a, \beta, \gamma$, and are capable, when the real parts of $a + \beta - 1$ and
$\gamma$ are positive, of being expressed in the form

$$a_v = \int_0^1 \left\{ \log \left( \frac{1}{u} \right) \right\} - (1 - u)^{\beta - 1} u^{\gamma - 1 + \nu} \phi (u) \, du,$$

where $\phi (u)$ is an arbitrary function of $u$ which we shall assume to be
regular at and in the immediate neighbourhood of all points of the straight
line $(0, 1)$.

Consider the contour integral

$$\int (\log u)^{a-1} (u-1)^{\beta-1} u^{\gamma-1+\nu} \phi (u) \, du,$$

$C$ denoting a loop starting from and ending at the origin and enclosing
the line $(0, 1)$. If this contour is so chosen that $\phi (u)$ is regular on and
within it, and those values of the many valued functions are taken which
are defined by the equations

$$(\log u)^{a-1} = e^{(a-1) \log \log u}, \quad (u-1)^{\beta-1} = e^{(\beta-1) \log (u-1)}, \quad u^{\gamma-1} = e^{(\gamma-1) \log u},$$

wherein $\log u$, $\log \log u$, $\log (u-1)$ are all real at the point where $C$ crosses
the line $(1, \infty)$, it is evident that (9) is an analytic function of $a, \beta,$ and $\gamma$.

* That is to say that $\phi (u)$ is regular within a region which includes the line, while the lower
limit of the distances between points on the line and points on the boundary of the region is
positive.
regular for all values of \( \alpha \) and \( \beta \) and all values of \( \gamma \) whose real part is positive. Moreover, if the real part of \( \alpha+\beta-1 \) is also positive, the contour \( C \) may be deformed into a contour \( C' \) consisting of the two sides of the line \((0, 1)\), and the contour integral is equal to

\[
2i \sin \{ (\alpha+\beta) \pi \} \int_{-1}^{1} \log \left( \frac{1}{u} \right) \left( \frac{1}{1-u} \right)^{\beta-1} u^{\gamma-1} du = 2i \sin \{ (\alpha+\beta) \pi \} \alpha, \]

since \( \log u/(u-1) \) is regular near \( u = 1 \). Hence (unless \( \alpha+\beta \) is an integer)

\[
\alpha = \frac{1}{2i \sin \{ (\alpha+\beta) \pi \}} \int_{C} (\log u)^{\alpha-1} (u-1)^{\beta-1} u^{\gamma-1} \phi(u) du.
\]

provided the real part of \( \gamma \) is positive.

Now suppose that \( x \) has any value other than a real value greater than unity. Then the contour \( C \) can be so chosen that the point \( u = 1/x \) lies outside it. But in the particular case in which \( |x| < 1 \)

\[
f(x) = \sum_{0}^{\infty} a_{\nu} x^{\nu} = \frac{1}{2i \sin \{ (\alpha+\beta) \pi \}} \int_{C} (\log u)^{\alpha-1} (u-1)^{\beta-1} u^{\gamma-1} \phi(u) \frac{w^{-1} du}{1-xu}.
\]

This equation therefore holds for all values of \( x \) save real values greater than unity.

We now draw another contour \( C_{1} \) starting from and ending at the origin and enclosing the contour \( C \) and the point \( u = 1/x \), but no other singularities of \( \phi(u) \). Then it is obvious from Cauchy's theorem that

\[
f(x) = \frac{1}{2i \sin \{ (\alpha+\beta) \pi \}} \left[ \int_{C_{1}} (\log u)^{\alpha-1} (u-1)^{\beta-1} u^{\gamma-1} \phi(u) \frac{du}{1-xu} \right. \\
\left. - 2\pi i \left( \log \frac{1}{x} \right)^{\alpha-1} (\frac{1}{x} - 1)^{\beta-1} x^{\gamma} \phi \left( \frac{1}{x} \right) \right]
\]

