MECHANICAL VIBRATIONS

(MECHANICAL ENGINEERING)

(A MODERN APPROACH)

A text book for B.E./B. Tech students
for
The Indian Universities

By

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DHANPAT RAI & CO. (P) LTD.
(EDUCATIONAL & TECHNICAL PUBLISHERS)
1682, NAI SARAK, DELHI - 110 006
(Phone : 326 5367, 325 0251)

ACKNOWLEDGEMENT

A work of this nature while entailing a log of personal effort, can not be completed without the help of some external sources. I will be failing my duty of an author if I do not acknowledge with gratitude the invaluable contribution of these sources.

I am thankful to the 'Punjab University', Chandigarh, 'M.D. University', Rohtak, 'The University of Roorkee', Roorkee, 'The Institution of Engineers (India)', and various Indian and Foreign authors for getting useful resource material for the book.

It would have been very difficult to write this book without the enthusiastic support, advice and inspiration given to me by Dr. R.P. Jain, Director-Principal, C.R. State College of Engineering, Murthal (Sonepat), Haryana, I am very grateful for his help.

My thanks are due to Dr. R.A. Khan, Professor & Head, Mech. Engg. Deptt., Jamia Millia Islamia, New Delhi for incorporation of S.I. units and other vexing problems during the preparation of this book.

I express my gratitude to Professor Rajnish Prakash, Principal, Punjab Engineering College, Chandigarh for his valuable suggestions and inspiration. It will not be out of place here to express thanks to my wife Shashi for her unfailing patience and cheerfulness during the long job of preparing the manuscript.

I thank my student Mr. Parag Jain, S/o Sh. S.K. Jain, a student of Final year Mechanical Engineering, for his never ending support during the preparation of the book. The keen interest evinced and the hard work done by him are worth mentioning. His efforts at checking the proof-reading and the painstaking attention shown by him while preparing the manuscript are acknowledged. Many problems were solved and new topics were added after healthy discussions on the subject matter with him. He was quick to point out errors and his suggestions regarding making the text simpler and clarity in presentation with emphasis on better understanding of core topics are also acknowledged. His indepth understanding of the subject matter as a student are exemplary. His being in touch with the subject provided great help in presenting the text in a manner which will be more acceptable to the student community. I also thank my wife Mrs. Shashi Singh and brother Mr. Amar Singh for checking the proof of this text.

I also wish to thank my Publishers M/s. Dhanpat Rai and Co. (P) Ltd., Delhi, for valuable assistance during planning and preparation of the book and bringing out this volume in such a short time.

Finally, there are many others who have encouraged, inspired and helped me to complete this work. I gratefully acknowledge their contribution.

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1

Elements of Vibration

1.1. INTRODUCTION

This chapter gives the brief history of vibration in a simple and systematic way. The various vibration terms and definitions are discussed. Many items related to vibration, types of vibration, parts of vibratory system, harmonic motion analysis, etc. are discussed in this chapter. In the end some solved numerical problems are presented.

1.2. HISTORY OF VIBRATION

The discovery of musical instruments such as drums, whistles etc. made the vibration known and more interesting to the scientists and engineers. It was known since long that sound is related to vibration ; but no mathematical relation was available. Galileo (1564-1642), an Italian mathematician, studied the oscillations of strings and simple pendulum. He developed mathematical relationship between the length of a pendulum and its frequency and discussed the term resonance. Then Galileo and Hooke developed relationship between the frequency and pitch of sound.

Sir Isaac Newton (1642-1727), an English mathematician, made a lot of scientific contribution towards dynamics by introducing the definitions of force, mass, momentum and three laws of motion.

Daniel Bernoulli (1700-1782) developed the equation of motion for vibrations of beams and studied the vibrating strings and discovered the principle of superposition of harmonics in free vibration.

L. Euler (1707-1783) worked on the bending vibrations of a rod and studied the dynamics of a vibrating ring. J.B.J. Fourier (1768-1830) was a French mathematician who made valuable contribution to the development of vibration theory. He has shown that any periodic function can be represented by a series of sines and cosines. This work of Fourier helps in analysing the experimentally obtained vibration plots analytically.

Lagrange (1736-1813), an Italian mathematician, worked on theoretical mechanics and developed a very important equation known as Lagrange's equation. This equation is very useful in deriving the equations of motion for a vibrating system.

Lord Rayleigh (1842-1919), an English physicist, has computed the approximate natural frequencies of vibrating bodies using an energy approach. The method derived by him is useful in developing the equations of motion and the technique is known as Rayleigh's method.

J.H. Poincare (1854-1912), a French mathematician, contributed a lot in the field of pure and applied mathematics. His work on non-linear vibrations is outstanding. S.P. Timoshenko (1878-1972), a Russian engineer, worked in the field of elasticity, strength of materials and vibrations. He studied the vibrations in beams and his work is known as Timoshenko Beam Theory.

Frahm discovered the importance of torsional vibrations in the design of shaft and developed some vibratory instruments in 1909 such as Frahm's Reed Tachometer for measuring the frequency of vibration and dynamic vibration absorber.

A lot of work has been done in vibration by many authors. About thirty years back, the vibration analysis of complex multidegree of freedom systems was very difficult. But now with the help of finite element method and other advanced techniques the engineers are able to use computers to conduct numerically detailed vibration analysis of complex mechanical systems even having thousands degree of freedom.

1.3. BASIC CONCEPTS OF VIBRATION

With the discovery of musical instruments like drums, the vibration became a point of interest for scientists and since then there has been much investigation in the field of vibration. All bodies having mass and elasticity are capable of vibration. The mass is inherent of the body and elasticity causes relative motion among its parts. When body particles are displaced by the application of external force, the internal forces in the form of elastic energy are present in the body. These forces try to bring the body to its original position. At equilibrium position, the whole of the elastic energy is converted into kinetic energy and body continues to move in the opposite direction because of it. The whole of the kinetic energy is again converted into elastic or strain energy due

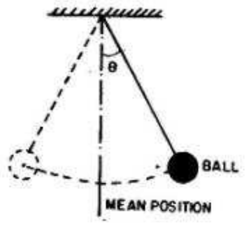


Fig. 1.1. Simple pendulum.

to which the body again returns to the equilibrium position. In this way, vibratory motion is repeated indefinitely and exchange of energy takes place. Thus, any motion which repeats itself after an interval of time is called vibration or oscillation. The swinging of simple pendulum as shown in figure 1.1 is an example of vibration or oscillation as the motion of ball is to and fro from its mean position repeatedly. The main reasons of vibration are as follows :

1. Unbalanced centrifugal force in the system. This is caused because of non-uniform material distribution in a rotating machine element.
2. Elastic nature of the system.
3. External excitation applied on the system.
4. Winds may cause vibrations of certain systems such as electricity lines, telephone lines, etc.

1.4. IMPORTANCE OF VIBRATION STUDY IN ENGINEERING

The structures designed to support the high speed engines and turbines are subjected to vibration. Due to faulty design and poor manufacture there is unbalance in the engines which causes excessive and unpleasant stresses in the rotating system because of vibration. The vibration causes rapid wear of machine parts such as bearings and gears. Unwanted vibrations may cause loosening of parts from the machine. Because of improper design or material distribution, the wheels of locomotive can leave the track due to excessive vibration which results in accident or heavy loss. Many buildings, structures and bridges fall because of vibration. If the frequency of excitation coincides with one of the natural frequencies of the system, a condition of resonance is reached, and dangerously large oscillations may occur which may result in the mechanical failure of the system.

Sometimes because of heavy vibrations proper readings of instruments cannot be taken. Excessive vibration is dangerous for human beings. Thus keeping in view all these devastating effects, the study of vibration is essential for a mechanical engineer to minimize the vibrational effects over mechanical components by designing them suitably.

Vibration can be used for useful purposes such as vibration testing equipments, vibratory conveyors, hoppers, sieves and compactors. Vibration is found very fruitful in mechanical workshops such as in improving the efficiency of machining, casting, forging and welding techniques, musical instruments and earthquakes for geological research. It is useful for the propagation of sound.

Thus undesirable vibrations should be eliminated or reduced upto certain extent by the following methods :

1. Removing external excitation, if possible.

2. Using shock absorbers.
3. Dynamic absorbers.
4. Resting the system on proper vibration isolators.

1.5. DEFINITIONS

Periodic motion. A motion which repeats itself after equal intervals of time.

Time period. Time taken to complete one cycle.

Frequency. Number of cycles per unit time.

Amplitude. The maximum displacement of a vibrating body from its equilibrium position.

Natural frequency. When no external force acts on the system after giving it an initial displacement, the body vibrates. These vibrations are called free vibrations and their frequency as natural frequency. It is expressed in rad/sec or Hertz.

Fundamental Mode of Vibration. The fundamental mode of vibration of a system is the mode having the lowest natural frequency.

Degree of freedom. The minimum number of independent coordinates required to specify the motion of a system at any instant is known as degrees of freedom of the system.

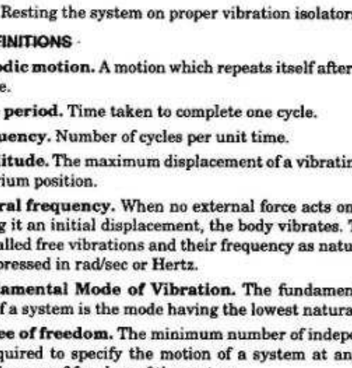


Fig. 1.2. Finite degree of freedom.

In general, it is equal to the number of independent displacements that are possible. This number varies from zero to infinity. The one, two and three degrees of freedom systems are shown in figure 1.2. In single degree of freedom there is only one independent coordinate (x_1) to specify the configuration as shown in figure 1.2 (*a*). Similarly, there are two (x_1, x_2). and three coordinates (x_1, x_2 and x_3) for two and three degrees of freedom systems as shown in figure 1.2 (*b*) and 1.2 (*c*)

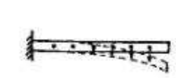


Fig. 1.3. Infinite degree of freedom.

respectively. A cantilever beam as shown in figure 1.3 has infinite degree of freedom.

Simple Harmonic Motion. The motion of a body to and fro about a fixed point is called simple harmonic motion. The motion is periodic and its acceleration is always directed towards the mean position and is proportional to its distance from mean position. The motion of a simple pendulum as shown in figure 1.1 is simple harmonic in nature.

Let a body having simple harmonic motion is represented by the equation

$$x = A \sin \omega t \qquad ...(1.5.1)$$

$$\dot{x} = A\omega \cos \omega t \qquad ...(1.5.2)$$

$$\ddot{x} = -A\omega^2 \sin \omega t$$

$$\ddot{x} = -\omega^2 x \qquad ...(1.5.3)$$

where x, $\dot{x}$ and $\ddot{x}$ represent the displacement, velocity and acceleration of the body respectively.

Damping. It is the resistance to the motion of a vibrating body. The vibrations associated with this resistance are known as damped vibrations.

Phase difference. Suppose there are two vectors x_1 and x_2 having frequencies ω rad/sec each. The vibrating motions can be expressed as

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin (\omega t + \phi) \qquad ...(1.5.4)$$

In the above equation the term ϕ is known as the phase difference.

Resonance. When the frequency of external excitation is equal to the natural frequency of a vibrating body, the amplitude of vibration becomes excessively large. This concept is known as resonance.

Mechanical systems. The systems consisting of mass, stiffness and damping are known as mechanical systems.

Continuous and Discrete Systems. Most of the mechanical systems include elastic members which have infinite number of degree of freedom. Such systems are called continuous systems. Continuous systems are also known as distributed systems. Cantilever, simply supported beam etc. are the examples of such systems.

Systems with finite number of degrees of freedom are called discrete or lumped systems.

1.6. PARTS OF A VIBRATING SYSTEM

A vibratory system basically consists of three elements, namely the mass, the spring and damper. In a vibrating body there is exchange of energy from one form to another. Energy is stored by mass in the form of kinetic energy ($1/2\ m\dot{x}^2$), in the spring in the form of potential energy ($1/2\ kx^2$) and dissipated in the damper in the form of heat energy which opposes the motion of the system. Energy enters the system with the application of external force known as excitation. The excitation disturbs the mass from its mean position and the mass goes up and down from the mean position. The kinetic energy is converted into potential energy and potential energy into kinetic energy. This sequence goes on repeating and the system continues to vibrate. At the same time damping force $c\dot{x}$ acts on the mass and opposes its motion. Thus some energy is dissipated in each cycle of vibration due to damping. The free vibrations die out and the system remains at its static equilibrium position. A basic vibratory system is shown in figure 1.4.

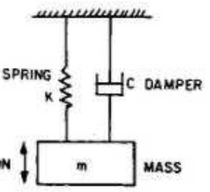


Fig. 1.4. Vibrating System.

The equation of motion of such a vibratory system can be written as

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad \text{...(1.6.1)}$$

where $c\dot{x}$ = damping force

kx = spring force

$m\ddot{x}$ = inertia force

1.7. METHODS OF VIBRATION ANALYSIS

There are various methods by means of which we can derive the equations of motion of a vibratory system. Some of the methods are discussed here.

1.7.1. Energy Method

According to this method the sum of the energies associated with the system is constant.

Kinetic energy + Potential energy = constant

$$(\text{K.E.} + \text{P.E.}) = \text{constant}$$

$$\frac{d}{dt}\left(\frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2\right) = 0$$

$$m\dot{x}\,\ddot{x} + kx\dot{x} = 0$$

or

$$m\ddot{x} + kx = 0 \qquad \text{...(1.7.1.1)}$$

This is the equation of motion.

If the motion is simple harmonic given as

$$x = A \sin \omega t$$

So

$$\ddot{x} = -A\omega^2 \sin \omega t$$

Then

$$-mA\omega^2 \sin \omega t + kA \sin \omega t = 0 \qquad \text{...(1.7.1.2)}$$

Thus

$$\omega = \sqrt{\frac{k}{m}} \text{ rad/sec}$$

or

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ Hz} \qquad \text{...(1.7.1.3)}$$

1.7.2. Rayleigh's Method

This method is the extension of energy method. The method is based on the principle that the total energy of a vibrating system is equal to the maximum potential energy.

At any moment total energy is either the kinetic energy or potential energy or the sum of the both. Let us say the total energy is kinetic energy which is expressed as

$$(\text{K.E.})_{max} = \left(\frac{1}{2} m\dot{x}^2\right)_{max} = \frac{1}{2} m(\omega A)^2$$

$$(\text{P.E.})_{max} = \left(\frac{1}{2} kx^2\right)_{max} = \frac{1}{2} kA^2$$

So

$$m(\omega A)^2 = kA^2$$

$$m\omega^2 = k$$

$$\omega = \sqrt{k/m}$$

$$= \frac{1}{2\pi}\sqrt{k/m} \text{ Hz} \qquad \text{...(1.7.2.1)}$$

1.7.3. Equilibrium Method

According to this method the algebraic sum of the forces and moments acting on the system must be zero. If the external force acting

on the system is F, spring force kx, damping force $c\dot{x}$ and inertia force $m\ddot{x}$, then the equation of motion can be written as

$$m\ddot{x} + c\dot{x} + kx = F \qquad \text{...(1.7.3.1)}$$

The above three methods will be discussed in detail later on.

1.8. TYPES OF VIBRATION

Some of the important types of vibration are as follows :

1.8.1. Free and Forced Vibration

After disturbing the system the external excitation is removed, then the system vibrates on its own. This type of vibration is known as free vibration. Simple pendulum is one of the examples.

The vibration which is under the influence of external force is called forced vibration. Machine tools, electric bells etc. are the suitable examples.

1.8.2. Linear and Non-linear Vibration

If in a vibratory system mass, spring and damper behave in a linear manner, the vibrations caused are known as linear in nature. Linear vibrations are governed by linear differential equations. They follow the law of superposition. Mathematically speaking, if a_1 and a_2 are the solutions of equations (1.8.2.1) and (1.8.2.2) respectively, then $(a_1 + a_2)$ will be the solution of equation (1.8.2.3).

$$m\ddot{x} + c\dot{x} + kx = F_1(t) \qquad \text{...(1.8.2.1)}$$

$$m\ddot{x} + c\dot{x} + kx = F_2(t) \qquad \text{...(1.8.2.2)}$$

$$m\ddot{x} + c\dot{x} + kx = F_1(t) + F_2(t) \qquad \text{...(1.8.2.3)}$$

On the other hand, if any of the basic components of a vibratory system behaves non-linearly, the vibration is called non-linear. Linear vibration becomes non-linear for very large amplitude of vibration. It does not follow the law of superposition.

1.8.3. Damped and Undamped Vibration

If the vibratory system has a damper, the motion of the system will be opposed by it and the energy of the system will be dissipated in friction. This type of vibration is called damped vibration.

On the contrary, the system having no damper is known as undamped vibration.

1.8.4. Deterministic and Random Vibration

If in the vibratory system the amount of external excitation is known in magnitude, it causes deterministic vibration. Contrary to it the non-deterministic vibrations are known as random vibrations.

1.8.5. Longitudinal, Transverse and Torsional Vibrations

Figure 1.5 represents a body of mass m carried on one end of a weightless spindle, the other end being fixed. If the mass m moves up and down parallel to the spindle axis, it is said to execute longitudinal vibrations as shown in figure 1.5 (a).

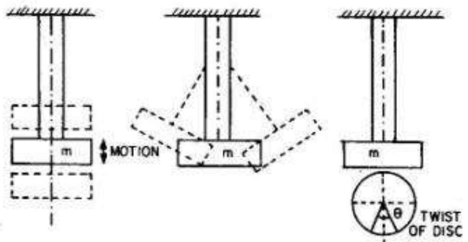


Fig. 1.5. Vibrations in spindle.

When the particles of the body or shaft move approximately perpendicular to the axis of the shaft, as shown in figure 1.5 (b), the vibrations so caused are known as transverse.

If the spindle gets alternately twisted and untwisted on account of vibratory motion of the suspended disc, it is called to be undergoing torsional vibrations as shown in figure 1.5 (c).

1.8.6. Transient Vibration

In ideal systems the free vibrations continue indefinitely as there is no damping. The amplitude of vibration decays continuously because of damping (in a real system) and vanishes ultimately. Such vibration in a real system is called transient vibration.

1.9. PERIODIC AND HARMONIC MOTION

The motion which repeats itself after an equal interval of time is known as periodic motion. The equal interval is called time period. If we consider a motion of the type $x_1 = A_1 \sin \omega t$, here ω is the natural frequency and the motion will be repeated after $2\pi/\omega$ time. The harmonic motion is one of the form of periodic motion. The harmonic motion is represented in terms of circular sine and cosine functions. All harmonic motions are periodic in nature but vice-versa is not always true. In the equation $x_1 = A_1 \sin \omega t$, x_1 is the displacement and A_1 the amplitude. The velocity and acceleration are $\dot{x}_1 = \frac{dx_1}{dt} = A_1\, \omega \cos \omega t$ and $\ddot{x}_1 = -\omega^2 x_1$

respectively. Thus the acceleration in a simple harmonic motion is always proportional to its displacement and directed towards a particular fixed point. It is shown that when harmonic motions of same period are added, the resultant harmonic motion of same period is obtained.

Addition of Harmonic Motion

When we add two harmonic motions of the same frequency, we get the resultant motion as harmonic. Let us have two harmonic motions of amplitudes A_1 and A_2, the same frequency ω and phase difference ϕ as

$$x_1 = A_1 \sin \omega t \qquad ...(1.9.1)$$

$$x_2 = A_2 \sin (\omega t + \phi) \qquad ...(1.9.2)$$

The resultant motion is given by adding the above equations

$$x = x_1 + x_2 = A_1 \sin \omega t + A_2 \sin (\omega t + \phi)$$

$$= A_1 \sin \omega t + A_2 (\sin \omega t \cos \phi + \sin \phi \cos \omega t)$$

$$= \sin \omega t (A_1 + A_2 \cos \phi) + A_2 \cos \omega t \sin \phi \qquad ...(1.9.3)$$

Assuming $A_1 + A_2 \cos \phi = A \cos \theta$

$$A_2 \sin \phi = A \sin \theta \qquad ...(1.9.4)$$

Now equation (1.9.3) can be written as

$$x = A \sin \omega t \cos \theta + A \sin \theta \cos \omega t$$

$$= A \sin (\omega t + \theta) \qquad ...(1.9.5)$$

The above equation shows that the resultant displacement is also simple harmonic motion of amplitude A and phase θ. To find out the value of A, squaring and adding equation (1.9.4), we get

$$A^2 = (A_1 + A_2 \cos \phi)^2 + A_2^2 \sin^2 \phi$$

$$= A_1^2 + A_2^2 \cos^2\phi + 2A_1A_2 \cos \phi + A_2^2 \sin^2\phi$$

$$= A_1^2 + A_2^2 + 2A_1A_2 \cos \phi$$

or

$$A = (A_1^2 + A_2^2 + 2A_1A_2 \cos \phi)^{1/2} \qquad ...(1.9.6)$$

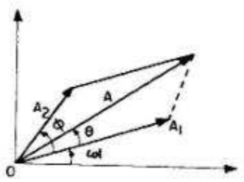


Fig. 1.6. Addition of two harmonic motions.