\[
= - \frac{\pi}{\sin \{ (\alpha+\beta) \pi \}} \left( \log \frac{1}{x} \right)^{\alpha-1} (1-x)^{\beta-1} x^{1-\beta} \phi \left( \frac{1}{x} \right) \\
+ \frac{1}{2i \sin \{ (\alpha+\beta) \pi \}} \int_{C_{1}} (\log u)^{\alpha-1} (u-1)^{\beta-1} u^{\gamma-1} \phi(u) \frac{du}{1-xu}.
\]
Now when \( x \) tends to unity the second term on the right-hand side of (6) tends to the finite limit

\[
A = \frac{i}{2 \sin \left( \frac{(\alpha + \beta) \pi}{2} \right)} \int_{(C)} (\log u)^{\alpha - 1} (u - 1)^{\beta - 1} u^{-\gamma - 1} \phi(u) \, du.
\]

Hence

\[
f(x) = -\frac{\pi}{\sin \left( \frac{(\alpha + \beta) \pi}{2} \right)} \left( \log \frac{1}{x} \right)^{\alpha - 1} (1-x)^{\beta - 1} x^{1-\beta-\gamma} \phi \left( \frac{1}{x} \right) + A + \epsilon_x
\]

where \( \epsilon_x \) is a quantity whose limit is zero when \( x \) tends to 1 along any path which has no point in common with the straight line \((1, \infty)\), the many valued functions being fixed by the equations

\[
\left( \log \frac{1}{x} \right)^{\alpha - 1} = e^{(\alpha - 1) \log \log (1/x)} \quad (1-x)^{\beta - 1} = e^{(\beta - 1) \log (1-x)},
\]

\[
x^{1-\beta-\gamma} = e^{(1-\beta-\gamma) \log x}
\]

where \( \log \log (1/x), \log (1-x), \) and \( \log x \) are real when \( 0 < x < 1 \), the real part of \( \gamma \) is positive, and \( \alpha + \beta \) is not an integer.

2. I shall now illustrate the application of the general formula (7) by considering some interesting special cases.

(i.) Suppose \( \alpha = 1, \phi(u) \equiv 1 \). Then equation (2) becomes

\[
a_x = \int_0^1 (1-u)^{\beta - 1} u^{1+\nu} \, du
\]

\[
= \frac{\Gamma(\beta) \Gamma(\gamma+\nu)}{\Gamma(\beta+\gamma+\nu)} = \frac{\Gamma(\beta) \Gamma(\gamma)}{\Gamma(\beta+\gamma)} \frac{\gamma(\gamma+1) \ldots (\gamma+\nu-1)}{(\beta+\gamma)(\beta+\gamma+1) \ldots (\beta+\gamma+\nu-1)},
\]

and

\[
f(x) = \frac{\Gamma(\beta) \Gamma(\gamma)}{\Gamma(\beta+\gamma)} \left( 1 + \frac{\gamma}{\beta+\gamma} x + \ldots \frac{\gamma(\gamma+1)}{(\beta+\gamma)(\beta+\gamma+1)} x^2 + \ldots \right).
\]

Hence, if

\[
F(x) = 1 + \frac{\gamma}{\beta+\gamma} x + \frac{\gamma(\gamma+1)}{(\beta+\gamma)(\beta+\gamma+1)} x^2 + \ldots,
\]

\[
F(x) - \frac{\Gamma(\beta+\gamma) \Gamma(1-\beta)}{\Gamma(\gamma)} (1-x)^{\beta - 1} x^{1-\beta-\gamma}
\]

tends to a finite limit when \( x \) approaches unity along any point lying within or along the circle of convergence. This may be verified by means of the relations which hold between the particular solutions of the hypergeometric equation.
Moreover the equation (5) furnishes the analytic continuation of the function \(f(x)\) for a region exterior to the circle of convergence; a region whose extent depends on the extent of the region in which \(\phi(u)\) is regular. In the present case \(\phi(u)\) is a constant, and, if \(f(x)\) is made uniform by a cut along \((1, \infty)\), the branch of \(f(x)\) thus defined is regular all over the plane except at \(x = 1\), and its singularity there is completely specified by the expression

\[
\frac{1}{\Gamma(\beta+\gamma) \Gamma(1-\beta)/\Gamma(\gamma)} (1-x)^{\beta-1} x^{1-\beta-\gamma}.
\]

The restriction that the real part of \(\gamma\) must be positive is easily removed by the help of the recurrence formula

\[
f_{\beta, \gamma}(x) - xf_{\beta, \gamma+1}(x) = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta) \Gamma(\gamma)}
\]

except in the trivial case in which \(\gamma\) is a negative integer or zero. We also supposed that \(\beta\) was not an integer. If \(\beta\) (but not \(\beta+\gamma\)) is zero or a negative integer, \(F(x)\) reduces to a sum of binomial expansions. If \(\beta\) is a positive integer greater than unity, \(F(x)\) remains convergent for \(x = 1\); if \(\beta = 1\), the infinite term is easily found to be

\[
\gamma \log \left| \frac{1}{1-x} \right|.
\]

(ii.) Suppose \(\beta = 1\), \(\phi(u) \equiv 1\). Then

\[
a_\nu = \int_0^1 \left\{ \log \left( \frac{1}{u} \right) \right\} \nu^{-1} u^{\nu-1+\nu} du = \frac{\Gamma(a)}{(\gamma+\nu)^a}
\]

and

\[
f(x) = \Gamma(a) \left\{ \frac{1}{\gamma^a} + \frac{x}{(\gamma+1)^a} + \frac{x^2}{(\gamma+2)^a} + \ldots \right\},
\]

\((\gamma+\nu)^a\) being equal to \(\exp \{a \log (\gamma+\nu)\}\) where the logarithm has its principal value. Hence

\[8\]

\[
\frac{1}{\gamma^a} + \frac{x}{(\gamma+1)^a} + \frac{x^2}{(\gamma+2)^a} + \ldots - \Gamma(1-a) \left( \log \frac{1}{x} \right)^{a-1} x^{-\gamma}
\]

tends for \(x = 1\) to the finite limit

\[
\frac{\Gamma(1-a)}{2\pi i} \int_{C_i} (\log u)^{a-1} u^{\nu-1} du \frac{1}{1-u}.
\]

In this case, too, the restriction as to the real part of \(\gamma\) is easily removed. The restriction that \(\alpha\) is not to be an integer is more fundamental.
To consider this case we must go back to the equation

\[ f(x) = \frac{\pi}{\sin \alpha \pi} \left\{ \log \left( \frac{1}{x} \right) \right\}^{a-1} x^{-\gamma} + \frac{i}{2 \sin \alpha \pi} \int_{c_i} (\log u)^{a-1} \frac{u^{\gamma-1} du}{1-xu}. \]

Suppose that \( a = k + \varepsilon \), where \( k \) is a positive integer and \( \varepsilon \) is small. It is easy to see that we are at liberty to expand in powers of \( \varepsilon \) and equate coefficients. Thus we obtain

\[ \frac{i}{2\pi} \int_{c_i} (\log u)^{k-1} \frac{u^{\gamma-1} du}{1-xu} = \left\{ \log \left( \frac{1}{x} \right) \right\}^{k-1} x^{-\gamma} \]

and

\[ \Gamma(k) \left\{ \frac{1}{\gamma^k} + \frac{x}{(\gamma+1)^k} + \ldots \right\} = (-)^k x^{-\gamma} \left( \log \left( \frac{1}{x} \right) \right)^{k-1} \log \log \left( \frac{1}{x} \right) \]

\[ + \frac{(-)^k i}{2\pi} \int_{c_i} (\log u)^{k-1} \log \log \left( \frac{1}{u} \right) \frac{u^{\gamma-1} du}{1-xu}. \]

The equations (9) and (10) may be used to study the behaviour of \( f(x) \) all over the plane. I do not propose to go into this, as the results have, in this case, been already obtained by other methods. But there is one point to which it is worth while to call attention.