The resultant phase difference can be determined from the equation (1.9.4) as

$$\tan \theta = \frac{A_2 \sin \phi}{A_1 + A_2 \cos \phi}$$

$$\theta = \tan^{-1} \frac{A_2 \sin \phi}{A_1 + A_2 \cos \phi} \qquad ...(1.9.7)$$

The graphical method for the addition of two simple harmonic motions is shown in figure (1.6).

1.10. ORTHOGONAL FUNCTIONS

Consider the set of functions $f_1(x), f_2(x)... f_n(x)... f_m(x)$ defined such that

$$\int_a^b f_n(x) f_m(x) \, . \, dx = 0 \qquad \text{if } (m \neq n) \qquad ...(1.10.1)$$

and

$$\int_a^b f_n(x) f_m(x) \, . \, dx = \lambda \qquad \text{if } (m = n) \qquad ...(1.10.2)$$

where λ is non-zero quantity and

$$m = 1, 2, 3.....$$

$$n = 1, 2, 3.....$$

The above functions are termed as orthogonal functions.

Certain relations of Fourier series are orthogonal in nature such as

$$\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, . \, d\theta = \begin{bmatrix} 0 & n \neq m \\ \pi & n = m \end{bmatrix} \qquad ...(1.10.3)$$

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, . \, d\theta = \begin{bmatrix} 0 & n \neq m \\ \pi & n = m \end{bmatrix} \qquad ...(1.10.4)$$

$$\int_{-\pi}^{\pi} \sin n\theta \cos m\theta \, . \, d\theta = \begin{bmatrix} 0 & n \neq m \\ \pi & n = m \end{bmatrix} \qquad ...(1.10.5)$$

1.11. SINUSOIDAL MOTION

This is periodic vibratory motion and is referred as simple harmonic motion. It can be shown mathematically as

$$x = A \cos (\omega t + \phi)$$

or $= A \sin (\omega t + \phi)$ ...(1.11.1)

where A is the amplitude of vibration,

ω angular frequency,

ϕ and α phase angles, and

x displacement

If the phase angle is zero, the above equation can be expressed as

$$x = A \sin \omega t \qquad ...(1.11.2)$$

The velocity of such a vibratory motion can be determined as $\dot{x} = dx/dt$ and the acceleration as $\ddot{x} = d^2x/dt^2$

So

$$\dot{x} = A\omega \cos \omega t \qquad ...(1.11.3)$$

[differentiating equation (1.11.2) w.r.t. time]

$$\ddot{x} = -A\omega^2 \sin \omega t$$

$$= -\omega^2 x \qquad ...(1.11.4)$$

The above two equations are widely used in vibration analysis.

1.12. FOURIER SERIES AND HARMONIC ANALYSIS

J. Fourier, a French mathematician, developed a periodic function in terms of series of sines and cosines. With the help of this mathematical series known as Fourier Series, the vibration results obtained experimentally can be analysed analytically. If $x(t)$ is a periodic function with period T, the Fourier Series can be written as

$$x(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t + ...$$

$$+ b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t + ...$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \qquad ...(1.12.1)$$

where $\omega = 2\pi/T$ is the fundamental frequency and $a_0, a_1, a_2...$ $b_1, b_2, b_3....$ are constant coefficients. The term $(a_1 \cos \omega t + b_1 \sin \omega t)$ is called the Fundamental or First Harmonic. The term $(a_2 \cos 2\omega t + b_2 \sin 2\omega t)$ is called the second Harmonic and so on.

1. $$\int_\alpha^{\alpha+2\pi} \cos nx \, dx = \left| \frac{\sin nx}{n} \right|_\alpha^{\alpha+2\pi} = 0 \qquad (n \neq 0)$$

2. $$\int_\alpha^{\alpha+2\pi} \sin nx \, dx = -\left| \frac{\cos nx}{n} \right|_\alpha^{\alpha+2\pi} = 0 \qquad (n \neq 0)$$

3. $$\int_\alpha^{\alpha+2\pi} \cos mx \cos nx \, dx \qquad (n \neq 0)$$

$$= \frac{1}{2} \int_\alpha^{\alpha+2\pi} [\cos (m+n) x + \cos (m-n) x] \, dx$$

$$= \frac{1}{2} \left| \frac{\sin (m+n) x}{m+n} + \frac{\sin (m-n) x}{m-n} \right|_\alpha^{\alpha+2\pi} = 0 \qquad (m \neq 0)$$

4. $$\int_\alpha^{\alpha+2\pi} \cos^2 nx \, dx = \left| \frac{x}{2} + \frac{\sin 2nx}{4n} \right|_\alpha^{\alpha+2\pi} = \pi \qquad (n \neq 0)$$

5. $$\int_\alpha^{\alpha+2\pi} \sin mx \cos nx \, dx$$

$$= -\frac{1}{2} \left[\frac{\cos (m-n) x}{m-n} + \frac{\cos (m+n) x}{m+n} \right] = 0 \qquad (m \neq 0)$$

6. $$\int_\alpha^{\alpha+2\pi} \sin nx \cos nx \, dx = \left| \frac{\sin^2 nx}{2n} \right|_\alpha^{\alpha+2\pi} = 0$$

7. $$\int_\alpha^{\alpha+2\pi} \sin mx \sin nx \, dx$$

$$= \frac{1}{2} \left| \frac{\sin (m-n) x}{m-n} - \frac{\sin (m+n) x}{m+n} \right|_\alpha^{\alpha+2\pi} = 0 \qquad (m \neq n)$$

8. $$\int_\alpha^{\alpha+2\pi} \sin^2 nx \, dx = \left| \frac{x}{2} - \frac{\sin 2nx}{4\pi} \right|_\alpha^{\alpha+2\pi} = \pi \qquad (n \neq 0)$$

Determination of a_0

Integrate both sides of equation (1.12.1) over any interval of length $T = 2\pi/\omega$. All the integrals on the right hand side of the above equation are zero except the one containing a_0, that is

$$\int_0^{2\pi/\omega} x(t) \, dt = \int_0^{2\pi/\omega} \frac{a_0 \, dt}{2} = \frac{a_0}{2} \frac{2\pi}{\omega} = \frac{a_0 \pi}{\omega}$$

So

$$a_0 = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t) \, dt \qquad ...(1.12.2)$$

Determination of a_n

To find a_n, multiply both sides of equation (1.12.1) by cos ωnt and integrate over any interval of time $T = 2\pi/\omega$

$$\int_0^{2\pi/\omega} x(t)\cos(n\omega t)\,dt = \int_0^{2\pi/\omega} a_n \cos^2(n\omega t)\,dt$$

$$= \int_0^{2\pi/\omega} a_n\left(\frac{1+\cos 2n\omega t}{2}\right)dt$$

$$= a_n\frac{\pi}{\omega}$$

$$a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\cos(n\omega t)\,dt \qquad ...(1.12.3)$$

Similarly, we can find b_n by multiplying sin $(n\omega t)$ both sides

So
$$b_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\sin(n\omega t)\,dt \qquad ...(1.12.4)$$

The above mathematical analysis is known as harmonic analysis.

Numerical Method For Practical Harmonic Analysis

In practice, the function is often given not by a formula but by a graph or by a table of corresponding values. In such cases, the integrals to determine a_0, a_n and b_n cannot the evaluated. Thus the following alternative forms of these integrals are used :

Since the mean value of a function $y = f(x)$ over the range

$$(a, b) \text{ is } \frac{1}{b-a}\int_a^b f(x)\,dx$$

Thus the mean value of a function $y = x(t)$ over the range

$$\left(0, \frac{2\pi}{\omega}\right) \text{is } \frac{\omega}{2\pi}\int_0^{2\pi/\omega} x(t)\,dt$$

the above integrals become

$$a_0 = 2\times\frac{\omega}{2\pi}\int_0^{2\pi/\omega} x(t)\,dt$$

$$= 2\left[\text{ mean value of } x(t) \text{ in } \left(0, \frac{2\pi}{\omega}\right)\right]$$

$$a_n = 2\times\frac{\omega}{2\pi}\int_0^{2\pi/\omega} x(t)\cos(n\omega t)\,dt$$

$$= 2\left[\text{ mean value of } x(t)\cos(n\,\omega\,t) \text{ in } \left(0, \frac{2\pi}{\omega}\right)\right]$$

$$b_n = 2\times\frac{\omega}{2\pi}\int_0^{2\pi/\omega} x(t)\sin(n\,\omega\,t)\,dt$$

$$= 2\left[\text{ mean value of } x(t)\sin(n\,\omega\,t) \text{ in } \left(0, \frac{2\pi}{\omega}\right)\right]$$

1.13. WORK DONE BY A HARMONIC FORCE

Let a harmonic force $F = F_0 \sin \omega t$ is acting on a vibrating body having motion $x = x_0 \sin(\omega t - \phi)$. The work done by the force during a small displacement dx is Fdx. So the work done in one cycle

$$W = \int_0^T F\frac{dx}{dt}\,dt$$

$$= \int_0^T \left[F_0 \sin\omega t\,\frac{d}{dt}x_0\sin(\omega t - \phi)\right]dt$$

$$= \int_0^T F_0\sin\omega t\, x_0\omega\cos(\omega t - \phi)\,dt$$

$$= x_0F_0\omega\int_0^T \sin\omega t\cos(\omega t - \phi)\,dt$$

$$= x_0F_0\omega\int_0^T\left[\frac{\sin 2\omega t\cos\phi}{2} + \frac{\sin\phi(1-\cos 2\omega t)}{2}\right]dt$$

Putting $T = 2\pi/\omega$

$$W = \pi F_0 x_0 \sin\phi \qquad ...(1.13.1)$$

In the above equation if $\phi = 0$, the work done will be zero. It means force and displacement should not be in phase to get the work done.

1.14. BEATS

When two harmonic motions pass through some point in a medium simultaneously, the resultant displacement at that point is the vector sum of the displacements due to two component motions. This superposition of motion is called interference. The phenomenon of beat occurs as a result of interference between two waves of slightly different frequencies moving along the same straight line in the same direction.

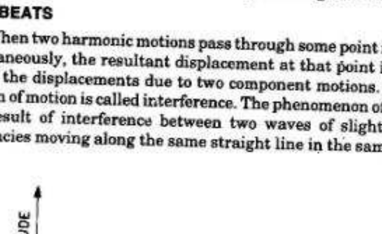


Fig. 1.7. Beats.

Consider that at particular time, the two wave motions are in the same phase. At this stage the resultant amplitude of vibration will be maximum. On the other hand, when the two motions are not in phase with each other, they produce minimum amplitude of vibration.

Again after some time the two motions are in phase and produce maximum amplitude and then minimum amplitude. This process goes on repeating and the resultant amplitude continuously keeps on changing from maximum to minimum. This phenomenon is known as beat.

Let us consider two waves of the same amplitude A and slightly different frequencies ω_1 and ω_2. If x_1 and x_2 are the displacements of these waves at any time t, then

$$x_1 = A\sin\omega_1 t \qquad ...(1.14.1)$$

$$x_2 = A\sin\omega_2 t \qquad ...(1.14.2)$$

The resultant displacement x at any time is given by adding the above two equations

$$x_1 + x_2 = x = A(\sin\omega_1 t + \sin\omega_2 t)$$

$$= 2A\sin\frac{(\omega_1+\omega_2)t}{2}\cos\frac{(\omega_1-\omega_2)t}{2}$$

$$x = B\sin\left(\frac{\omega_1+\omega_2}{2}\right)t \qquad ...(1.14.3)$$

where
$$B = 2A\cos\left(\frac{\omega_1-\omega_2}{2}\right)t$$

Equation (1.14.3) represents a simple harmonic motion whose amplitude is B. The maximum value of B is $2A$ and minimum zero. The frequency of beat is $(\omega_1 - \omega_2)/2\pi$ Hz. See figure 1.7.

Differentiating equation (1.14.3) w.r.t time, we get

$$\frac{dx}{dt} = 2A\sin\left(\frac{\omega_1+\omega_2}{2}\right)t\left[-\left(\frac{\omega_1-\omega_2}{2}\right)\sin\left(\frac{\omega_1-\omega_2}{2}\right)t\right]$$
$$+ 2A\cos\left(\frac{\omega_1-\omega_2}{2}\right)t\left[-\left(\frac{\omega_1+\omega_2}{2}\right)\cos\left(\frac{\omega_1+\omega_2}{2}\right)t\right]$$

The term $\frac{dx}{dt}$ is called the slope of the beat.

The existence of beats can also be shown mathematically. Let $\omega_2 - \omega_1 = \Delta\omega$ = a very small value ...(1.14.4)

($\because$ beats occur only when ω_1 and ω_2 are slightly different)

Then the resultant displacement x at any time is given by adding the equations (1.14.1) and (1.14.2)

$$x = x_1 + x_2$$
$$= A\sin\omega_1 t + A\sin(\omega_1 + \Delta\omega)t$$
$$= A\sin\omega_1 t + A[\sin\omega_1 t\cos\Delta\omega t + \sin\Delta\omega t\cos\omega_1 t]$$
$$= (A + A\cos\Delta\omega t)\sin\omega_1 t + (A\sin\Delta\omega t)\cos\omega_1 t$$

Let $x = X\sin(\omega_1 t + \phi)$

$$X\sin\omega_1 t\cos\phi + X\sin\phi\cos\omega_1 t$$
$$= (A + A\cos\Delta\omega t)\sin\omega_1 t + (A\sin\Delta\omega t)\cos\omega_1 t$$

Equating the terms of sin $\omega_1 t$ and cos $\omega_1 t$ on both sides, we get

$$X\cos\phi = A + A\cos\Delta\omega t$$
$$X\sin\phi = A\sin\Delta\omega t$$

Adding the squares of above two eqns., we get the amplitude of the resultant motion.

i.e.
$$X = \sqrt{(A + A\cos\Delta\omega t)^2 + (A\sin\Delta\omega t)^2}$$
$$= \sqrt{2A^2 + 2A^2\cos\Delta\omega t}$$
$$= A\sqrt{2(1+\cos\Delta\omega t)} \qquad ...(1.14.5)$$

If the amplitudes of the two sinusoidal motions are approximately equal then,

$$x_1 = A \sin \omega_1 t \qquad ...(1.14.6)$$

$$x_1 = B \sin \omega_2 t \qquad ...(1.14.7)$$

Resultant displacement x at any time is given by adding Eqns. (1.14.6) and (1.14.7)

$$x = A \sin \omega_1 t + B \sin \omega_2 t$$

Applying eqn. (1.14.4), we get

$$x = A \sin \omega_1 t + B \sin (\omega_1 + \Delta\omega) t \qquad (\because\ \omega_2 = \omega_1 + \Delta\omega)$$

On a similar analysis like above we get the amplitude of resultant motion as

$$X = \sqrt{(A + B \cos \Delta \omega t)^2 + (B \sin \Delta \omega t)^2}$$

$$= \sqrt{A^2 + B^2 + 2AB \cos \Delta \omega t} \qquad ...(1.14.8)$$

This expression is seen to vary between $(A + B)$ and $(A - B)$ with a frequency $\Delta\omega$ which is the difference of natural frequency of the beats phenomena.

Under the conditions of the two frequencies being slightly different from each other, the phase difference between the two sinusoidal motions keeps on shifting slowly and continuously. At the moment when these motions, represented by rotating vectors, are in phase with each other, the amplitude of the resultant motion is maximum and equal to the sum of amplitude of individual motion i.e. $(A + B)$. At a moment, when they are out of phase, the resultant amplitude is equal to the difference of the individual amplitude i.e. $(A - B)$.

Thus the resultant amplitude continuously keeps on changing from maximum of $(A + B)$ to minimum of $(A - B)$ with a frequency equal to the difference between the individual component frequencies. This is the BEATS PHENOMENA.

The frequency of the beats i.e. $\Delta\omega$ should be small in order to experience the phenomenon. The amplitudes A & B should be approximately equal to get clear and distinct beats.

This can be shown mathematically from equation 1.14.8.

Case-I. When the two sinusoidal motions are in phase, then phase difference $\Delta\omega = 0$

$$\text{Resultant amplitude} = \sqrt{A^2 + B^2 + 2AB \cos 0^\circ}$$

$$= \sqrt{A^2 + B^2 + 2AB} \qquad (\because\ \cos 0^\circ = 1)$$

$$= (A + B)$$

Case-II. When two sinusoidal motions are out of phase then phase difference $\Delta\omega = 180^\circ$

$$\text{Resultant amplitude} = \sqrt{A^2 + B^2 + 2AB \cos 180^\circ}$$

$$= \sqrt{A^2 + B^2 - 2AB}$$

$$= (A - B) \qquad [\because\ \cos 180^\circ = -1]$$

1.15. REPRESENTATION OF HARMONIC MOTION IN COMPLEX FORM

Suppose a vector X be represented as a complex number

$$X = x + i\,y \qquad ...(1.15.1)$$

where $i = \sqrt{-1}$

and x and y denote the real and imaginary components of X, respectively, refer figure 1.8 A. x and y are known as the real and imaginary parts of vector X. If the vector makes angle θ with the x-axis, it can be written as

$$X = A \cos \theta + i\, A \sin \theta$$

$$= Ae^{i\theta} \qquad ...(1.15.2)$$

where A is the modulus or the absolute value of the vector X.

The relation shown by equation (1.15.2) is known as **Euler's formula.**

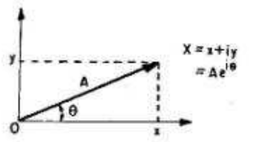


Fig. 1.8 A.

We can find the value of θ as

$$\theta = \tan^{-1} \frac{y}{x} \qquad ...(1.15.3)$$

Velocity can be determined by differentiating equation (1.15.2) **with respect** to time as

$$\dot{x} = \frac{dX}{dt} = i\, A\omega e^{i\omega t} \qquad (\text{since } \theta = \omega t)$$

$$= i\omega\, Ae^{i\omega t}$$

$$= i\omega\, X \qquad ...(1.15.4)$$

This is known as velocity vector.

Again differentiating, we get

$$\ddot{x} = i^2 \omega^2 Ae^{i\omega t}$$

$$= -\omega^2 Ae^{i\theta}$$

$$= -\omega^2 X \qquad ...(1.15.5)$$

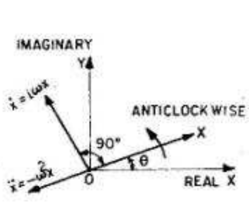


Fig. 1.8 B.

This is known as acceleration vector and its amplitude is $\omega^2 X$. In figure 1.8 B, it is shown that velocity vector leads the displacement by 90° and the acceleration vector leads the displacement by 180°. All the vectors with constant angular velocity rotate in the same direction (anticlockwise).

SOLVED EXAMPLES

Example 1.1. *Add the following harmonic motions analytically and check the solution graphically.* (P.U., 89)

$$x_1 = 4 \cos (\omega t + 10^\circ)$$

$$x_2 = 6 \sin (\omega t + 60^\circ)$$

Solution. The frequency is same for both x_1 and x_2, so we express the sum as

$$x = A \sin(\omega t + \alpha)$$

$$x = x_1 + x_2$$

$$A (\sin \omega t \cos \alpha + \cos \omega t \sin \alpha) = 4 \cos (\omega t + 10^\circ) + 6 \sin (\omega t + 60^\circ)$$

$$= 4 \cos \omega t \cos 10^\circ - 4 \sin \omega t \sin 10^\circ + 6 \sin \omega t \cos 60^\circ + 6 \cos \omega t \sin 60^\circ$$

$$\sin \omega t\, (A \cos \alpha) + \cos \omega t\, (A \sin \alpha) = \sin \omega t\, (-4 \sin 10^\circ + 6 \cos 60^\circ) + \cos \omega t\, (4 \cos 10^\circ + 6 \sin 60^\circ)$$

$$= \sin \omega t\, (-.6945 + 3) + \cos \omega t\, (3.9392 + 5.1961)$$

$$= \sin \omega t\, (2.305) + \cos \omega t\, (9.135)$$

By equating the corresponding coefficients of $\cos \omega t$ and $\sin \omega t$ on both sides, we obtain

$$A \cos \alpha = 2.305$$

$$A \sin \alpha = 9.135$$

$$A = \sqrt{(2.305)^2 + (9.135)^2} = \sqrt{(88.7612)}$$

$$= 9.42$$

$$\tan \alpha = \frac{9.135}{2.305} = 3.963$$

$$\alpha = \tan^{-1} (3.963)$$

$$\therefore \qquad \alpha = 75.838^\circ$$

$$\text{So} \qquad x = 9.42 \sin (\omega t + 75.838^\circ)$$

Graphical Method

For adding the two motions graphically. Let us put the two equations as

$$x_2 = 4 \cos (\omega t + 10^\circ) = 4 \sin (\omega t + 100^\circ)$$

$$x_1 = 6 \sin (\omega t + 60^\circ)$$

Since both the equations are in the same form, the vector diagram can be drawn as shown in figure 1.9.

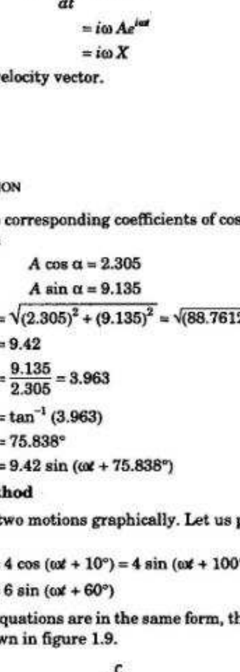


Fig. 1.9.

The sum of the vectors as obtained by measurement is 9.4 and at an angle of 76°.

$\therefore\ x = 9.4 \sin (\omega t + 76^\circ)$ which agrees closely with the analytical results.

Procedure

(*i*) Draw OX and OY axes.

(*ii*) Draw vector x_1 equal to $OB = 6 \sin(\omega t + 60°)$ *i.e.* the length of x_1 is 6 unit and it makes an angle 60° with OX axis.

(*iii*) Draw vector $x_2 = 4 \sin(\omega t + 100°)$ *i.e.* the length of x_2 is 4 unit and it makes angle 100° with OX or 10° with OY axis. It is represented by OD.

(*iv*) From B draw a line parallel to OD and from D draw a line parallel to OB. Both the lines intersect at C. Now OC is the required resultant motion which is equal to 9.4 units and makes an angle 76° with OX.

Example 1.2. *Split the harmonic motion $x = 10 \sin(\omega t + \pi/6)$, into two harmonic motions one having a phase angle of zero and the other of 45°.* (P.U., 90)

Solution.

Graphical Method (Refer figure 1.10)

1. Draw OX and OY axes.
2. Draw $OA = 10 \sin(\omega t + 30°)$.
3. OA makes 30° with OX.
4. Draw OC making 45° with OX.
5. Complete the parallelogram with arms OC and OB.
6. Measurement of OB gives $x_1 = 3.6 \sin \omega t$ and the measurement of OC gives the value of x_2 *i.e.* $x_2 = 7.1 \sin(\omega t + 45°)$

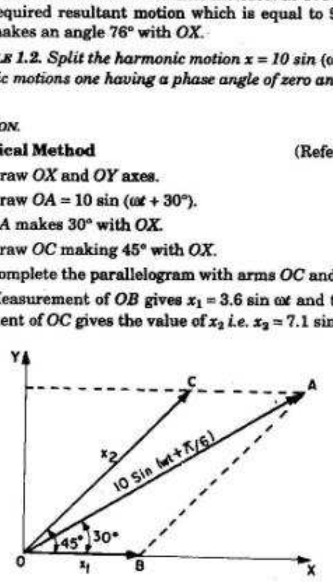


Fig. 1.10.

Analytical Method

Let the equations are $x_1 = a \sin \omega t$

and $$x_2 = b \sin(\omega t + 45°)$$

Since $$x = x_1 + x_2$$

$$10 \sin(\omega t + \pi/6) = a \sin \omega t + b \sin(\omega t + 45°)$$

$$10 \sin \omega t \cos \pi/6 + 10 \cos \omega t \sin \pi/6$$

$$= a \sin \omega t + b \sin \omega t \cos 45° + b \cos \omega t \sin 45°$$

Comparing the coefficients of $\sin \omega t$ and $\cos \omega t$ both sides, we obtain the values of a and b as

$$\sin \omega t\,(10 \cos \pi/6) = (a + b \cos 45°) \sin \omega t$$

$$\cos \omega t\,(10 \sin \pi/6) = (b \sin 45°) \cos \omega t$$

Solving these equations $a = 3.67$

$$b = 7.07$$

So the equations of harmonic motions can be written as

$$x_1 = 3.67 \sin \omega t$$

and $$x_2 = 7.07 \sin(\omega t + 45°).$$

Example 1.3. *Show that the resultant motion of three harmonic motions given below is zero.*

$$x_1 = a \sin \omega t$$

$$x_2 = a \sin(\omega t + 2\pi/3)$$

$$x_3 = a \sin(\omega t + 4\pi/3)$$ (P.U., 89 ; M.D.U., 90)

Solution. The resultant motion is given as

$$x = x_1 + x_2 + x_3$$

$$= a \sin \omega t + a \sin(\omega t + 2\pi/3) + a \sin(\omega t + 4\pi/3)$$

$$= a \sin \omega t + a \sin \omega t \cos 2\pi/3 + a \cos \omega t \sin 2\pi/3$$

$$+ a \sin \omega t \cos 4\pi/3 + a \cos \omega t \sin 4\pi/3$$

$$= a \sin \omega t + a \sin \omega t\,(-1/2) + a \cos \omega t\,(.866)$$

$$+ a \sin \omega t\,(-1/2) + a \cos \omega t\,(-.866)$$

$$x = a \sin \omega t - a \sin \omega t + .866\, a \cos \omega t - .866\, a \cos \omega t$$

$$= 0$$

Hence, the resultant motion is zero.

Example 1.4. *A force $P_0 \sin \omega t$ acts on a displacement $x_0 \sin(\omega t - \pi/3)$. If*

$$P_0 = 100\,N, \quad x_0 = 0.02\,m, \quad \omega = 2\pi\, rad/sec$$

Find the work done during (i) the first cycle (ii) the first second (iii) the first quarter second.

Solution. Work done $= \int_0^{t_1} P.\frac{dx}{dt}\,dt$

$$= \int_0^{t_1} \left[P_0 \sin \omega t \frac{d}{dt} x_0 \sin(\omega t - \pi/3)\right] dt$$

$$= P_0 x_0 \omega \int_0^{t_1} \sin \omega t \cos(\omega t - \pi/3)\, dt$$

$$= P_0 x_0 \omega \int_0^{t_1} \sin \omega t\,(\cos \omega t \cos \pi/3 + \sin \omega t \sin \pi/3)\, dt$$

$$= P_0 x_0 \omega \int_0^{t_1} \left[\frac{2 \sin \omega t \cos \omega t}{4} + \frac{\sqrt{3}}{4}(1 - \cos 2\omega t)\right] dt$$

$$= P_0 x_0 \omega \int_0^{t_1} \left[\frac{\sin 2\omega t}{4} + \frac{\sqrt{3}}{4}(1 - \cos 2\omega t)\right] dt$$

$$= P_0 x_0 \omega \left[-\frac{\cos 2\omega t}{4.2\omega} + \frac{\sqrt{3}}{4} t \frac{-\sqrt{3}}{4} \frac{\sin 2\omega t}{2\omega}\right]_0^{t_1}$$

$$= \frac{P_0 x_0 \omega}{4}\left[-\frac{\cos 2\omega t}{2\omega} + \sqrt{3}t - \frac{\sqrt{3}}{2\omega} \sin 2\omega t\right]_0^{t_1}$$

$$= \frac{100 \times .02 \times 2\pi}{4}\left[\frac{-\cos 2\omega t}{2\omega} + \sqrt{3}t - \frac{\sqrt{3}}{2\omega} \sin 2\omega t\right]_0^{t_1}$$

(*i*) Work done during cycle $T = \frac{2\pi}{\omega}$

$$= \pi\,[\sqrt{3}] = 5.44 \text{ N–m}$$

(*ii*) Putting $t = 1$

$$\pi\left[-\frac{\cos 2 \times 2\pi}{2 \times 2\pi} + \sqrt{3} - \frac{\sqrt{3}}{4\pi} \sin 4\pi + \frac{1}{4\pi}\right]$$

$$= \pi\left[-\frac{1}{4\pi} + \sqrt{3} + \frac{1}{4\pi}\right] = \pi\sqrt{3}$$

$$= 5.44 \text{ N–m}$$

(*iii*) Putting $t = 1/4$

$$= \pi\left[-\frac{\cos 4\pi \times \frac{1}{4}}{4\pi} + \sqrt{3} \times \frac{1}{4} - \frac{\sqrt{3}}{4\pi} \sin 4\pi \times \frac{1}{4}\right]$$

$$= 1.609 \text{ N-m}$$

Example 1.5. *A body describes simultaneously two motions,*

$$x_1 = 3 \sin 40t, \quad x_2 = 4 \sin 41t$$

What is the maximum and minimum amplitude of combined motion and what is the beat frequency ?

Solution. If a body is subjected to two harmonic motions given by

$$x_1 = a \sin \omega_1 t$$

$$x_2 = b \sin \omega_2 t$$

Maximum amplitude is $(a + b)$ and minimum amplitude is $(a - b)$. So in the present problem Max. amplitude $= 3 + 4 = 7$ and minimum amplitude $= 4 - 3 = 1$.

Beat frequency $= \frac{\omega_1 - \omega_2}{2\pi}$

$$= \frac{41 - 40}{2\pi} = \frac{1}{2\pi} \text{ Hz.}$$

Example 1.6. *Develop the Fourier Series for the curve shown in figure 1.11.*

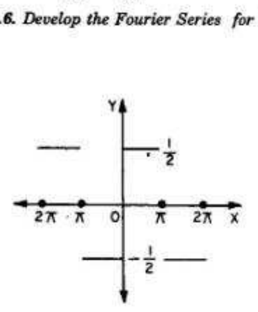


Fig. 1.11.

Solution. The function is defined as

$$x(t) = -1/2, \qquad -\pi < x < 0$$

$$= 1/2, \qquad 0 < x < \pi$$

It is seen that the graph is symmetrical about the origin and hence the function is odd. Therefore, $a_0 = a_n = 0$ and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{+\pi} \frac{1}{2}\sin nx\,dx$$

Since $x(t)\sin nx$ is an even function.

So, $$b_n = \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{1}{n\pi}(1-\cos n\pi) = 0 \text{ for } n \text{ being even}$$

$$= \frac{2}{n\pi} \text{ for } n \text{ being odd.}$$

So $$x(t) = \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right]$$

EXAMPLE 1.7. *The rectilinear motion of a point is given by* $\alpha = -9x$ *where* α *and* x *are the acceleration and displacement of simple harmonic motion and the amplitude is 2 inch. Find (a) the period and frequency (b) displacement, velocity and acceleration after 21.5 seconds.*

SOLUTION. As the motion is harmonic the equation of displacement can be written as

$$x = X\sin\omega t$$
$$\dot{x} = \omega X\cos\omega t$$
$$\ddot{x} = -\omega^2 X\sin\omega t$$

As per the problem $\ddot{x} = -9x$

So $$-9x = -\omega^2 x$$

$\therefore$ $$\omega = 3$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{3} = 2.09 \text{ second}$$

$$f = \frac{1}{T} = \frac{1}{2.09} = 0.478 \text{ cycles/sec}$$

Now the equation can be written

$$x = X\sin 3t$$

The value of amplitude is 2 *i.e.* $X = 2$

So $$x = 2\sin 3t$$
$$x = 2\sin(3\times 21.5) = 1.97 \text{ inch}$$
$$\dot{x} = 3\,(2\cos 3t) = 6\cos(3\times 21.5) = 0.35 \text{ inch/sec}$$
$$\ddot{x} = -\omega^2 x = -9\times 1.97$$
$$= -17.73 \text{ in/sec}^2$$

EXAMPLE 1.8. *Add two harmonic motions expressed by the following equations :*

$$x_1 = 3\sin(\omega t + 30^\circ)\ ;\quad x_2 = 2\cos(\omega t - 15^\circ)$$

and express the result in the form $x = A\sin(\omega t + \phi)$. (Roorkee Uni., 70)

SOLUTION. $$x_1 = 3(\sin\omega t\cos 30^\circ + \cos\omega t\sin 30^\circ)$$
$$x_2 = 2(\cos\omega t\cos 15^\circ + \sin\omega t\sin 15^\circ)$$

Adding the motions, we get

$$x_1 + x_2 = 2.598\sin\omega t + 1.5\cos\omega t + 1.93\cos\omega t + .5176\sin\omega t$$
$$x = 3.11\sin\omega t + 3.43\cos\omega t = A\sin(\omega t + \phi)$$
$$= A(\sin\omega t\cos\phi + \cos\omega t\sin\phi)$$

Comparing the results, we get

$$A\cos\phi = 3.11$$
$$A\sin\phi = 3.43$$
$$\tan\phi = \frac{3.43}{3.11}$$
$$\phi = 47.8^\circ$$
$$A = \sqrt{3.11^2 + 3.43^2}$$
$$A = 4.63$$

Now equation can be written as

$$x = 4.63\sin(\omega t + 47.8^\circ).$$

EXAMPLE 1.9. *A periodic motion observed on the oscilloscope is illustrated in figure 1.12. Represent this motion by harmonic series.* (P.U., 91)

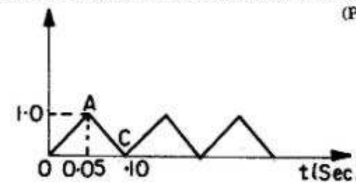


Fig. 1.12.

SOLUTION. The periodic function in terms of sine and cosine series can be written as

$$x(t) = \frac{a_0}{2} + a_1\cos\omega t + a_2\cos 2\omega t + a_3\cos 3\omega t + \ldots$$
$$+ b_1\sin\omega t + b_2\sin 2\omega t + b_3\sin 3\omega t + \ldots$$

The equation for the curve shown for one cycle is

For OA $x(t) = 20t$ $\qquad 0 \le t \le 0.05$

For AC $\quad = -20t + 2$ $\qquad 0.05 \le t \le 0.1$

The time period of motion is $= 0.10$

$$\omega = \text{frequency} = \frac{2\pi}{.10} = 20\pi$$

a_0, using equation (1.12.2) can be expressed as

$$\frac{a_0}{2} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} x(t)\,dt$$

$$= \frac{\omega}{2\pi}\left[\int_0^{0.05}(20t)\,dt + \int_{0.05}^{0.1}(-20t+2)\,dt\right]$$

$$= \frac{2\pi}{2\pi}\left[100t^2\Big|_0^{.05} + (-100t^2 + 20t)\Big|_{.05}^{0.1}\right]$$

$$= 1\,[.05\times .05\times 100 - 100\,(.1^2 - .05^2) + 20\,(.1 - .05)]$$

$$a_0 = 1\,[.25 - 1 + .25 + 1] = 0.5$$

a_n can be determined from equation (1.12.3) as

$$a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\cos(n\omega t)\,dt$$

$$= \frac{20\pi}{\pi}\left[\int_0^{.05} 20t\cos(20\pi nt)\,dt + \int_{.05}^{.10}(-20t+2)\cos(20\pi nt)\,dt\right]$$

$$= 20\left[20\left(\frac{-t}{20\pi n}\sin 20\pi nt - \frac{\cos 20\pi nt}{(20\pi n)^2}\right)\Bigg|_0^{.05}\right.$$

$$+ -20\left(\frac{-t}{20\pi n}\sin 20\pi nt - \frac{\cos 20\pi nt}{(20\pi n)^2}\right)\Bigg|_{.05}^{0.10}$$

$$\left. + 20\left(\frac{\sin 20\pi nt}{20\pi n}\right)\Bigg|_{.05}^{0.10}\right]$$

$$\left(\begin{array}{l}\text{Since } \omega = 20\pi \\ \text{and } \int t\cos nt\,dt = -\frac{t}{n}\sin nt - \frac{\cos nt}{n^2}\end{array}\right)$$

$$= -20\left[\frac{20}{400\,\pi^2 n^2}(-\cos n\pi + 1 - \cos n\pi + \cos 2\pi n)\right]$$

$$= -20\left[\frac{20}{400\,\pi^2 n^2}(-2\cos n\pi + 1 + \cos 2\pi n)\right]$$

$$= -\frac{4}{\pi^2 n^2}, \qquad \text{for odd values of } n$$

$$= 0, \qquad \text{for even values of } n$$

b_n can be determined from equation (1.12.4) as

$$b_n = \omega/\pi\int_0^{2\pi/\omega} x\,.\,(t)\sin(n\omega t)\,dt$$

$$\int t\sin nt\,dt = -\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt$$

$$\int \cos nt\,dt = \frac{\sin nt}{n}$$

$$b_n = 20\left[20\left\{\frac{-t}{20\pi n}\cos 20\pi nt + \frac{1}{(20\pi n)^2}\sin 20\pi nt\right\}\Bigg|_0^{.05}\right]$$

$$+ 20\left[-20\left\{\frac{-t}{20\pi n}\cos 20\pi nt + \frac{1}{(20\pi n)^2}\sin 20\pi nt\right\}\Bigg|_{.05}^{.10}\right]$$

$$+ 20\left[20\,\frac{(\sin 20\pi nt)}{20\pi n}\Bigg|_{.05}^{0.10}\right]$$

So we get $b_n = 0$

Thus harmonic series can be shown as

$$x(t) = 0.25 - \frac{4}{\pi^2}\left[\frac{\cos 20\pi t}{(1)^2} + \frac{1}{(3)^2}\cos 60\pi t + \frac{1}{(5)^2}\cos 100\pi t + \ldots\right]$$

Solution by Numerical Method

Let the Fourier series upto the third harmonic representing $x(t)$ in $\left(0, \frac{2\pi}{\omega}\right)$ be

$$x(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t$$
$$+ b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t \ldots$$

Let us now divide the time period into 12 equal intervals of 30° each in the range $(0, 2\pi)$

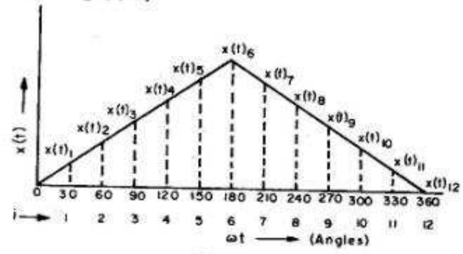


Fig. 1.13.

No. of elements in one cycle $= 2p = 12$

Now we can derive table 1.9.1.