The integral

\[ \frac{i}{2 \sin \alpha \pi} \int_{c_i} (\log u)^{a-1} \frac{u^{\gamma-1} du}{1-xu} \]

is plainly expansible, in the neighbourhood of \( x = 1 \), in the power series

\[ \frac{i}{2 \sin \alpha \pi} \sum_{0}^{\infty} (x-1)^{\nu} \int_{c_i} (\log u)^{a-1} \frac{u^{\gamma-1+\nu}}{(1-u)^{\nu+1}} du. \]

In particular the value of the integral (11) can be assigned without calculation when \( \nu = 0 \). For it is plainly an analytic function of \( a \) regular for all non-integral values of \( a \) at any rate. But, if the real part of \( a \) is greater than unity, the value of the integral is clearly

\[ \frac{1}{\gamma^a} + \frac{1}{(\gamma+1)^a} + \ldots = \zeta(a, \gamma), \]

the well known generalised Riemann zeta function.

It follows that

\[ \lim_{x \to 1} \left\{ \sum_{0}^{\infty} \frac{x^{\nu}}{(\gamma+\nu)^x} - \Gamma(1-a) \left( \log \left( \frac{1}{x} \right) \right)^{a-1} x^{-\gamma} \right\} = \zeta(a, \gamma). * \]

It is interesting to note that the function \( \zeta(a, \gamma) \), which may, for all values

* It was first proved by Appell (Comptes Rendus, t. LXXXVII.) that, if

\[ f_s(x) = \frac{1}{2} x^{-s} x^s, \]
of $a$, be defined as the finite term* in the asymptotic expansion of

$$\sum_{1}^{n} (\gamma + \nu)^{-a}$$

in powers of $1/n$, appears also as the finite term in the expansion of

$$\sum \frac{x^r}{(\gamma + \nu)^a}$$

in powers of $(1 - x)$.

(iii.) Suppose that $a = 1$ and $\phi(u) = (1 - tu)^{-s}$, $t$ being any quantity whose modulus is less than unity. Then

$$a_n = \int_{0}^{1} (1 - u)^{\beta - 1} u^{\gamma - 1 + r} (1 - tu)^{-s} \, du = \frac{\Gamma(\gamma + \nu) \Gamma(\beta)}{\Gamma(\beta + \gamma + \nu)} F(\delta, \gamma + \nu, \beta + \gamma + \nu, t).$$

Hence the term which is not regular at $x = 1$ in the case of the series

$$1 + \frac{\gamma}{\beta + \gamma} F(\delta, \gamma + 1, \beta + \gamma + 1, t) x$$

$$+ \frac{\gamma(\gamma + 1)}{(\beta + \gamma)(\beta + \gamma + 1)} F(\delta, \gamma + 2, \beta + \gamma + 2, t) x^2 + \ldots$$

is

$$\frac{\Gamma(\beta + \gamma) \Gamma(1 - \beta)}{\Gamma(\gamma)} (1 - x)^{\beta - 1} x^{1 - \beta - \gamma} (x - t)^{-t}.$$

(iv.) Suppose that $\beta = 1$, and that $\phi(u)$ is an integral function of

$$\log \frac{1}{u},$$

such that $\phi(u)$ remains continuous even for $u = 0$, provided $\log (1/u)$ approaches infinity by a path which does not recede indefinitely from the real axis in the plane of $\log (1/u)$, i.e., provided $u$ approaches 0 by a path

then

$$\lim_{x \to 1} \left[ f_1(x) \int \Gamma(1 - s) \left\{ \log \left( \frac{1}{x} \right) \right\}^{s-1} \right] = 1 \quad [R(s) < 1],$$

when $x$ tends to 1 along the line $(0, 1)$. Various generalisations of this result have been given by Appell, Cesaro, Pringsheim, Le Roy, and Lindelof (Cesaro, \textit{Rend. della R. Accademia di Napoli}, December, 1893; Pringsheim, \textit{Acta Math.}, Vol. xxviii.; Le Roy, \textit{Bulletin des Sc. Math.}, 1900-1, and \textit{Annales de la Faculte des Sciences de Toulouse}, 1900; Lindelof, \textit{Acta Soc. Fennica}, t. xxxi., No. 3).