From the table :

$$\Sigma x(t)_i = 60$$
$$\Sigma x(t)_i \cos \theta_i = -24.876$$
$$\Sigma x(t)_i \sin \theta_i = 0$$
$$\Sigma x(t)_i \cos 2\theta_i = 0$$
$$\Sigma x(t)_i \sin 2\theta_i = 0$$
$$\Sigma x(t)_i \cos 3\theta_i = -3.320$$
$$\Sigma x(t)_i \sin 3\theta_i = 0$$

$\therefore \quad \frac{a_0}{2} = 2 \times$ mean value of $x(t)$

$$= \frac{2}{2} \times \frac{60}{12} = 5.0$$

$a_1 = 2 \times$ mean value of $x(t)_1 \cos(\omega t)_1$

$$= 2 \times \frac{-24.876}{12} = -4.146$$

$b_1 = 2 \times$ mean value of $x(t)_1 \sin(\omega t)_1$

$$= 2 \times \frac{0}{12} = 0$$

$a_2 = 2 \times$ mean value of $x(t)_2 \cos(2\omega t)_2$

$$= 2 \times \frac{0}{12} = 0$$

$b_2 = 2 \times$ mean value of $x(t)_2 \sin(2\omega t)_2$

$$= 2 \times \frac{0}{12} = 0$$

$a_3 = 2 \times$ mean value of $x(t)_3 \cos(3\omega t)_3$

$$= 2 \times \frac{-3.320}{12} = -0.553$$

$b_3 = 2 \times$ mean value of $x(t)_3 \sin(3\omega t)_3$

$$= 2 \times \frac{0}{12} = 0$$

$\therefore$ Fourier series

$$x(t) = 5.0 - 4.146 \cos 20\pi t - 0.553 \cos 60\pi t + \ldots$$

Example 1.10. *A harmonic motion is given by $x(t) = 10 \sin(30t - \pi/3)$ mm where t is in seconds and phase angle in radians. Find (i) frequency and the period of motion, (ii) the maximum displacement, velocity and acceleration.* (P.U., 92)

Solution. Let us assume a solution of the form

$$x = A \sin(\omega t - \phi)$$

Maximum velocity $\dot{x} = \omega A$

Maximum Acceleration $\ddot{x} = -\omega^2 A$

where A = max. displacement and the frequency is ω.

By comparing our equation with the given equation, we get

$$\omega = 30 \text{ rad/sec}$$
$$\phi = \pi/3$$
$$A = 10 \text{ mm}$$

Max. velocity $\quad \dot{x} = \omega A$

$$= 30 \times 10 = 300 \text{ mm/sec}$$

Acceleration $\quad \ddot{x} = -\omega^2 A$

$$= -(30)^2 \times 10$$
$$= 9000 \text{ mm/sec}^2$$

Period of motion $\quad = \frac{2\pi}{\omega} = \frac{2\pi}{30} = 0.209$ sec.

Table 1.9.1

Element No. i	$\theta_i = \omega_i$ $J = 1$ t/p	*cos* θ_i	*sin* θ_i	*cos* $2\theta_i$	*sin* $2\theta_i$	*cos* $3\theta_i$	*sin* $3\theta_i$	$x(t)_i$	$x(t)_i \times \cos\theta_i$	$x(t)_i \times \sin\theta_i$	$x(t)_i \times \cos 2\theta_i$	$x(t)_i \times \sin 2\theta_i$	$x(t)_i \times \cos 3\theta_i$	$x(t)_i \times \sin 3\theta_i$
1	30	0.866	0.500	0.500	0.866	0	1	1.67	1.446	0.835	0.835	1.446	0.000	1.670
2	60	0.500	0.866	− 0.500	0.866	− 1	0	3.33	1.665	2.885	− 1.665	2.884	− 3.330	0.000
3	90	0.000	1.000	− 1.000	0.000	0	− 1	5.00	0.000	5.000	− 5.000	0.000	0.000	− 5.000
4	120	− 0.500	0.866	− 0.500	− 0.866	1	0	6.61	− 3.335	5.776	− 3.335	5.776	6.670	0.000
5	150	− 0.866	0.500	0.500	− 0.866	0	1	8.33	− 7.214	4.165	4.165	− 7.219	0.000	8.330
6	180	− 1.00	0.000	1.000	0.000	− 1	0	10.00	− 10.00	0.000	10.000	− 0.000	− 10.000	0.000
7	210	− 0.866	− 0.500	0.500	0.866	0	− 1	8.33	− 7.214	− 4.165	4.165	7.219	0.000	− 8.330
8	240	− 0.500	− 0.866	0.500	0.866	1	0	6.67	− 3.335	− 5.776	− 3.335	5.776	6.670	0.000
9	270	0.000	− 1.000	− 1.000	0.000	0	1	5.00	0.000	− 5.000	− 5.000	0.000	0.000	5.000
10	300	0.500	− 0.866	− 0.500	− 0.866	− 1	0	3.33	1.665	− 2.884	− 1.665	− 2.884	− 3.330	0.000
11	330	0.866	− 0.500	0.500	− 0.866	0	− 1	1.67	1.446	− 0.835	0.835	− 1.446	0.000	− 1.670
12	360	1.000	0.000	1.000	0.000	1	0	0.00	0.000	0.000	0.000	0.000	0.000	0.000
								$\Sigma x(t)_1 = 60$	$\Sigma x(t)_i \cos\theta_i = -24.876$	$\Sigma x(t)_i \sin\theta_i = 0$	$\Sigma x(t)_i \cos 2\theta_i = 0$	$\Sigma x(t)_i \sin 2\theta_i = 0$	$\Sigma x(t)_i \cos 3\theta_i = -3.320$	$\Sigma x(t)_i \sin 3\theta_i = 0$

Example 1.11. *If there is a non-zero number y such that*

$$\phi(t + y) = \phi(t)$$

What is the type of motion $\phi(t)$? State its most important characteristic. (P.U., 93)

Solution. This type of motion is called the periodic motion. The important characteristic of this type of motion is its repeatability or periodicity. Here, y is called the period of motion. It means that the motion is repeated itself after an interval of time y.

Example 1.12. *Represent the periodic motions given in figure 1.14 by harmonic series.* (P.U., 87, 88)

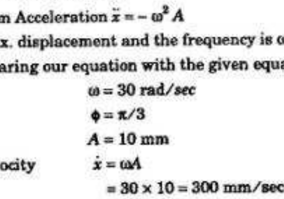


Fig. 1.14.

Solution. The equation of line AB is given by

$$x(t) = -20t + 2 \qquad 0 \le t \le 0.2$$

The time period of motion = 0.2

So $\quad \omega = \frac{2\pi}{T} = \frac{2\pi}{0.2} = 10\pi$

$$a_0 = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} x(t).\, dt = \frac{10\pi}{2\pi} \int_0^{0.2} (-20t + 2)\, dt$$

$$= 5\left(\frac{-20t^2}{2} + 2t\right)\Bigg|_0^{0.2} = 5(-10(.2)^2 + 2 \times .2)$$

$$= 5(-.4 + .4) = 0$$

$$a_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t) \,.\, \cos(n\omega t)\, dt = \frac{10\pi}{\pi} \int_0^{0.2} (-20t + 2) \cos(10\pi n t)\, dt$$

$$a_n = 10\left[-20\left\{\frac{t \sin 10\pi n t}{10\pi n} + \frac{\cos 10\pi n t}{(10\pi n)^2}\right\}\Bigg|_0^{0.2} + 2 \mid \sin 10\pi n t \mid_0^{0.2}\right]$$

$$= 10\left[-20\left\{0+\frac{\cos 10\pi nt}{100\pi^2n^2}\right|_0^{0.2}\right]$$

$$a_n = 0$$

and $$b_n = \frac{\omega}{\pi}\left[\int_0^{2\pi/\omega}(-20t+2)\sin(n\omega t)\,dt\right]$$

$$= 10\left[\int_0^{0.2}(-20t\,.\,\sin 10\pi nt\,dt + 2\sin 10\pi nt)\,dt\right]$$

$$= 10\left[\left\{-20\left|\frac{-t\cos 10\pi nt}{10\pi n}\right|_0^{0.2} + \int_0^{0.2}\frac{\cos 10\pi nt}{10\pi n}\,dt\right. + 2\left|\frac{\cos 10\pi nt}{10\pi n}\right|_0^{0.2}\right\}\right]$$

$$= 10\left[-20\left(\frac{.2\cos 2\pi n}{10\pi n}+\frac{2\cos 2\pi n-2}{10\pi n}\right)\right]$$

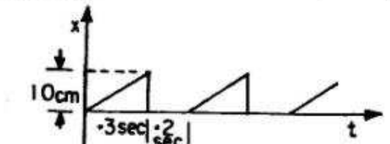
$$= 10\left[\frac{-20\times .2\times(-1)^2}{10\pi n}+\frac{2\times(-1)^2\times -2}{10\pi n}\right]$$

$$= 10\left[\frac{4}{10\pi n}\right] = \frac{4}{\pi n}$$

Thus harmonic series can be written as

$$x(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 10\pi nt.$$

Example 1.13. *Represent the periodic motions given in figure 1.15 by harmonic motion.* (P.U., 88)



Solution. Time period of motion = 0.50 sec

frequency $\omega = \frac{2\pi}{0.5} = 4\pi$ rad/sec

Equation of curve for one cycle

$$x(t) = \frac{100t}{3} \quad 0 \le t \le 0.3$$
$$= 0 \quad 0.3 \le t \le 0.5$$

$$a_0 = \frac{\omega}{2\pi}\int_0^{0.3}x(t)\,dt = \frac{4\pi}{2\pi}\int_0^{0.3}\frac{100t}{3}\,dt = 2\left(\frac{100}{3}\right)\left[\frac{t^2}{2}\right]_0^{.30} = 3,\ \frac{a_0}{2}=\frac{3}{2}=1.50$$

$$a_n = \frac{\omega}{\pi}\int_0^{0.3}x(t)\cos(n\omega t)\,dt = 4\int_0^{0.3}\frac{100\,t}{3}\cos 4\pi nt\,dt$$

$$= \frac{400}{3}\int_0^{0.3}t\cos 4\pi nt\,dt = \frac{400}{3}\left[\frac{t\sin 4\pi nt}{4\pi n}+\frac{\cos(4\pi nt)}{16\pi^2n^2}\right]_0^{0.30}$$

$$= \frac{400}{3}\left[\frac{0.3\sin(1.2\pi n)}{4\pi n}+\frac{\cos(1.2\pi n)-1}{16\pi^2n^2}\right]$$

$$= 10\,\frac{\sin(1.2\pi n)}{\pi n}+\frac{8.33}{\pi^2n^2}\{\cos(1.2\pi n)-1\}$$

$$b_n = \frac{\omega}{t}\int_0^t x(t)\sin(n\omega t)\,dt = \frac{400}{3}\,[t\sin(4\pi nt)\,dt]$$

$$= \frac{400}{3}\left[-\frac{t\cos 4\pi nt}{4\pi n}+\frac{\sin(4\pi nt)}{16\pi^2n^2}\right]_0^{0.3}$$

$$= \frac{400}{3}\left[-\frac{.3\cos(1.2\pi n)}{4\pi n}+\frac{\sin(1.2\pi n)}{16\pi^2n^2}\right]$$

$$= -\frac{10\cos(1.2\pi n)}{\pi n}+\frac{8.33}{\pi^2n^2}\sin(1.2\pi n)$$

Thus $x(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\omega t + b_n\sin n\omega t)$

$$= \frac{3}{2}+\sum_{n=1}^{\infty}\left[\frac{10}{\pi n}\sin(1.2\pi n)+\frac{8.33}{\pi^2n^2}\{\cos(1.2\pi n)-1\}\right]\cos 4\pi nt$$

$$+\sum_{n=1}^{\infty}\left[\frac{-10}{\pi n}\cos(1.2\pi n)+\frac{8.33}{\pi^2n^2}\sin(1.2\pi n)\right]\sin 4\pi nt$$

Example 1.14. *Represent* $17\,e^{-i3.74}$ *in rectangular form.* (P.U.,94)

Solution. $X = 17\,e^{-i3.74}$

here $\theta = \tan^{-1}\frac{y}{x}$

$X = Ae^{+i\theta} = A(\cos\theta + i\sin\theta)$

and $X = A\,e^{-i\theta} = A(\cos\theta - i\sin\theta)$

Given $\theta = 3.74$ (radians)

$= \frac{3.74}{3.14}\times 180° = 214.39°$

Thus $X = A(\cos 214.39° - i\sin 214.39°)$

$= 17\,[-0.82 - i(-.564)]$

$= -13.94 + i\,9.58$ (since $A = 17$)

Example 1.15. *Represent* $3 + i\,6$ *in exponential form.*

Solution. $X = 3 + i\,6$

$A = \sqrt{3^2+6^2} = 6.70$

$\theta = \tan^{-1} 6/3$

$\theta = 63.43° = 1.106$ radian

$X = Ae^{i\theta}$

$= 6.7e^{i\,1.106}$.

Example 1.16. *A force* $P_0\sin\omega t$ *acts on a displacement* $x_0\sin(\omega t - \pi/6)$

where $P_0 = 25$ *N,* $x_0 = 0.05$ *m*

and $\omega = 20\pi$ *rad/sec*

What is the work done during

(i) the first second ?

(ii) the first 1/40 second ? (P.U., 94)

Solution. We know that work done is given by

$$= \int_0^{t_1}P\frac{dx}{dt}\,dt$$

$$= P_0x_0\omega\int_0^{t_1}\sin\omega t\cos(\omega t - \pi/6)\,dt$$

$$= P_0x_0\omega\int_0^{t_1}[\sin\omega t\,\{\cos\omega t\cos\pi/6 + \sin\omega t\sin\pi/6\}]\,dt$$

$$= P_0x_0\omega\int_0^{t_1}[\sin\omega t\cos\omega t\times .866 + \frac{1}{2}\sin^2\omega t]\,dt$$

$$= \frac{P_0x_0\omega}{2}\int_0^{t_1}\left[2\sin\omega t\cos\omega t\times .866 + \frac{2\sin^2\omega t}{2}\right]dt$$

$$= \frac{P_0x_0\omega}{2}\int_0^{t_1}\left[\sin 2\omega t\times .866 + \frac{1-\cos 2\omega t}{2}\right]dt$$

$$= \frac{P_0x_0\omega}{2}\left[-\frac{\cos 2\omega t}{2\omega}\times .866 + \frac{t}{2} - \frac{\sin 2\omega t}{2\times 2\omega}\right]_0^t$$

$\omega = 20\pi$, $x_0 = 0.05m$, $P_0 = 25$ N

$$\frac{P_0\,x_0\,\omega}{2}\,[4.01\times10^{-3} + .5 + 7.2\times10^{-3} - 6.47\times10^{-3}] = 19.8\text{ N–m}$$

(*ii*) When the time $t_1 = 1/40$ sec

$$\frac{P_0\,x_0\,\omega}{2} = \frac{25\times .05\times 20\pi}{2} = \frac{78.5}{2} = 39.25$$

$$\frac{P_0\,x_0\,\omega}{2}\left[\frac{-\cos 40\pi t}{40\pi}\times .866 + \frac{t}{2} - \frac{\sin 40\pi t}{80\pi} + \frac{.866}{40\pi}\right]t = 1/40\text{ sec}$$

$$= 39.25\left[-\frac{\cos\pi}{40\pi}\times .866 + \frac{1}{80} - \frac{\sin\pi}{80\pi} + \frac{.866}{40\pi}\right]$$

$$= 0.48\text{ N–m}$$

Example 1.17. *A harmonic motion given by the equation* $x = 5\sin(3t + \phi)$ *is to be split into two components such that one leads it by 30° and the other lags it by 80°. Find the components.*

Solution. Let the required components are given by $x_1 = A_1\sin(3t + \phi - 80°)$ and $x_2 = A_2\sin(3t + \phi + 30°)$. We can solve the equation graphically as shown in figure 1.16.

Procedure :

1. Draw $OA = 5$ cm showing $5\sin(3t+\phi)$ in any direction as shown.
2. Draw OB' from O making angle 80° (lag) with OA.
3. Draw OC' from O making angle 30° (leading) with OA.

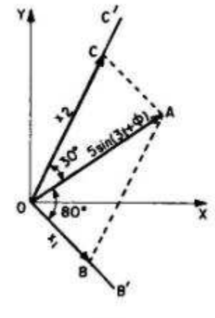


Fig. 1.16.

4. Resolve OA along OB' and OC' as OB and OC respectively.
5. OB and OC are the required components.
6. By measurement $OB = 2.66 \sin(3t + \phi - 80°)$ *i.e.* $A_1 = 2.66$

$$OC = 5.24 \sin(3t + \phi + 30°) \text{ i.e. } A_2 = 5.24$$

EXAMPLE 1.18. *A body is subjected to two harmonic motions as given below :*

$$x_1 = 15 \sin(\omega t + \pi/6) \text{ and } x_2 = 8 \cos(\omega t + \pi/3)$$

What harmonic motion should be given to the body to bring it to equilibrium ? (M.D.U., 95; P.U., 99)

SOLUTION. Let $A \sin(\omega t + \phi)$ extra motion be given to the body.

Then $\quad A \sin(\omega t + \phi) + x_1 + x_2 = 0$

Expanding various terms, we get

$$A \sin \omega t \cos \phi + A \cos \omega t \sin \phi$$
$$+ 15 \sin \omega t \cos \pi/6$$
$$+ 15 \cos \omega t \sin \pi/6$$
$$+ 8 \cos \omega t \cos \pi/3$$
$$- 8 \sin \omega t \sin \pi/3 = 0$$

$$\sin \omega t(A \cos \phi + 6.07) + \cos \omega t(A \sin \phi + 11.5) = 0$$

The coefficients of $\sin \omega t$ and $\cos \omega t$ are equated to zero.

Thus $\quad A \cos \phi = -6.07$

$$A \sin \phi = -11.5$$

so $\quad \tan \phi = \dfrac{11.5}{6.07} = 1.894$

$$\phi = 62.17°, 242.17°$$

Now $A^2 \sin^2 \phi + A^2 \cos^2 \phi = (11.5)^2 + (6.07)^2 = 169.0$

$$A^2 = 169$$
$$A = 13$$

The equation of harmonic motion can be written as

$$13 \sin(\omega t + 242.17°)$$

Problems

1. Represent the periodic motion given in figure 1.1 P, by harmonic motion. (A.M.I.E., 94)

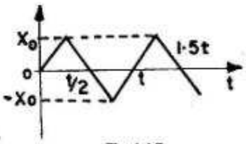


Fig. 1.1 P.

2. A body is subjected to the two harmonic motions as

$$x_1 = 15 \sin(\omega t + \pi/6)$$
$$x_2 = 8 \cos(\omega t + \pi/3)$$

What extra motion should be given to the body to bring it to the static equilibrium ? (P.U., 90)

3. Represent the periodic motion given in figure 1.2 P, by harmonic series. (P.U. 85)

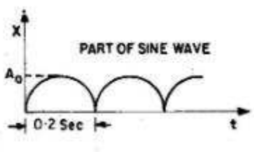


Fig. 1.2 P.

4. Represent the periodic motions given in figure 1.3 P by harmonic series.

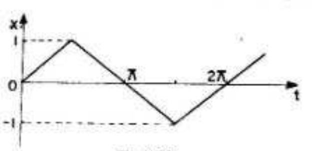


Fig. 1.3 P.

5. Split up the harmonic motion $x = 10 \sin(\omega t + \pi/6)$ into two harmonic motions one having a phase angle of zero and the other of 46°. (P.U., 90)
6. Add the following vectors analytically

$$x_1 = 4 \cos(\omega t + 10°); \quad x_2 = 6 \sin(\omega t + 60°)$$

Check the solution graphically. (P.U., 89)

7. A harmonic displacement is given by $x(t) = 6 \sin(20t + \pi/3)$ mm, where t is in seconds and phase angle in radians. Find (*i*) frequency and period of motion, (*ii*) the maximum displacement velocity and acceleration.
8. The displacement of the slider in the slider crank mechanism is given by

$$x = 24 \cos \theta\, \pi t + 3/2 \cos 16\, \pi t$$

Plot a displacement versus time diagram. What is the acceleration of the piston at $t = 1/8$ sec. (P.U., 92)

9. Represent the following complex numbers in exponential form
(*i*) $-3 + j4$ $\quad$ (*ii*) $-3 - j4$ (P.U., 99)
10. Find the sum of two harmonic motions of equal amplitude but of slightly different frequencies. Discuss the beat phenomenon that result from this sum. (P.U., 99)
11. Show that two simple harmonic motions (SHM) with frequency p and $2p$ when added will result in a periodic function of frequency p. Generalize the above for n number of harmonic functions with frequencies $p, 2p, np$, etc. (P.U., 96)
12. Express $f(x) = x$ as a half-range sine series in $0 < x < 2$.
13. Find the half-range cosine series for the function $f(x) = (x - 1)^2$ in the interval $0 < x < 1$.

Hence prove that $\pi^2 = 8\left(\dfrac{1}{(1)^2} + \dfrac{1}{(3)^2} + \dfrac{1}{(5)^2} + \ldots\right)$

14. The following values of y give the displacement in inches of a certain machine part for the rotation x of the flywheel. Expand y in the form of a Fourier series :

$x = 0$	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$
$y = 0$	9.2	14.4	17.8	17.3	11.7

15. Obtain the constant term and the coefficients of the first sine and cosine terms in the Fourier expansion of y as given in the following table :

x	0	1	2	3	4	5
y	9	18	24	28	26	20

2

Undamped Free Vibrations

2.1. INTRODUCTION

When the elastic system vibrates because of inherent forces and no external force is included, it is called free vibration. If during vibrations there is no loss of energy due to friction or resistance it is known as undamped vibration. Free vibrations which occur in absence of external force are easy to analyse for single degree of freedom systems; that is a system where only one coordinate is required to describe the motion. Other motions also may occur; but they can be neglected for analysis.

A vibratory system having mass and elasticity with single degree of freedom is the simplest case to analyse. For example, a single cylinder engine with a flywheel offers very simple mathematical solution and good results for practical purpose.

Determination of natural frequencies to avoid resonance is essential in machine elements. In the present chapter the development of equations of motion is discussed and natural frequencies of the systems are determined. The effects of damping on natural frequencies have been neglected. The methods for determining the natural frequency of the system are discussed and used for certain physical systems. In the end, some numerical problems are solved.

2.2. DERIVATION OF DIFFERENTIAL EQUATION

The equations of motion for a single degree of freedom system can be found by employing many methods, but in this chapter the discussion will limit to three methods *i.e.* Newton's method, Energy method and Rayleigh's method.

2.2.1. Newton's Method

Spring-mass system in vertical position

Consider a spring-mass system of figure 2.1 constrained to move in a rectilinear manner along the axis of the spring. Spring of constant stiffness k which is fixed at one end carries a mass m at its free end. The body is displaced from its equilibrium position vertically downwards. This equilibrium position is called static equilibrium. The free body diagram of the system is shown in figure 2.2.

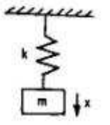


Fig. 2.1.

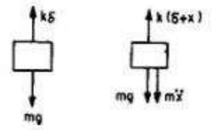


Fig. 2.2.

In equilibrium position, the gravitational pull W, is balanced by a force of spring, such that

$$mg = W = k\delta$$

where δ is the static deflection of the spring. Since the mass is displaced from its equilibrium position by a distance x and then released, so after time t,

$$\text{Restoring force} = W - k(\delta + x)$$

$$m\ddot{x} = W - k\delta - kx$$

$$= -kx \qquad (\because W = k\delta)$$

$$m\ddot{x} + kx = 0 \qquad ...(2.2.1)$$

where $\ddot{x} = \dfrac{d^2x}{dt^2}$ is the acceleration of mass, m. This is recognised as the equation for simple harmonic motion.

The solution is $x = A\cos\omega_n t + B\sin\omega_n t$ ...(2.2.2)

where A and B are constants which can be found by considering the initial conditions, and ω_n is the circular frequency of the motion. Substituting equation (2.2.2) in (2.2.1), one gets

$$-\omega_n^2(A\cos\omega_n t + B\sin\omega_n t) + (k/m)(A\cos\omega_n t + B\sin\omega_n t) = 0$$

Since $(A\cos\omega_n t + B\sin\omega_n t) \neq 0$

$$-\omega_n^2 + k/m = 0$$

$$\therefore \qquad \omega_n = \sqrt{k/m} \text{ rad/s}$$

The frequency of vibration, $f_n = \omega_n/2\pi$

or

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ Hz} \qquad ...(2.2.3)$$

Equation (2.2.1) can be obtained by 'D' Alembert's Principle also which states that the resultant force acting on a body along with the inertia force is zero, then the body will be in static equilibrium. Inertia force acting on the body is represented mathematically as

$$F_i = m \cdot f$$

where m = mass of the body, and

f = linear acceleration of the centre of mass

Assuming that the resultant force acting on the body is F, then the body will be in static equilibrium if

$$F + F_i = o$$

It is to be mentioned that the inertia force and accelerating force (m.f) are equal in magnitude but opposite in direction. The inertia force is an external force acting on the body. If we consider figure 2.1, the spring force of the body $k.x$ will be acting in the upward direction. The acceleration of the body is $\ddot{x}$ which acts in the downward direction. The accelerating force is acting downwards and so the inertia force will be acting in the upward direction. So the body will be in static equilibrium under the action of these two k.x and $m\ddot{x}$ forces. Mathematically, it can be written

$$m\ddot{x} + kx = 0$$

The same equation was obtained by Newton's method.

In eqn. (2.2.2), constants A and B can be determined by considering that the displacement x is x_0 at $t = 0$ and velocity $\dot{x}$ is v_0 at $t = 0$.

Applying these conditions in eqn. (2.2.2), we get

$$x = x_0 = A\cos 0 + B\sin 0, \;\Rightarrow\; x_0 = A \qquad (\text{at } t = 0)$$

and

$$\dot{x} = -A\,\omega_n\sin\omega_n t + B\,\omega_n\cos\omega_n t$$

$$v_0 = -x_0\,\omega_n \cdot 0 + B\,\omega_n \cdot \cos 0, \;\Rightarrow\; v_0 = B \cdot \omega_n \qquad (\text{at } t = 0)$$

The equation of displacement can be written as

$$x = x_0\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t \qquad (2.2.3\text{A})$$

$$\left(\because\; A = x_0 \text{ and } B = \frac{v_0}{\omega_n}\right)$$

Trigonometric Form

Equation (2.2.2) could be written in most general form of S.H.M. by assuming $A = C\cos\phi$, $B = C\sin\phi$

Then the equation (2.2.2) in trigonometric form can be written as

$$x = C\cos\phi\cos\omega_n t + C\sin\phi\sin\omega_n t$$

$$= C\cos(\omega_n t - \phi) \qquad ...(2.2.3\text{B})$$

Equation (2.2.3A) can be written after differentiation, as

$$\dot{x} = -x_0\,\omega_n\sin\omega_n t + v_0\cos\omega_n t$$

or

$$\frac{\dot{x}}{\omega_n} = -x_0\sin\omega_n t + \frac{v_0}{\omega_n}\cos\omega_n t$$

$$\frac{B}{A} = \frac{C\sin\phi}{C\cos\phi} = \tan\phi$$

or

$$\phi = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{v_0}{\omega_n x_0}\right)$$

and

$$B^2 + A^2 = C^2(\cos^2\phi + \sin^2\phi)$$

or

$$C = \sqrt{B^2 + A^2} = \sqrt{\left(\frac{v_0}{\omega_n}\right)^2 + x_0^2}$$

Eqn. (2.2.3B) can be written by substituting the value of C and ϕ in it

$$x = \sqrt{\left(\frac{v_0}{\omega_n}\right)^2 + x_0^2} \cdot \cos\left(\omega_n \cdot t - \tan^{-1}\frac{v_0}{\omega_n x_0}\right) \qquad ...(2.2.3\text{C})$$

Similarly, expression for $\dot{x}$ also can be obtained.

Spring-mass System in Horizontal Position

In the system shown in figure 2.3 a body of mass m is free to move on a fixed horizontal surface. The mass is supported on frictionless rollers. The spring of constant stuffiness k is attached to a fixed frame at one side and to mass m at other side.

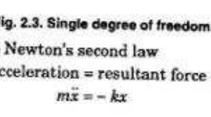
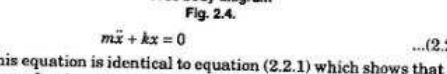


Fig. 2.3. Single degree of freedom systems.

According to Newton's second law

mass × acceleration = resultant force on the mass

$$m\ddot{x} = -kx$$



(a) Applied force (b) Effective force

Free body diagram

Fig. 2.4.

$$m\ddot{x} + kx = 0 \qquad ...(2.2.4)$$

This equation is identical to equation (2.2.1) which shows that the frequency of a given system is same whether it vibrates in a horizontal or vertical position.

2.2.2. Energy Method

Equation (2.2.4) can also be derived assuming the system to be a conservative one. In a conservative system the total sum of the energy is constant. In a vibratory system the energy is partly potential and partly kinetic. The kinetic energy T is because of velocity of the mass and potential energy V is stored in the spring because of its elastic deformation. According to conservation law of energy, we know

$$T + V = \text{constant} \qquad ...(2.2.5)$$

Differentiation of the above equation w.r.t. time, will be zero.

$$\frac{d}{dt}(T + V) = 0 \qquad ...(2.2.6)$$

Kinetic and potential energies for system shown in figure 2.1 are given by

$$T = \frac{1}{2}m\dot{x}^2 \qquad ...(2.2.7)$$

$$V = \frac{1}{2}kx^2 \qquad ...(2.2.8)$$

So

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$$

$$(m\dot{x}\ddot{x} + kx\dot{x}) = 0$$

or

$$m\ddot{x} + kx = 0 \qquad ...(2.2.9)$$

This is the same equation as obtained by Newton's method.

2.2.3. Rayleigh's Method

In deriving the expression for motion, it is assumed that the maximum kinetic energy at the mean position is equal to the maximum potential energy at the extreme position. The motion is assumed to be simple harmonic, then

$$x = A\sin\omega_n t \qquad ...(2.2.10)$$

where x = displacement of the body from mean position after time t.

A = maximum displacement from mean position to the extreme position

Differentiating equation (2.2.10) w.r.t., time we get

$$\dot{x} = \omega_n A\cos\omega_n t$$

Maximum velocity at the mean position,

$$\dot{x} = \omega_n A$$

So maximum kinetic energy at mean position

$$= 1/2\, m\dot{x}^2 = 1/2\, m\,\omega_n^2 A^2 \qquad ...(2.2.11)$$

and maximum potential energy at the extreme position

$$= \frac{1}{2}kA^2 \qquad ...(2.2.12)$$

Equating equation (2.2.11) and (2.2.12), we get

$$\frac{1}{2} m\omega_n^2 A^2 = \frac{1}{2} kA^2$$

$$\omega_n^2 = k/m$$

$$\therefore \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad ...(2.2.13)$$

Equation (2.2.13) is identical to equation (2.2.1) and (2.2.9). These methods are widely used for the determination of natural frequency of the system.

2.3. TORSIONAL VIBRATIONS

Suppose a system having a rotor of mass moment of inertia I connected to a shaft (at its end) of torsional stiffness k_T, is twisted by an angle θ as shown in figure 2.5

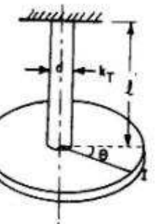
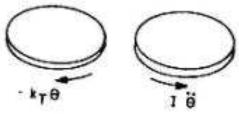


Fig 2.5.

The body is rotated through an angle θ and released, the torsional vibrations will result. The mass moment of inertia of the shaft about the axis of rotation is usually negligible compared with I. The free body diagram of general angular displacement θ is shown in figure 2.6. The equation of motion is written as

$$I\ddot{\theta} = -K_T\,\theta \qquad \text{(restoring torque)}$$

$$I\ddot{\theta} + K_T\theta = 0$$



Torques are balanced
Fig. 2.6

$$\text{or} \qquad \ddot{\theta} + \frac{k_T}{I}\theta = 0 \qquad ...(2.3.1)$$

$$\text{Putting} \qquad \omega_n^2 = \frac{k_T}{I} \qquad ...(2.3.2)$$

The equation (2.3.1) becomes

$$\ddot{\theta} + \omega_n^2\,\theta = 0 \qquad ...(2.3.3)$$

This equation is of the same form as equations (2.2.1) and (2.2.9)

The natural frequency can be determined as

$$\omega_n = \sqrt{\frac{k_T}{I}} \qquad ...(2.3.4)$$

The torsional stiffness k_T of shaft can be determined with the help of this relation *i.e.*,

$$\frac{T}{J} = \frac{G\theta}{l} \quad \text{or} \quad K_t = \frac{T}{\theta} = \frac{GJ}{l} \qquad \text{(From Strength of Materials)}$$

$$k_T = \frac{GJ}{l} = \frac{G}{l}\frac{\pi}{32}d^4$$

where d = dia. of shaft, l = length of shaft, $J = \pi/32\,d^4$ ...(2.3.5)

2.4. EQUIVALENT STIFFNESS OF SPRING COMBINATIONS

Certain systems have more than one spring. The springs are joined in series or parallel or both. They can be replaced by a single spring of the same stiffness as they all show the same stiffness jointly. See figure 2.7.

Springs in parallel Springs in series
Fig. 2.7

Springs in parallel

The deflection of individual spring is equal to the deflection of the system.

So $\qquad k_1x + k_2x = k_ex$

$$\therefore \qquad k_e = k_1 + k_2 \qquad ...(2.4.1)$$

where $\quad k_e$ = equivalent stiffness of the system

k_1, k_2 = stiffnesses of springs

x = deflection of the system

k_ex = Force on the system

Thus equivalent spring stiffness is equal to the sum of individual spring stiffnesses.

Springs in series

The total deflection of the system is equal to the sum of deflection of individual springs.

So

$$x = x_1 + x_2 + x_3 + ...$$

$$\frac{\text{Force}}{k_e} = \frac{\text{Force}}{k_1} + \frac{\text{Force}}{k_2} + \frac{\text{Force}}{k_3} + ...$$

$$\text{or} \qquad \frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + ... \qquad ...(2.4.2.)$$

Thus when springs are connected in series the reciprocal of equivalent spring stiffness is equal to the sum of the reciprocals of individual spring stiffnesses.

2.5. THE COMPOUND PENDULUM

The system which is suspended vertically and oscillates with a small amplitude under the action of the force of gravity, is known as compound pendulum. It is an example of undamped single degree of freedom system. See figure 2.8.

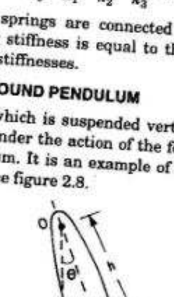


Fig. 2.8.

Let W = weight of the rigid body

$$m = \frac{W}{g}$$

O = point of suspension

k = radius of gyration about an axis through the centre of gravity G

h = distance of point of suspension from G

I = moment of inertia of the body about O.

$= mk^2 + mh^2$

If OG is displaced by an angle θ, restoring torque T.

$$T = -h\,W\sin\theta$$

$$= -mgh\sin\theta \qquad ...(2.5.1)$$

If θ is small $\sin\theta = \theta$, then equation (2.5.1) is written as

$$T = -mgh\,\theta \qquad ...(2.5.2)$$

Inertia torque is given as

$$= -I\ddot{\theta} \qquad ...(2.5.3)$$

Summing up all moments acting on the body

$$I\ddot{\theta} + mgh\,\theta = 0 \qquad ...(2.5.4)$$

The natural frequency ω_n can be determined as

$$\omega_n = \sqrt{\frac{mgh}{I}}$$

$$\text{Also} \qquad \omega_n = \sqrt{\frac{mgh}{mk^2 + mh^2}} = \sqrt{\frac{gh}{k^2 + h^2}}$$

and natural frequency in Hz

$$f_n = \frac{1}{2\pi}\sqrt{\frac{gh}{k^2 + h^2}}\ \text{Hz} \qquad ...(2.5.5)$$

2.6. TRANSVERSE VIBRATION'S OF BEAMS

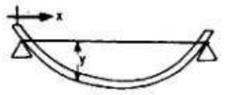
Beams are widely used for structural elements such as floor supports, parts of chassis, etc. Beams of different dimensions are used for different purposes. A vibrating beam is an elastic distributed mass system which has infinite degree of freedom and hence the same number of the natural frequency. But from practical point of view only a few of the lower natural frequencies are important.

Consider the beam shown in figure 2.9 (a and b). If the mass per unit length of the beam is m and y is the amplitude, the maximum kinetic energy is given as

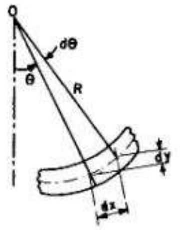
$$\text{K.E.} = \frac{1}{2}\,\omega_n^2 \int y^2\,dm \qquad ...(2.6.1)$$

where ω_n is the natural frequency of the beam. If the bending moment is M and slope of elastic curve is θ, the strain energy of the beam can be expressed as

$$\text{Strain energy} = \frac{1}{2}\int M\,d\theta \qquad ...(2.6.2)$$



Beam element
Fig. 2.9 (a)



Beam deflection
Fig. 2.9. (b)

Usually the deflection of the beam is small and for that the following relations are assumed

$$\theta = \frac{dy}{dx} \text{ and } R d\theta = dx \qquad ...(2.6.3)$$

So
$$\frac{1}{R} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2} \qquad ...(2.6.4)$$

From beam theory, we know that $\frac{M}{I} = \frac{E}{R}$ where R is the radius of curvature and EI the flexural rigidity.

Strain energy is expressed as $= \frac{1}{2}\int \frac{M}{R} dx$ using above relations and equation (2.6.3), we have strain energy as $\frac{1}{2}\int EI\left(\frac{d^2y}{dx^2}\right)^2 dx$ ...(2.6.5)

Since the maximum strain energy is equal to maximum kinetic energy, so

$$\frac{1}{2}\omega_n^2 \int y^2 dm = \frac{1}{2}\int EI\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\omega_n^2 = \frac{\int EI\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int y^2 dm} \qquad ...(2.6.6.)$$

where $dm = \rho A\, dx$

m = mass per unit length, $dm = m \,.\, dx$

This expression gives the lowest natural frequency of transverse vibration of a beam. The ratio ω_n^2 is called Rayleigh's quotient.

2.7. BEAMS WITH SEVERAL MASSES

Let us consider a weightless beam carrying several masses on itself as shown in figure 2.9 (*c*). Let us assume three weights W_1, W_2 and W_3 which cause y_1, y_2 and y_3 the static displacements respectively. ACB represents the static deflection curve. Say ω_n is the natural frequency of vibration of the beam and it executes harmonic motion.

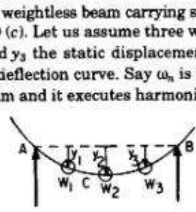


Fig. 2.9 (c)

The maximum velocity of each mass can be written in terms of frequency *i.e.* $\omega_n y_1$, $\omega_n y_2$, $\omega_n y_3$ and so on.

The system itself is the extension of Rayleigh's method to determine the natural frequency.

The maximum kinetic energy of the system

$$T_{max} = \frac{1}{2g}\left[W_1(\omega_n y_1)^2 + W_2(\omega_n y_2)^2 + W_3(\omega_n y_3)^3\right]$$

$$= \frac{\omega_n^2}{2g}\left[W_1 y_1^2 + W_2 y_2^2 + W_3 y_3^2\right] = \frac{\omega_n^2}{2g}\sum_{i=1}^{n} W_i y_i^2 \qquad ...(2.7.1)$$

The Potential Energy (V) of deformation due to bending is given as

$$V_{max} = \frac{1}{2}W_1 y_1 + \frac{1}{2}W_2 y_2 + \frac{1}{2}W_3 y_3$$

$$= \frac{1}{2}(W_1 y_1 + W_2 y_2 + W_3 y_3) = \frac{1}{2}\sum_{i=1}^{n} W_i y_i \qquad ...(2.7.2)$$

Since the system is conservative, so its maximum kinetic energy is equal to maximum potential energy.

So
$$T_{max} = V_{max}$$

$$\frac{\omega_n^2}{g}\sum_{i=1}^{n} W_i y_i^2 = \sum_{i=1}^{n} W_i y_i$$

Thus
$$\omega_n^2 = \frac{g\sum_{i=1}^{n} W_i y_i}{\sum_{i=1}^{n} W_i y_i^2} \qquad ...(2.7.3)$$

2.8 BIFILAR SUSPENSION

In bifilar suspension a weight W is suspended by two long flexible strings as shown in figure 2.9 (*d*).