The method used in this paper was devised with a view to proving the particular formula

$$\lim_{x \to 1} \left[ \frac{2 \pi x^n}{n!} - \Gamma(1 - s) \left\{ \log \left( \frac{1}{x} \right) \right\}^{s-1} \right] = \zeta(s),$$

which at the time I supposed to be new. As a matter of fact it was first proved by Lindelof in the memoir cited above (see his \textit{Calcul des Residus}, Gauthier-Villars, 1905); M. Lindelof's method is entirely different and rests on an application of a general summation formula due in principle to Abel. The function $2 \pi x/(\gamma + \nu)^a$ is an infinitely many valued function whose singular points are 0, 1, $\infty$; but the principal branch is regular at the first of these points, as is obvious.

which does not wind an infinite number of times round the origin. Suppose further that

\[ a_\nu = \sum_{p=0}^{\infty} \frac{\Gamma(a+p)}{(\gamma+\nu)^{a+p}} c_p = \int_0^1 \left\{ \log \left( \frac{1}{u} \right) \right\}^{a-1} u^{\gamma-1+\nu} \phi(u) \, du, \]

the series being convergent for all values of \( \nu \). I shall not delay to discuss what hypotheses as to the nature of the coefficients \( c_p \) are necessary to ensure the truth of this last equation; it is enough for my present purpose to say that such functions \( \phi(u) \) certainly exist. For instance, M. le Roy* has proved that

\[
\sum_{n=0}^{\infty} \frac{x^n}{1+(n+1)^2} = \int_0^1 \sin \left( \log \frac{1}{u} \right) \frac{du}{1-xu},
\]

\[
\sum_{n=0}^{\infty} \frac{e^{-a(n+1)}}{n+1} x^n = \int_0^1 \left\{ 2\sqrt{a} \log \left( \frac{1}{u} \right) \right\} \frac{du}{1-xu},
\]

of which the first corresponds to the case in which \( a = \gamma = 1 \), \( \phi(u) = \sin \left\{ \log (1/u) \right\} \).

Then, for example,

\[
\sum_{n=0}^{\infty} \frac{x^n}{1+(n+1)^2} - \frac{1}{x} \sin \left( \log \frac{1}{x} \right) \log \log \left( \frac{1}{x} \right)
\]

is regular in the neighbourhood of \( x = 1 \); a result which may be also deduced from (10).

It would be easy to multiply examples: but I prefer to indicate shortly how the same method may be generalised so as to apply to functions of several variables.

3. Suppose that we have a double power series

\[ f(x, y) = \sum \sum a_{\mu, \nu} x^\mu y^\nu \]

convergent for \( |x| < 1, \ |y| < 1 \);† and suppose that

\[ a_{\mu, \nu} = \int_0^1 \left\{ \log \left( \frac{1}{u} \right) \right\}^{a-1} (1-u)^{\beta-1} u^{\gamma-1+\omega+\omega'} \phi(u) \, du \]

(at any rate for some values of the parameters \( a, \beta, \gamma \), \( \omega, \omega' \) being two quantities whose real parts we shall, for convenience, suppose positive.

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* Annales de la Faculté des Sciences de Toulouse, 1900, p. 342.
† That is to say, that (1, 1) form one pair of rayons associés : v. Borel, Leçons sur les Séries à Termes positifs, p. 86.
Proceeding as in § 1, we find

\[
(16) \quad f(x, y) = \frac{1}{2i \sin \left(\frac{(\alpha + \beta)\pi}{2}\right)} \int_c (\log u)^{\alpha - 1} (u - 1)^{\beta - 1} \frac{u^{\gamma - 1} \phi(u) du}{(1 - xu^\gamma)(1 - yu^\gamma)}.
\]

The argument is now exactly similar to our previous argument, except that, instead of having to consider only the one pole \( u = 1/x \), we have to consider two poles, viz. those points

\[
u = x^{-1/\omega}, \quad u = y^{-1/\omega},
\]

whose amplitudes are very small when \( x \) and \( y \) are both nearly equal to unity.