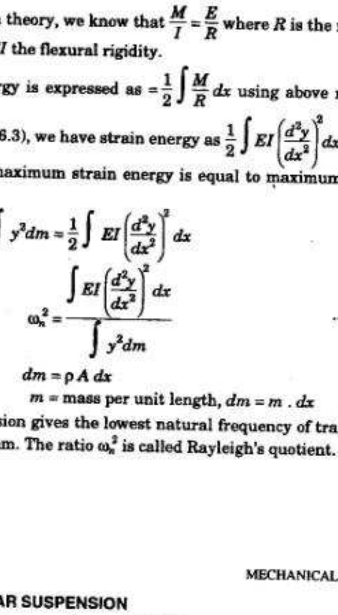


Fig. 2.9 (d). Bifilar Suspension.

Initially the support and the bar AB are parallel. The bar AB is given a slight twist θ and then released. Let the strings are displaced by angles ϕ_1 and ϕ_2. If l_1 and l_2 are the distances of two ends from the centre of gravity G, then tensions T_A and T_B can be written as

$$T_A = \frac{Wl_2}{l_1 + l_2} \text{ and } T_B = \frac{Wl_1}{l_1 + l_2} \qquad ...(i)$$

Since the angles ϕ_1 and ϕ_2 are small, so the effects of vertical accelerations can be neglected. Only the horizontal components of tensions will be considered which are given as

$T_A \,.\, \phi_1$ and $T_B \,.\, \phi_2$ and both are perpendicular to $A'B'$.

From the geometry of figure

$$l_1\,\theta = l\,\phi_1 \text{ and } l_2 \,.\, \theta = l \,.\, \phi_2$$

or
$$\phi_1 = l_1 \,.\, \theta/l, \quad \phi_2 = \frac{l_2 \,.\, \theta}{l} \qquad ...(ii)$$

The resisting torque T for the system, can be written as

$$T = T_A \,.\, \phi_1 \,.\, l_1 + T_B \,.\, \phi_2 \,.\, l_2$$

$$= \frac{Wl_2}{l_1 + l_2} \cdot \frac{l_1\,\theta}{l} \,.\, l_1 + \frac{Wl_1}{l_1 + l_2} \cdot \frac{l_2 \,.\, \theta}{l} \,.\, l_2$$

(Substituting T_A and T_B from (*i*) and ϕ_1 and ϕ_2 from eqn. (*ii*))

$$= \frac{Wl_1\, l_2\,\theta}{(l_1 + l_2)\, l}(l_1 + l_2)$$

$$= \frac{Wl_1\, l_2\,\theta}{l} \qquad ...(iii)$$

We know that $T = I \,.\, \alpha$

where T = Torque

I = moment of inertia $= \frac{W}{g}k^2$

α = angular acceleration

k = radius of gyration

So
$$\alpha = \frac{T}{I}$$

$$= \frac{Wl_1\, l_2\,\theta}{l \,.\, \frac{W}{g}k^2} = \frac{g l_1\, l_2\,\theta}{l \,.\, k^2} \qquad ...(iv)$$

We also know that

$$\omega^2 = \frac{\text{Angular acceleration}}{\text{Angular displacement}}$$

where ω = Angular velocity of bar AB

$$\omega^2 = \frac{g \,.\, l_1 l_2 \,.\, \theta}{l \,.\, k^2 \,.\, \theta}$$

or
$$k^2 = \frac{g \,.\, l_1 \,.\, l_2}{l \,.\, \omega^2} \qquad ...(v)$$

Thus radius of gyration k can be determined with the help of eqn. (*v*). Moment of inertia of the bar can be determined as given below

$$I = \frac{W}{g}k^2 = \frac{W}{g}\frac{g \,.\, l_1 \,.\, l_2}{l \,.\, \omega^2} = \frac{W \,.\, l_1 \,.\, l_2}{l \,.\, \omega^2}$$

This bifilar suspension is a method or device by means of which we can find the moment of inertia of the bar suspended by two strings.

2.9 TRIFILAR SUSPENSION

In trifilar suspension there is a disc usually circular or triangular type suspended by three wires of equal length. The support and disc both are horizontal. The wires are in equilateral position i.e. 120° apart from each other. The method is used to find the mass moment of inertia of complicated shapes. Refer figure 2.9 (*e*).

In this case with an angular displacement θ in horizontal plane, each string will be displaced by an angle ϕ in the vertical plane.

Then tension in each wire, $T = \frac{W}{3}$

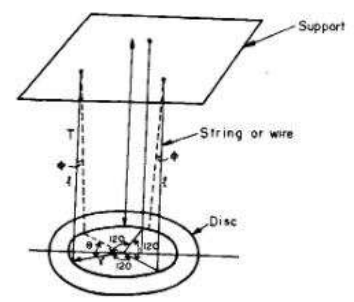


Fig. 2.9 (e). Trifilar Suspension.

where W is the weight of the disc.

and $$r\,.\,\theta = l\,.\,\phi$$

or $$\phi = \frac{r\,.\,\theta}{l}$$

Horizontal component of each tension $= T\,.\,\phi$

or $$= \frac{W}{3}\cdot\left(\frac{r\,.\,\theta}{l}\right)$$

Total torque (restoring) $= W\left(\frac{r\,.\,\theta}{l}\right).\,r = \frac{W\,.\,r^2\,.\,\theta}{l}$ ...(i)

Accelerating torque $= I\,.\,\alpha$

$$= \frac{W}{g}k^2\,.\,\alpha \qquad ...(ii)$$

We know that

Accelerating torque = Restoring torque

$$\frac{W}{g}\,.\,k^2\,.\,\alpha = \frac{W}{l}r^2\,.\,\theta$$

or $$\alpha = \frac{r^2\,.\,\theta\,.\,g}{l\,.\,k^2}$$

where k is the radius of gyration.

But $$\frac{\text{Angular Acceleration}}{\text{Angular displacement}} = \omega^2$$

$$\frac{r^2\,.\,\theta\,.\,g}{l\,.\,k^2\,.\,\theta} = \omega^2$$

or $$1\Big/\frac{\omega^2\,.\,l}{r^2\,.\,g} = k^2$$

With the help of above expression, the value of radius of gyration can be determined and then the mass moment of inertia of the disc can be evaluated with the help of this relation :

$$I = \frac{W}{g}k^2$$

SOLVED EXAMPLES

EXAMPLE 2.1. *Determine the natural frequency of the mass m placed at one end of a cantilever beam of negligible mass as shown in figure 2.10.*

Fig. 2.10.

SOLUTION. The stiffness of beam is given as

$$k = 3EI/l^3$$

where EI is the flexural rigidity of the beam.

The general equation of motion for undamped free vibrations is given as

$$m\ddot{x} + kx = 0$$

Substituting the value of k in the above equation, we get

$$m\ddot{x} + \frac{3EI}{l^3}x = 0$$

So $$\omega_n = \sqrt{\frac{3EI}{l^3 m}} \text{ rad/sec}$$

or $$f_n = \frac{1}{2\pi}\sqrt{\frac{3EI}{ml^3}} \text{ Hz}$$

EXAMPLE 2.2. *Find the natural frequency of the system shown in figure 2.11.*

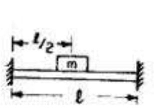


Fig. 2.11.

SOLUTION. The deflection at the centre of a bar fixed at both ends with load F at the centre is given as

$$\text{deflection} = Fl^3/192\,EI$$
$$\text{stiffness} = \text{load/deflection}$$
$$= 192\,EI/l^3$$

The general equation for undamped free vibration is written as

$$m\ddot{x} + kx = 0$$

$$\omega_n = \sqrt{k/m} = \sqrt{\frac{192\,EI}{ml^3}} \text{ rad/sec}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{192EI}{ml^3}} \text{ Hz}$$

EXAMPLE 2.3. *Determine the effect of the mass of the spring on the natural frequency of the system shown in figure 2.12.* (M.D.U., 94)

SOLUTION. Let x be the displacement of mass m and so the velocity will be $\dot{x}$. The velocity of spring element at a distance y from the fixed end may be written as $\frac{\dot{x}\,y}{l}$ where l is the total length of spring.

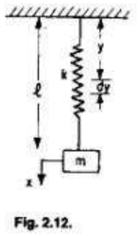


Fig. 2.12.

The kinetic energy of spring element dy is written as

$$\frac{1}{2}(\rho\,dy)\left(\frac{y}{l}\dot{x}\right)^2$$

where ρ being the mass of spring per unit length.

Total kinetic energy of the system

$$\text{K.E.} = \frac{1}{2}m\dot{x}^2 + \int_0^l \frac{1}{2}(\rho\,dy)\left(\frac{y}{l}\dot{x}\right)^2$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\rho\,\dot{x}^2\frac{l}{3} = \frac{1}{2}m\dot{x}^2 + \frac{1}{6}m_s\dot{x}^2$$

where mass of the spring $m_s = \rho l$

Potential energy of the system $= \frac{1}{2}kx^2$

Total energy of the system = K.E. + P.E.

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m_s\frac{\dot{x}^2}{3} + \frac{1}{2}kx^2 = \text{Constant}$$

Differentiating the above equation with respect to time

$$m\ddot{x} + \frac{m_s\ddot{x}}{3} + kx = 0$$

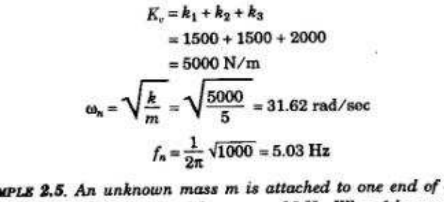
$$\left(m + \frac{m_s}{3}\right)\ddot{x} + kx = 0$$

So $$\omega_n = \sqrt{\frac{k}{m + m_s/3}} \text{ rad/sec}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m + m_s/3}} \text{ Hz}$$

EXAMPLE 2.4. *Find the natural frequency of the system shown in figure 2.13.*

Given $k_1 = k_2 = 1500$ N/m, $k_3 = 2000$ N/m, $m = 5$ kg.



Free body diagram
Fig. 2.13

SOLUTION. Given $m = 5$ kg

Equivalent stiffness in parallel

$$K_e = k_1 + k_2 + k_3$$
$$= 1500 + 1500 + 2000$$
$$= 5000 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5000}{5}} = 31.62 \text{ rad/sec}$$

$$f_n = \frac{1}{2\pi}\sqrt{1000} = 5.03 \text{ Hz}$$

EXAMPLE 2.5. *An unknown mass m is attached to one end of a spring of stiffness k having natural frequency of 6 Hz. When 1 kg mass is attached with m the natural frequency of the system is lowered by 20%. Determine the value of unknown mass m and stiffness k.* (P.U., 98)

SOLUTION. $$f_1 = 6 \text{ Hz} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad ...(1)$$

$$f_2 = 6 \times \frac{80}{100}\ \text{Hz} = \frac{1}{2\pi}\sqrt{\frac{k}{m+1}} \qquad ...(2)$$

So
$$\frac{f_1}{f_2} = \frac{6}{4.8} = \frac{\sqrt{k/m}}{\sqrt{k/m+1}}$$

$$(1.25)^2 = \frac{k/m}{k/m+1} = \frac{m+1}{m}$$

$$m = 1.777 \text{ kg}$$

Stiffness from equation (1) can be determined as

$$6 = \frac{1}{2\pi}\sqrt{k/m}$$

$$(6 \times 2\pi)^2 = \frac{k}{m}$$

$$k = 2523.94 \text{ N/m}$$

EXAMPLE 2.6. *A spring-mass system ($k_1 - m_1$) has a natural frequency f_1. Calculate the value of k_2 another spring which when connected to k_1 in parallel increases the frequency by 30%.*

SOLUTION. $f_1 = \sqrt{k_1/m_1}$

another spring is connected to k_1 in parallel, so $ke = k_1 + k_2$

$$f_2 = \frac{130}{100} f_1 = \sqrt{\frac{k_1 + k_2}{m_1}}$$

$$\frac{f_1}{f_2} = \frac{1}{1.3} = \sqrt{\frac{k_1}{m_1}\,\frac{m_1}{k_1 + k_2}}$$

$$\frac{1}{1.3} = \sqrt{\frac{k_1}{k_1 + k_2}}$$

$$1.69\, k_1 = k_1 + k_2$$

$$k_2 = .69\, k_1$$

EXAMPLE 2.7. *A simple U tube manometer filled with liquid is shown in figure 2.14. Calculate the frequency of resulting motion if the minimum length of a manometer tube is .15 m.*

SOLUTION. Let us apply energy method to find the frequency. The liquid column is displaced from equilibrium position by a distance x. If ρ and

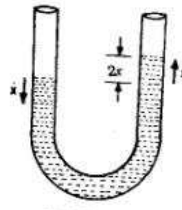


Fig. 2.14.

A are the mass density of liquid and cross section area of the tube respectively, then the mass of the liquid column is ρAl. With $\dot{x}$ as velocity of liquid, the kinetic energy is given as

$$\text{K.E.} = \frac{1}{2}(\rho Al)\dot{x}^2$$

$$\text{P.E.} = (\rho Axg)x = \rho Agx^2$$

Total energy of the system is constant

$$\frac{1}{2}\rho Al\dot{x}^2 + \rho Agx^2 = C$$

Differentiating it with respect to time, we get

$$\rho Al\dot{x}\ddot{x} + 2\rho Agx\dot{x} = 0$$

$$l\ddot{x} + 2gx = 0$$

So
$$\omega_n = \sqrt{\frac{2g}{l}} \text{ rad/sec}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{2 \times 9.81}{.15}}\ \text{Hz} = 1.82 \text{ Hz}$$

EXAMPLE 2.8. *An electric motor is supported by six springs of stiffness k each. The moment of inertia of the motor is I. Determine the natural frequency of the system. Refer figure 2.15.*

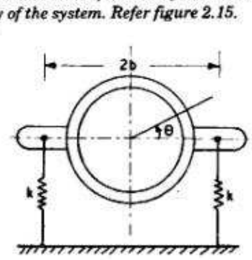


Fig. 2.15.

SOLUTION. The restoring torque is given as

$$= \text{spring force} \times \text{displacement}$$

$$= 6\,(kb\theta)b$$

$$I\ddot{\theta} = -6kb^2\theta$$

$$I\ddot{\theta} + 6kb^2\theta = 0$$

$$\ddot{\theta} + \frac{6kb^2\theta}{I} = 0$$

So
$$\omega_n = \sqrt{\frac{6kb^2}{I}} = b\sqrt{\frac{6k}{I}} \text{ rad/sec}$$

EXAMPLE 2.9. *Determine the natural frequency of the spring mass pulley system shown in figure 2.16.* (P.U., 89)

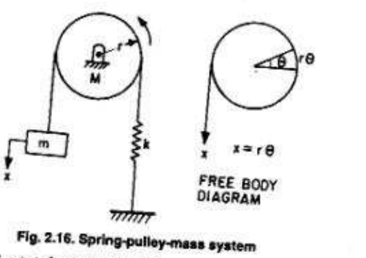


Fig. 2.16. Spring-pulley-mass system

SOLUTION. The total energy T of the system

T = kinetic energy of the mass m + kinetic energy of pulley M + potential energy of spring k

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}kx^2 = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}kr^2\theta^2 = \text{constant}$$

From free body diagram it is clear that at any moment $x = r\theta$

Moment of inertia of pulley $I = 1/2\, Mr^2$

Differentiating total energy equation with respect to time, we get

$$mr^2\dot{\theta}\ddot{\theta} + I\dot{\theta}\ddot{\theta} + kr^2\theta\dot{\theta} = 0$$

$$mr^2\ddot{\theta} + I\ddot{\theta} + kr^2\theta = 0$$

$$(mr^2 + 1/2\, Mr^2)\ddot{\theta} + kr^2\theta = 0$$

$$\ddot{\theta} + \frac{kr^2}{mr^2 + \frac{1}{2}Mr^2}\theta = 0$$

$$\omega_n = \sqrt{\frac{2kr^2}{2mr^2 + Mr^2}} = \sqrt{\frac{2k}{2m + M}} \text{ rad/sec}$$

EXAMPLE 2.10. *A circular cylinder of mass 4 kg and radius 15 cm. is connected by a spring of stiffness 4000 N/m as shown in figure 2.17. It is free to roll on horizontal rough surface without slipping, determine the natural frequency.* (P.U. 93, P.U., 99)

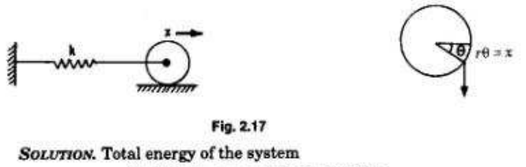


Fig. 2.17

SOLUTION. Total energy of the system

T = K.E. due to translatory motion

+ K.E. due to rotary motion + P.E. of spring

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}kx^2$$

$$= \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}\cdot\frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}kr^2\theta^2 \qquad (\because x = r\theta)$$

$$T = \frac{3}{4}mr^2\dot{\theta}^2 + \frac{1}{2}kr^2\theta^2 = \text{constant}$$

Differentiating T with respect to time, we get

$$0 = \frac{3}{4}\cdot 2\,mr^2\dot{\theta}\ddot{\theta} + kr^2\theta\dot{\theta} = 0$$

$$\frac{3}{2}mr^2\ddot{\theta} + kr^2\theta = 0$$

$$\omega_n = \sqrt{\frac{kr^2}{3/2\, mr^2}} = \sqrt{\frac{2k}{3m}} \text{ rad/sec}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{2 \times 4000}{3 \times 4}} = 4.11 \text{ Hz}$$

EXAMPLE 2.11. *Find the natural frequency of the system shown in figure 2.18.* (P.U. Aero 94)

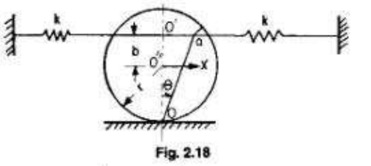


Fig. 2.18

SOLUTION. K.E. $= \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2$

$$= \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{4}mr^2\dot{\theta}^2 = \frac{3}{4}mr^2\dot{\theta}^2$$

From the geometry of figure

$$x = o'a$$
$$= oo' \times \theta$$
$$= (oo'' + o'o'') \times \theta$$
$$x = (r + b)\,\theta \qquad (\theta, \text{ being small})$$
$$\text{P.E.} = \frac{1}{2} kx^2 + \frac{1}{2} kx^2$$
$$= 2 \cdot \frac{1}{2} k\,[(r + b)\theta]^2 = k\,(r + b)^2\,\theta^2$$

Now, energy is conserved, so

$3/4\ mr^2\dot{\theta}^2 + k(r + b)^2\,\theta^2$ is constant, so differentiating on with respect to time gives

$$3/4\ mr^2\ddot{\theta} + k(r + b)^2\,\theta = 0$$

Hence the frequency of vibration is $\frac{1}{2\pi}\sqrt{\frac{4k(r+b)^2}{3mr^2}}$ Hz

***Example* 2.12.** *Find the natural frequency of the system shown in figure 2.19.*

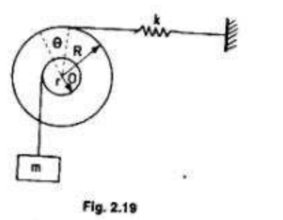


Fig. 2.19

Solution. Let x_2 = deflection of spring = $R \cdot \theta$

Spring force $= kx_2 = k.R.\theta$

$x_1 = r \cdot \theta$ = downward movement of mass m

Total Kinetic energy = K.E. of the mass + K.E. of rotating element

$$= \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} I\dot{\theta}^2$$

Potential energy of spring $= \frac{1}{2} kx_2^2$

Total energy $= \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} kx_2^2$

By energy method, we get

$$\frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} kx_2^2 = \text{constant}$$

Differentiating the above expression w.r.t θ, we get

$$\frac{d}{d\theta}\left[\frac{1}{2} mr^2\,\dot{\theta}^2 + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} kR^2\,\theta^2\right] = \frac{d}{d\theta}\,(\text{Contt})$$
$$mr^2\,\dot{\theta}\,\ddot{\theta} + I\,\dot{\theta}\,\ddot{\theta} + kR^2\,\theta\,\dot{\theta} = 0$$

or

$$(mr^2 + I)\,\ddot{\theta} + kR^2\,\theta = 0$$
$$\ddot{\theta} + \left(\frac{kR^2}{mr^2 + I}\right)\theta = 0$$
$$\omega_n = \sqrt{\frac{kR^2}{mr^2 + I}} \quad \text{or } f_n = \frac{1}{2\pi}\sqrt{\frac{kR^2}{mr^2 + I}}$$

***Example* 2.13.** *Find the natural frequency of the system in figure 2.20 assuming the bar CD to be weightless and rigid.*

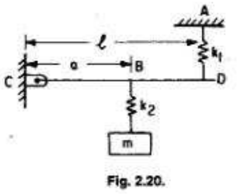


Fig. 2.20.

Solution. To find the natural frequency the value of equivalent spring at A is to be determined.

Let us consider a force P at D, so the force at point B can be written as Pl/a.

$$(F_B \cdot a = P \cdot l$$
$$F_B = \frac{P \cdot l}{a}\ )$$

The deflection at point B can be expressed as

$$\delta_B = \frac{F_B}{k_2} = \frac{Pl}{ak_2}$$
$$P \cdot \delta_D = F_B \cdot \delta_B$$
$$\delta_D = \frac{F_B \cdot \delta_B}{P}$$
$$= P\left(\frac{l}{a}\right)^2 \frac{1}{k_2} = \frac{Pl^2}{a^2 k_2}$$

For this value of deflection the corresponding deflection at point D

$\frac{P}{k_2}\,(l/a)^2$ and stiffness $k_2(a/l)^2 = k_d$

k_d and k_1 are connected in series, so their equivalence can be written as

$$k = \frac{k_d \cdot k_1}{k_d + k_1} = \frac{k_2\,(a/l)^2\,k_1}{k_2(a/l)^2 + k_1}$$

Natural frequency can be written as

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2 (a/l)^2}{m[k_1 + k_2(a/l)^2]}}\ \text{Hz}$$

***Example* 2.14.** *Calculate the natural frequency of the system shown in figure 2.21 if the mass of the rod is negligible compared to the mass m.*

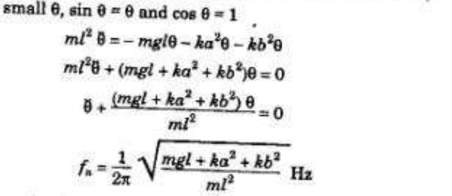


Fig. 2.21

Solution. From the torque equation, we get

$$I\ddot{\theta} = -\text{ Restoring torque}$$
$$ml^2\ddot{\theta} = -(mg\sin\theta)l - ka\sin\theta\,(a\cos\theta) - kb\sin\theta\,(b\cos\theta)$$

For small θ, $\sin\theta = \theta$ and $\cos\theta = 1$

$$ml^2\,\ddot{\theta} = -\,mgl\theta - ka^2\theta - kb^2\theta$$
$$ml^2\ddot{\theta} + (mgl + ka^2 + kb^2)\theta = 0$$
$$\ddot{\theta} + \frac{(mgl + ka^2 + kb^2)\,\theta}{ml^2} = 0$$

So

$$f_n = \frac{1}{2\pi}\sqrt{\frac{mgl + ka^2 + kb^2}{ml^2}}\ \text{Hz}$$

If no spring is used, the system becomes a simple pendulum.

***Example* 2.15.** *Determine the natural frequency of the system shown in figure 2.22 where*

I = moment of inertia of rocker about its axis

k_s = spring stiffness

k_p = push rod stiffness

m_s = mass of spring

m_v = mass of valve

and m_p = mass of push rod.

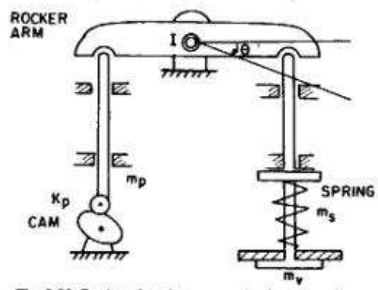


Fig. 2.22. Push rod rocker arm and valve of engine.

Solution. Let us consider the system to be conservative for which the sum of potential and kinetic energy is constant.

$$\text{K.E.} = \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} m_v(a\dot{\theta})^2 + \frac{1}{2}\frac{m_s}{3}(a\dot{\theta})^2 + \frac{1}{2}\frac{m_p}{3}(b\dot{\theta})^2$$
$$= \frac{1}{2}\,(I + m_v a^2 + \frac{1}{3} m_s a^2 + \frac{1}{3} m_p b^2)\dot{\theta}^2$$
$$\text{P.E.} = \frac{1}{2} k_s(a\theta)^2 + \frac{1}{2} k_p(b\theta)^2$$
$$= \frac{1}{2}(k_s a^2 + k_p b^2)\theta^2$$

Total energy = K.E. + P.E.

Differentiating the total energy with respect to time, we get

$$(I + m_v a^2 + \frac{1}{3} m_s a^2 + \frac{1}{3} m_p b^2)\ddot{\theta} + (k_s a^2 + k_p b^2)\theta = 0$$

This equation is of the form

$$I\ddot{\theta} + k\theta = 0$$

So the natural frequency of the system can be determined as

$$f_n = \frac{1}{2\pi}\sqrt{\frac{(k_s a^2 + k_p b^2)}{\ldots}}\ \text{Hz.}$$

***Example 2.16.** A mass is suspended from a spring system as shown in figure 2.23. Determine the natural frequency of the system.* (P.U., 89)

$k_1 = 5000\ N/m,\ k_2 = k_3 = 8000\ N/m,\ m = 25\ kg$

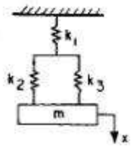


Fig. 2.23

Solution. Since springs k_2 and k_3 are connected in parallel, so their equivalence k is given as $k = k_2 + k_3$. Again k and k_1 are connected in series, so the equivalence k_e is given as

$$\frac{1}{k_e} = \frac{1}{k} + \frac{1}{k_1} = \frac{1}{k_2 + k_3} + \frac{1}{k_1}$$

$$= \frac{1}{2 \times 8000} + \frac{1}{5000}$$

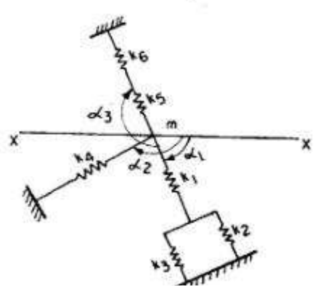
$$k_e = 3809.52\ \text{N/m}$$

The natural frequency

$$\frac{\omega_n}{2\pi} = f = \frac{1}{2\pi}\sqrt{k_e/m} = \frac{1}{2\pi}\sqrt{\frac{3809.52}{25}}$$

$$f_n = 1.96\ \text{Hz}$$

***Example 2.17.** A mass m guided in x – x direction is connected by a spring configuration as shown in figure 2.24. Set up the equation of mass m. Write down the expression for equivalent spring constant.* (P.U., 93)



Solution. Springs k_2 and k_3 are connected in parallel, their equivalence k_{e23}

$$k_{e23} = k_2 + k_3$$

k_{e23} is connected to k_1 in series

$$k_{e\,123} = \frac{k_1 k_{e23}}{k_1 + k_{e23}} = \frac{k_1(k_2 + k_3)}{k_1 + k_2 + k_3}$$

Springs k_5 and k_6 are connected in series, their equivalence is given as

$$k_{e\,56} = \frac{k_5 k_6}{k_5 + k_6}$$

Springs k_{e123}, k_4 and k_{e56} make angles α_1, α_2 and α_3 respectively with the direction of motion, so equivalent stiffness k_e can be determined as

$$k_e = k_{e123}\cos^2\alpha_1 + k_4\cos^2\alpha_2 + k_{e56}\cos^2\alpha_3$$

$$= \frac{k_1(k_2 + k_3)}{k_1 + k_2 + k_3}\cos^2\alpha_1 + k_4\cos^2\alpha_2 + \frac{k_5 k_6}{k_5 + k_6}\cos^2\alpha_3$$

The equation of motion of mass m can be written as $m\ddot{x} + k_e x = 0$

natural frequency, $\omega_n = \sqrt{\frac{k_e}{m}}$

Note : If any spring makes angle α with direction of motion of mass m, as shown here in the figure, the displacement x of mass deforms the spring by $x\cos\alpha$ along its axis. The force along the spring axis will be $kx\cos\alpha$. Again the component of force along the direction of motion will be $kx\cos^2\alpha$.

***Example 2.18.** A spring mass system has spring constant of k kg/cm and the weight of mass W kg. It has natural frequency of vibration as 12 c.p.s. An extra 2 kg weight is coupled to W and natural frequency reduces by 2 c.p.s. Find k and W.* (P.U., 92)

Solution. Given $\quad f_1 = \frac{1}{2\pi}\sqrt{\frac{k}{W/g}} = 12$

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{k}{\frac{W+2}{g}}} = 12 - 2 = 10$$

or

$$\frac{kg}{W} = (12)^2 \cdot 4\pi^2$$

$$\frac{kg}{W+2} = 100 \cdot 4\pi^2$$

$$\frac{W+2}{W} = \frac{144}{100} = 1.44$$

$$W = 2/.44 = 4.54\ \text{kg}$$

and $\quad \frac{kg}{W} = (12)^2 \times 4\pi^2$

$$k = \frac{(12)^2 \times 4\pi^2 \times 4.54}{981} = 26.28\ \text{kg/cm}$$

***Example 2.19.** A body weighing 5 kg is hung on two helical springs in parallel. One spring is elongated 1 cm by a force of 0.3 kg; the other spring requires a force of 0.2 kg for an elongation of 1 cm. Calculate the natural frequency of vibration.* (Uni. Roorkee, 67)

Solution. $\quad k_1 = 0.3$ kg/cm, $\ k_2 = 0.2$ kg/cm

The springs are connected in parallel

$$k_e = k_1 + k_2 = 0.3 + .2 = 0.5\ \text{kg/cm}$$

Natural frequency, $f_n = \frac{1}{2\pi}\sqrt{\frac{k_e}{m}}$

$$= \frac{1}{2\pi}\sqrt{\frac{.50}{5} \times 981} = 1.58\ \text{Hz}$$

***Example 2.20.** Determine the equations of motion for the systems shown in figure 2.25.* (P.U, 85)

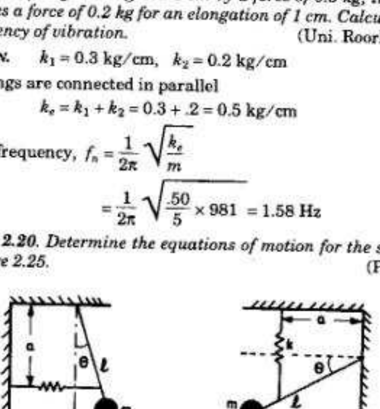


Fig. 2.25.

Solution. We know that

(a) $\quad J\ddot{\theta} = \text{torque}$

$$ml^2\ddot{\theta} = -mgl\sin\theta - (ka\sin\theta)(a\cos\theta)$$

$$ml^2\ddot{\theta} + (mgl + ka^2)\,\theta = 0$$

when θ is very small $\sin\theta = \theta$, $\cos\theta = 1$

So $\quad \omega_n = \sqrt{\frac{mgl + ka^2}{ml^2}}$ rad/sec

(b) $\quad ml^2\ddot{\theta} = -(ka\sin\theta)(a\cos\theta) - mgl(1 - \cos\theta)$

$$ml^2\ddot{\theta} + ka^2\theta = 0$$

Natural frequency, $\omega_n = \sqrt{\frac{ka^2}{ml^2}}$ rad/sec

***Example 2.21.** Find out the equation of motion for the vibratory system shown in figure 2.26.* (P.U., 94)

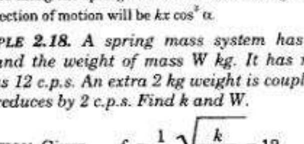


Fig. 2.26.

Solution. The equation of motion can be written as

$$(ka)a\theta + (mb)b\ddot{\theta} = F_0\sin\omega t \cdot b$$

$$ka^2\theta + mb^2\ddot{\theta} = F_0\sin\omega t \cdot b$$

It can be solved for natural frequency.

***Example 2.22.** An indicator mechanism is shown in figure 2.27. The arm pivoted at point O has a mass moment of inertia I. Find the natural frequency of the system. In the figure symbols have their usual meanings.* (P.U., 94)

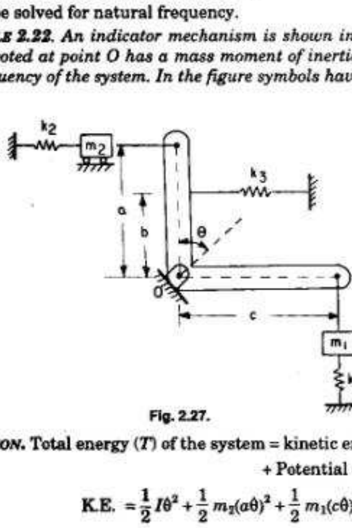


Fig. 2.27.

Solution. Total energy (T) of the system = kinetic energy (K.E.) + Potential energy (P.E.)

$$\text{K.E.} = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m_2(a\dot{\theta})^2 + \frac{1}{2}m_1(c\dot{\theta})^2$$

$$\text{P.E.} = \frac{1}{2}k_2(a\theta)^2 + \frac{1}{2}k_3(b\theta)^2 + \frac{1}{2}k_1(c\theta)^2$$

$$T = \text{K.E.} + \text{P.E.}$$

$\frac{d(T)}{dt} = 0$ as the sum of energy is constant.

It gives,

$$(I + m_2a^2 + m_1c^2)\ddot{\theta} + (k_2a^2 + k_1c^2 + k_3b^2)\theta = 0$$

It is of the form

$$I\ddot{\theta} + k_t\theta = 0$$

So natural frequency of the system can be expressed as

$$\omega_n = \sqrt{\frac{(k_2a^2 + k_1c^2 + k_3b^2)}{(I + m_2a^2 + m_1c^2)}} \text{ rad/sec}$$

EXAMPLE 2.23. Calculate the natural frequency of vibration of a two rotor system as shown in figure 2.28. Neglect the weight of the shaft. (P.U., 93)

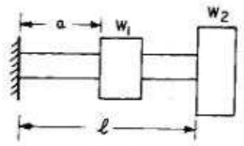


Fig. 2.28

Given

$W_1 = 60$ kg, $W_2 = 92$ kg, $l = 25.6$ cm, $a = 15$ cm, $I = 6.25$ cm^4, $E = 2 \times 10^6$ kgf/cm^2

SOLUTION. From strength of materials (static deflection relation), we know that

$$y_1 = \frac{W_1 a^3}{3EI} + \frac{W_2a^2(3l - a)}{6EI}$$

$$y_2 = \frac{W_2l^3}{3EI} + \frac{W_1a^2(3l - a)}{6EI}$$

$$y_1 = \frac{60 \times (15)^3}{3 \times 2 \times 10^6 \times 6.25} + \frac{92 \times (15)^2(3 \times 25.6 - 15)}{6 \times 2 \times 10^6 \times 6.25}$$

$$= 5400 \times 10^{-6} + 17056.8 \times 10^{-6}$$

$$= 22.456 \times 10^{-3} \text{ cm.}$$

$$y_2 = \frac{92 \times 25.6^3}{3 \times 2 \times 10^6 \times 6.25} + \frac{60 \times 15^2(76.8 - 15)}{6 \times 2 \times 10^6 \times 6.25}$$

$$= 52.28 \times 10^{-3} \text{ cm.}$$

Natural frequency ω_n is given by (see article 2.7)

$$\omega_n^2 = \frac{g\sum W y_i}{\sum W y_i^2} = \frac{g(W_1y_1 + W_2y_2)}{W_1y_1^2 + W_2y_2^2}$$

$$= \frac{980[60 \times 22.456 \times 10^{-3} + 92 \times 52.28 \times 10^{-3}]}{60 \times (22.456 \times 10^{-3})^2 + 92 \times (52.28 \times 10^{-3})^2}$$

So $\omega_n^2 = 21419.07$

$\omega_n = 146.35$ rad/sec

and $f_n = \dfrac{\omega_n}{2\pi} = 23.29$ Hz

EXAMPLE 2.24. Find the natural frequency of vibration of the system for small amplitudes. If k_1, k_2, a and b are fixed, determine the value of b for which the system will not vibrate. Find maximum acceleration of the mass. The system is shown in figure 2.29. (P.U., M.E., 94)

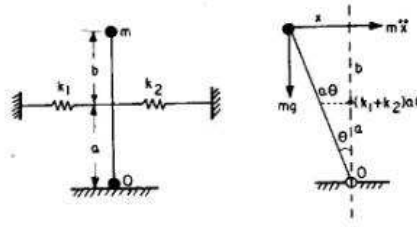


Fig. 2.29.

SOLUTION. At any moment $x = (a + b)\theta$

$\ddot{x} = (a + b)\ddot{\theta}$

The various forces are

$m\ddot{x} = m(a + b)\ddot{\theta}$	(inertia)
$kx = (k_1 + k_2)a\theta$	(spring)
$mg = mg$	(due to gravity)

Taking moments about 0, we get

$$m(a + b)\ddot{\theta}(a + b) + (k_1 + k_2)a^2\theta - (a + b)mg\theta = 0$$

$$(a + b)^2m\ddot{\theta} + [(k_1 + k_2)a^2 - (a + b)mg]\theta = 0$$

or

$$m\ddot{\theta} + \left\{\frac{(k_1 + k_2)a^2 - (a + b)mg}{(a + b)^2}\right\}\theta = 0$$

So the natural frequency of the system

$$\omega_n = \sqrt{\frac{(k_1 + k_2)a^2 - (a + b)mg}{m(a + b)^2}}$$

$$= \sqrt{\frac{(k_1 + k_2)a^2}{m(a + b)^2} - \frac{g}{a + b}} \text{ rad/sec}$$

When $\omega_n = 0$, the system will not vibrate.

Thus

$$0 = \frac{(k_1 + k_2)a^2}{m(a + b)^2} - \frac{g}{a + b}$$

$$\frac{(k_1 + k_2)a^2}{m(a + b)} = g$$

$$(k_1 + k_2)a^2 = mga + mgb$$

$$b = \frac{(k_1 + k_2)a^2 - mga}{mg} = \left[\frac{a(k_1 + k_2)}{mg} - 1\right]a$$

Thirdly, the maximum acceleration of the mass

$$= -\omega^2 \text{ (amplitude)} = -\omega^2(a + b)\theta$$

Putting the value of ω_n, we get

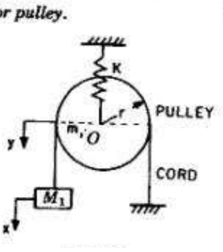
$$= \sqrt{\left(\frac{(k_1 + k_2)a^2}{(a + b)^2m} - \frac{g}{a + b}\right)^2}\,(a + b)\theta$$

$$= \left[\frac{g}{a + b} - \frac{(k_1 + k_2)a^2}{(a + b)^2m}\right](a + b)\theta$$

$$= g\theta - \frac{(k_1 + k_2)a^2}{(a + b)m}\theta$$

EXAMPLE 2.25. Using energy method find the natural frequency of system shown in figure 2.30. The cord may by assumed inextensible he spring mass pulley system and no slip.

Given $J = \frac{1}{2}mr^2$ for pulley. (P.U., Aero 93, 76)



SOLUTION. T = K.E. of mass + K.E. of pulley

$$= \frac{1}{2}M_1\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}J\dot{\theta}^2$$

For any small displacement θ,

$x = r\theta$ (where x = displacement of mass m at any instant)

and $y = x/2$ (y = vertical displacement of pulley centre)

$$= \frac{1}{2}M_1\dot{x}^2 + \frac{1}{2}m(\dot{x}/2)^2 + \frac{1}{2}\cdot\frac{1}{2}mr^2\frac{\dot{x}^2}{r^2}$$

$$= \frac{1}{2}M_1\dot{x}^2 + \frac{3}{8}m\dot{x}^2$$

The potential energy is given by

$$V = \text{P.E.}$$

$$= \frac{1}{2}ky^2$$

$$= \frac{1}{2}k\left(\frac{1}{2}x\right)^2$$

$$= \frac{1}{8}kx^2$$

The energy of the system is constant.