We deduce that

\[
(17) \quad f(x, y) + \frac{\pi}{\sin \left(\frac{(\alpha + \beta)\pi}{2}\right)} \left\{ \frac{\omega^{-\alpha} \left(\log \left(\frac{1}{x}\right)\right)^{\alpha - 1} (1 - x^{1/\omega})^{\beta - 1} x^{(1 - \beta - \gamma)\omega} \phi(x^{-1/\omega})}{1 - yx^{-\omega/\omega}} \right. \\
+ \left. \frac{\omega'^{-\alpha} \left(\log \left(\frac{1}{y}\right)\right)^{\alpha - 1} (1 - y^{1/\omega})^{\beta - 1} y^{(1 - \beta - \gamma)\omega} \phi(y^{-1/\omega})}{1 - xy^{-\omega/\omega}} \right\}
\]

tends to a finite limit when \( x \) and \( y \) tend together in any manner to unity, provided that the process to the limit is made in such a way that the denominators never vanish.

Let us suppose, for example, that \( \phi(u) \equiv 1 \) and \( \beta = 1 \). Then

\[
a_{\mu, \nu} = \int_0^1 \left(\log \left(\frac{1}{u}\right)\right)^\alpha u^{r-1+s+\omega+r} du = \frac{\Gamma(a)}{(\gamma + \omega\mu + \omega'\nu)^a}
\]

\[
f(x, y) = \Gamma(a) \sum \frac{x^\mu y^\nu}{(\gamma + \omega\mu + \omega'\nu)^a}.
\]

Then the part of \( \sum \frac{x^\mu y^\nu}{(\gamma + \omega\mu + \omega'\nu)^a} \) which is not regular near \((1, 1)\) is

\[
(18) \quad \Gamma(1-a) \left\{ \frac{\omega^{-\alpha} \left(\log \left(\frac{1}{x}\right)\right)^{\alpha - 1} x^{-\gamma}\omega}{1 - yx^{-\omega/\omega}} + \frac{\omega'^{-\alpha} \left(\log \left(\frac{1}{y}\right)\right)^{\alpha - 1} y^{-\gamma}}{1 - xy^{-\omega/\omega}} \right\}.
\]

Let us specialise further by supposing \( \omega = \omega' = 1 \). We obtain

\[
(19) \quad \Gamma(1-a) \left\{ \frac{x^{1-r} \left(\log \left(\frac{1}{x}\right)\right)^{\alpha - 1} - y^{1-r} \log \left(\frac{1}{y}\right)^{\alpha - 1}}{x - y} \right\}.
\]

This result may be verified. For

\[
f(x, y) = \sum_{\alpha, \nu} a_{\mu, r} x^\mu y^\nu = \frac{\Gamma(a)}{x - y} \sum_{\nu = 0}^\infty \frac{x^{1-r} - y^{1-r}}{(\gamma + k)^{\alpha}}.
\]
and it follows from § 2 that the irregular part of \( \sum x^{k+1}/(y+k)^a \) is
\[
\Gamma(1-a) x^{1-r} \left\{ \log \left( \frac{1}{x} \right) \right\}^{a-1}.
\]

It may further be shown that
\[
\lim_{x, y \to 1} \left\{ \sum \frac{x^m y^n}{(y+\omega_n + \omega'_n)^a} - \Phi(x, y) \right\} = \zeta_2(a, \omega, \omega'),
\]
\( \Phi(x, y) \) denoting the quantity (18) and \( \zeta_2(a, \omega, \omega') \) Mr. Barnes’ double zeta function, and the procedure to the limit being conditioned as specified above.

It is evident that the same method may be applied to the multiple series
\[
\sum \frac{x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}}{(y+v_1 \omega_1 + \ldots + v_k \omega_k)^a}
\]
in the neighbourhood of the point \((1, 1, 1, \ldots, 1)\). The same is true of such series as
\[
\sum \frac{\Gamma(y+v_1 \omega_1 + \ldots + v_k \omega_k)}{\Gamma(\beta+y+v_1 \omega_1 + \ldots + v_k \omega_k)} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k},
\]
the natural generalisation of the series considered in § 2 (i.). These multiple series are exceedingly interesting, and a large variety of results can be proved concerning them; my present object is merely to show as shortly as possible that by the method employed in this paper valuable information may be obtained concerning them, and I hope to return to the subject on some future occasion.