So

$$\frac{d}{dt}(T + V) = 0$$

$$M_1\dot{x}\ddot{x} + \frac{3}{4}m\dot{x}\ddot{x} + \frac{1}{4}kx\dot{x} = 0$$

$$M_1\ddot{x} + \frac{3}{4}m\ddot{x} + \frac{1}{4}kx = 0$$

$$(M_1 + \frac{3}{4}m)\ddot{x} + \frac{1}{4}kx = 0$$

So natural frequency is

$$\omega_n = \sqrt{\frac{1}{4}\frac{k}{\left(M_1 + \frac{3m}{4}\right)}} \text{ rad/sec}$$

$$= \sqrt{\frac{k}{(4M_1 + 3m)}} \text{ rad/sec}$$

EXAMPLE 2.26. Find the natural frequency of the system shown in figure 2.31.

$k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = k = 1000$ N/m. (P.U., ME Civil, 94)

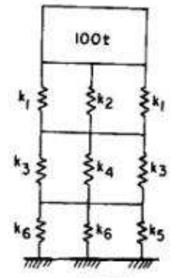


Fig. 2.31.

SOLUTION. Springs k_1, k_2 and k_1 are connected in parallel

Their equivalence, $k_{e1} = k_1 + k_2 + k_1 = 3k$

Similarly equivalence of k_3, k_4 and k_3

$$k_{e2} = 3k$$

Similarly $\qquad k_{e3} = 3k.$

Again k_{e1} k_{e2} and k_{e3} are in series, their equivalence, k_e.

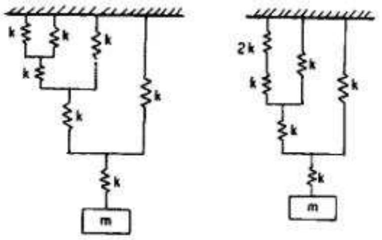
$$\frac{1}{k_e} = \frac{1}{3k} + \frac{1}{3k} + \frac{1}{3k} = k$$

Thus $\qquad k_e = k$

Natural frequency of the system

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{\frac{1000}{9.8} \times 100}} = 0.99 \text{ rad/sec}$$

EXAMPLE 2.27. *Find the natural frequency of the system shown in figure 2.32. $k = 2 \times 10^5$ N/m, $m = 20$ kg.* (P.U., Aero 93)



SOLUTION. The equivalence of spring in parallel $k_e = k_1 + k_2 + \ldots\ldots$

and in series $\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots\ldots$

See figure 2.33. $\quad k_e = \frac{13k}{21}$

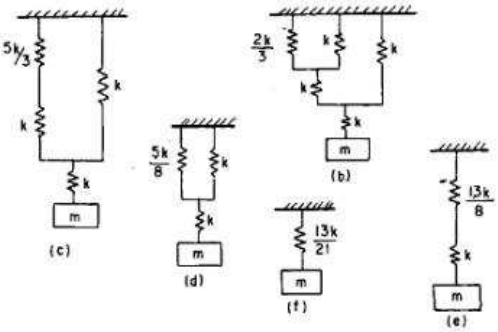


Fig. 2.33.

$$\text{So natural frequency } \omega_n = \sqrt{\frac{k_e}{m}} = \sqrt{\frac{2 \times 10^5 \times 13}{20 \times 21}}$$

$$= 78.68 \text{ rad/sec}$$

and $\qquad f_n = 12.5$ Hz

EXAMPLE 2.28. *A cylinder of diameter D and mass m floats vertically in a liquid of mass density ρ as shown in figure 2.34.*

It is depressed slightly and released. Find the period of its oscillation. What will be the frequency if salty liquid of specific gravity 1.2 is used ? (P.U., 90 ; M.D.U., 91)

SOLUTION. Let us assume x the displacement of the cylinder

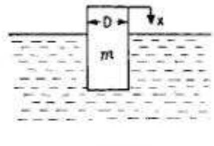
Restoring force = $\rho(Ax)$

According to Newton's law

$$m\ddot{x} = -\rho Ax$$

$$m\ddot{x} + \rho Ax = 0$$

$$\omega_n = \sqrt{\frac{\rho A}{m}}$$



Period of oscillation

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{\rho A}} \text{ sec}$$

Frequency when ρ is $1.2\,\rho$, $\omega_n = \sqrt{\frac{1.2\,\rho\,A}{m}}$ rad/sec where $A = \frac{\pi}{4}D^4$

EXAMPLE 2.29. *A pendulum consists of a stiff weightless rod of length l carrying a mass m on its end as shown in figure 2.35. Two springs each of stiffness k are attached to the rod at a distance a from the upper end. Determine the frequency for small oscillations.* (P.U. 88)

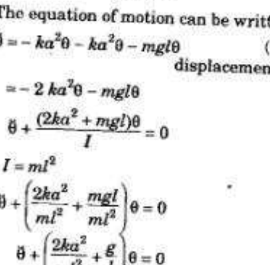


Fig. 2.35.

SOLUTION. The equation of motion can be written as

$$I\ddot{\theta} = -ka^2\theta - ka^2\theta - mgl\theta \qquad \text{(where } \theta = \text{ angular displacement of pendulum rod)}$$

$$= -2\,ka^2\theta - mgl\theta$$

$$\ddot{\theta} + \frac{(2ka^2 + mgl)\theta}{I} = 0$$

But $\qquad I = ml^2$

$$\ddot{\theta} + \left(\frac{2ka^2}{ml^2} + \frac{mgl}{ml^2}\right)\theta = 0$$

$$\ddot{\theta} + \left(\frac{2ka^2}{ml^2} + \frac{g}{l}\right)\theta = 0$$

$$\omega_n = \sqrt{\left(\frac{g}{l} + \frac{2ka^2}{ml^2}\right)} \text{ rad/sec}$$

EXAMPLE 2.30. *A torsion pendulum has to have a natural frequency f 5 Hz. What length of steel wire of diameter 2 mm should be used for his pendulum. The inertia of the mass fixed at the free end is 0.0098 kg-m². Take C = 0.85 × 10¹¹ N/m²* (P.U., 94)

SOLUTION. The natural frequency of pendulum is given as

$$\omega_n = \sqrt{\frac{k_t}{I}}$$

$$f_n = \frac{\omega_n}{2\pi}$$

$$\omega_n = f_n \cdot 2\pi = 5 \times 2\pi \text{ rad/sec} = 10\,\pi \text{ rad/sec}$$

$$d = .002 \text{ m}$$

We know that

$$\frac{T}{J} = \frac{G\theta}{l} \text{ or } \frac{T}{\theta} = \frac{GJ}{l} = k_t$$

$$J = \pi/32\,d^4$$

$$k_t = \frac{G.\pi/32\,d^4}{l} = \frac{.85 \times 10^{11} \times 3.14 \times (.002)^4}{32l} = \frac{3.336 \times 10^4}{l}$$

$$10\pi = \sqrt{\frac{G\,.\,\pi/32d^4}{lI}} = \sqrt{\frac{3.336 \times 10^4}{l \times .0098}}$$

$$l = 1.35 \text{ cm.}$$

EXAMPLE 2.31. *Derive the differential equation of motion for a spring controlled simple pendulum as shown in figure 2.36.*

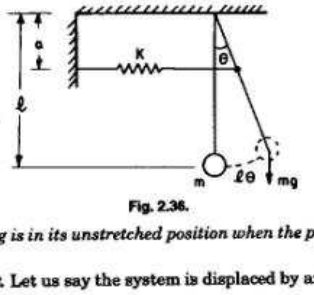


Fig. 2.36.

The spring is in its unstretched position when the pendulum rod is vertical. (P.U., 93)

SOLUTION. Let us say the system is displaced by an angle θ to the right.

We can write the equation as

$$I\ddot{\theta} = -mgl\,\theta - ka\,.\,a\theta$$

$$ml^2\ddot{\theta} + (mgl + ka^2)\theta = 0 \qquad (I = ml^2)$$

$$\ddot{\theta} + \left(\frac{mgl + ka^2}{ml^2}\right)\theta = 0$$

$$\ddot{\theta} + \left(\frac{g}{l} + \frac{ka^2}{ml^2}\right)\theta = 0$$

The frequency $\omega_n = \sqrt{\frac{g}{l} + \frac{ka^2}{ml^2}}$ rad/sec

***Example* 2.32**. *A 5 kg mass attached to the lower end of a spring, whose upper end is fixed, vibrates with a natural period of 0.45 sec. Determine the natural period when a 2.5 kg mass is attached to the mid point of the same spring with the upper and lower ends fixed.* (P.U., 92)

Solution. Natural frequency can be

$$\omega_n = \frac{2\pi}{T} = \frac{2\pi}{0.45}$$

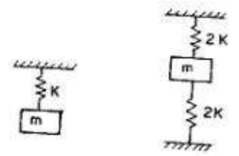
$$= 13.95 \text{ rad/sec}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 13.95$$

or $$(13.95)^2 = \frac{k}{m}$$

$$k = m(13.95)^2 = 5(13.95)^2$$

$$= 973 \text{ N/m}$$



First Postion

Second Position (2K and 2K in parallel)

when the spring is divided into two parts, its stiffness will be twice *i.e.* $2k$, But now two parts of the spring are in parallel, so $k_e = 4k$

Then $$\omega_n = \sqrt{\frac{4k}{m_1}} = \sqrt{\frac{4 \times 973}{2.5}} = 39.45 \text{ rad/sec}$$

Natural period $\frac{2\pi}{\omega_n} = \frac{2\pi}{39.45} = 0.159$ sec.

***Example* 2.33**. *A circular cylinder of mass m and radius r is connected by a spring of stiffness k on an inclined plane as shown in figure 2.37. If it is free to roll on the rough surface which is horizontal without slipping, determine its natural frequency.* (P.U., Aero 92)

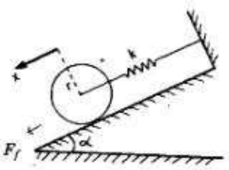


Fig. 2.37.

Solution. By applying Newton's second law, the equation of motion can be written as

$$m\ddot{x} = -kx + F_f$$

where $$F_f = \text{friction force}$$

$$m\ddot{x} + kx - F_f = 0 \qquad ...(1)$$

Also $$I\ddot{\theta} = -F_f \cdot r$$

$$\frac{1}{2} mr^2\ddot{\theta} = -F_f \cdot r$$

At any moment $(x = r\theta)$

$\therefore$ $$F_f = -\frac{1}{2} mr\,\ddot{\theta} \qquad (\ddot{x} = r\ddot{\theta})$$

$$= -\frac{1}{2} mr\frac{\ddot{x}}{r}$$

$$= -\frac{1}{2} m\ddot{x} \qquad ...(2)$$

Equation (1) can be written as

$$m\ddot{x} + kx + \frac{1}{2} m\ddot{x} = 0$$

$$\frac{3}{2} m\ddot{x} + kx = 0$$

$$\ddot{x} + \frac{2k}{3m} x = 0$$

So $$\omega_n = \sqrt{\frac{2k}{3m}} \text{ rad/sec}$$

***Example* 2.34**. *Consider the system shown in figure 2.38.*

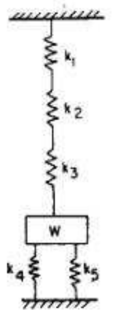


Fig. 2.38.

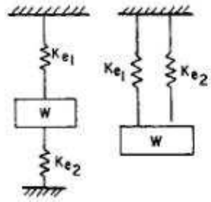


Fig. 2.39.

If $k_1 = 2\ kg_f/cm,\ k_2 = 1.5\ kg_f/cm$

$k_3 = 3.0\ kg_f/cm,\ k_4 = k_5 = .5\ kg_f/cm$

Find weight W if the system has a natural frequency of 10 Hz.

(P.U., Aero 91)

Solution. The springs k_1, k_2 and k_3 are in series, let k_{e1} be their equivalent stiffness

$$\frac{1}{k_{e1}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

$$= \frac{1}{2} + \frac{1}{1.5} + \frac{1}{3} = 1.499$$

$$k_{e1} = 0.667 \text{ kg}_f\text{/cm}$$

The two lower springs k_4 and k_5 are connected in parallel so their equivalence k_{e2}

$$k_{e2} = k_4 + k_5 = .5 + .5 = 1.0 \text{ kg}_f\text{/cm}$$

Again k_{e1} and k_{e2} are in parallel, so their equivalence k_e

$$k_e = k_{e1} + k_{e2} = .667 + 1.0 = 1.667 \text{ kg}_f\text{/cm}$$

$$f_n = 10 \text{ Hz}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_e}{m}}$$

$$f_n^2 = \frac{1}{4\pi^2}\frac{k_e}{m}$$

$$(10)^2 = \frac{1}{4\pi^2}\frac{1.667 \times 980}{W}$$

$$W = \frac{1.667 \times 980}{100 \times 4\pi^2} = 0.414 \text{ kg}$$

***Example* 2.35**. *The exhaust from a single cylinder four stroke diesel engine is connected to a silencer and the pressure therein is to be measured with a simple U tube manometer. Calculate the minimum length of a manometer tube so that the natural frequency of the oscillation of the liquid column will be 3.5 times slower than the frequency of the pressure fluctuations in the silencer for an engine speed of 600 r.p.m.*

(M.D.U., 91, 93)

Solution. We can find the frequency of pressure vibration

$$= \frac{N}{2} \times \text{speed of engine}$$

where $$N = \text{No. of cylinders}$$

$$\text{Frequency} = 1/2 \times 600 = 300 \text{ rpm.}$$

$$\omega = \frac{2\pi \times 300}{60} = 10\pi \text{ rad/sec}$$

Frequency in manometer

$$= \frac{10\pi}{3.5} = 8.97 \text{ rad/sec}$$

The kinetic and potential energies can be written as

$$\text{K.E.} = T = \frac{1}{2}\frac{\rho Al}{g}\dot{x}^2$$

$$V = \rho Ax^2$$

where ρ = weight per unit volume

A = cross section area

l = length of fluid column

Total energy $$T + V = \text{constant}$$

So $$\frac{d}{dt}(T + V) = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}\frac{\rho Al}{g}\dot{x}^2 + \rho Ax^2\right) = 0$$

$$\frac{\rho Al}{g}\dot{x}\ddot{x} + 2\rho Ax\dot{x} = 0$$

or $$\frac{l}{g}\ddot{x} + 2x = 0$$

$$\ddot{x} + 2x\frac{g}{l} = 0$$

Thus $$\omega_n = \sqrt{\frac{2g}{l}}$$

But we have $\omega_n = 8.97$ rad/sec

$$8.97 = \sqrt{\frac{2 \times 9.80}{l}}$$

$$l = 0.243 \text{ m}$$

***Example* 2.36**. *In figure 2.40 a thin semi-circular cylinder of mass M and radius R slides on the horizontal surface without slipping. Determine the natural frequency.*

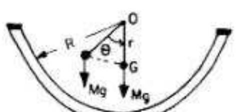


Fig. 2.40.

Solution. Let us say the distance between O and $C.G.$ is r.

We can find the equation of motion by equating the maximum potential energy of the system to its maximum

Let us assume the motion of the form

$$\theta = A \sin \omega t$$

where θ = small angular displacement

A = amplitude $\quad P.E. = Mgh \qquad (h = r - r\cos\theta)$

$$(\text{P.E.})_{max} = Mgr(1-\cos\theta) \qquad \text{(expanding } \cos\theta)$$

$$= Mgr\left[1-\left(1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}\cdots\right)\right]$$

$$= Mgr\,\frac{\theta^2}{2} \qquad \text{(leaving smaller terms)}$$

$$= Mgr \cdot \frac{A^2}{2} = MgA^2\frac{r}{2} \qquad (\theta_{max} = A)$$

$$(\text{K.E.})_{max} = \frac{1}{2}I_c\dot\theta^2 = \frac{1}{2}I_c\,\omega^2\,\theta^2 \qquad (\dot\theta = \omega\,\theta)$$

But
$$I_c = I_{C.G} + M(R-r)^2$$
$$= (I_0 - Mr^2) + M(R-r)^2$$
$$= (MR^2 - Mr^2) + M(R-r)^2$$
$$= M[R^2 - r^2 + (R-r)^2]$$
$$= M[R^2 - r^2 + R^2 + r^2 - 2Rr]$$
$$= 2MR(R-r)$$

Putting $\quad (\text{K.E.})_{max} = (\text{P.E.})_{max}$

$$\frac{\omega^2\theta_{max}^2}{2}\,2MR(R-r) = \frac{1}{2}MgrA^2$$

$$\omega^2 = \frac{1}{2}Mgr/MR(R-r) \qquad (\theta_{max} = A)$$

$$\omega = \sqrt{\frac{gr}{2R(R-r)}}$$

Example 2.37. *In figure 2.41, find the natural frequency of the system, if $m = 10$ kg attached at one end of weightless rod and $k = 1000$ N/m.*

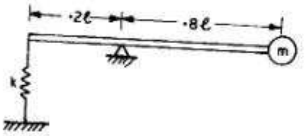


Fig. 2.41.

Solution. Let us assume very small angular displacement θ of the rod in the vertical plane.

Deflection of spring = $0.2\ l\theta$

Spring force $\quad = 0.2\,l\theta\,k$

The equation of motion can be written as

$$I\ddot\theta = -(0.2\,l)(0.2\,l\theta\,k) = -.04\,l^2k\theta$$

and
$$I = m(0.8\,l)^2 = 0.64\,ml^2$$

$$0.64\,ml^2\ddot\theta + .04\,l^2k\theta = 0$$

$$\ddot\theta + \frac{.04}{0.64}\frac{k\theta}{m} = 0$$

$$\ddot\theta + \frac{1}{16}\frac{k\theta}{m} = 0$$

So
$$\omega = \frac{1}{4}\sqrt{\frac{k}{m}} = \frac{1}{4}\sqrt{\frac{1000}{10}} = \frac{10}{4} = 2.5 \text{ rad/sec}$$

Example 2.38. *A bifiliar suspension consists of a thin cylindrical rod of mass m suspended symmetrically by two equal strings as shown in figure 2.42.*

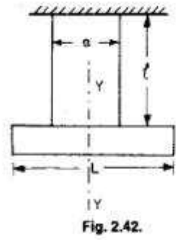


Fig. 2.42.

Find the period of motion for small angular oscillations of the rod about the vertical axis Y–Y. (P.U., 89)

Solution. The mass carried by each string in static position, $T = \dfrac{m \cdot g}{2}$

Let the rod is rotated by an angle θ in the horizontal plane and corresponding angular movement of the strings is ϕ in the vertical plane.

Then
$$l\phi = \frac{a}{2}\cdot\theta$$
$$\phi = \frac{a\theta}{2l}$$

The equation of motion can be written as

$$I\ddot\theta = -2T\sin\phi \cdot \frac{a}{2}$$

$$= -\frac{2mg}{2}\,\phi\cdot\frac{a}{2} = -\phi\,\frac{amg}{2}$$

$$= -\frac{amg}{2}\cdot\frac{a\theta}{2l} \qquad (\text{when } \phi \text{ is very small } \sin\phi = \phi)$$

$$= -\frac{a^2mg\theta}{4l}$$

But
$$I = \frac{mL^2}{12}$$

$$\ddot\theta + \frac{a^2m\theta g}{4lI} = 0$$

$$\ddot\theta + \frac{a^2m\theta \times 12\,g}{4l \cdot mL^2} = 0$$

$$\ddot\theta + \frac{3a^2g}{lL^2}\,\theta = 0$$

So
$$\omega_n = \sqrt{\frac{3a^2g}{lL^2}} = \frac{a}{L}\sqrt{\frac{3g}{l}}$$

Period of motion $= \dfrac{2\pi}{\omega_n} = \dfrac{2\pi L}{a}\sqrt{\dfrac{l}{3g}}$ sec.

Example 2.39. *The cylinder of mass m and radius r rolls without slipping on a circular surface of radius R. Determine the natural frequency for small oscillations about the lowest point. Refer figure 2.43.* (P.U., 91)

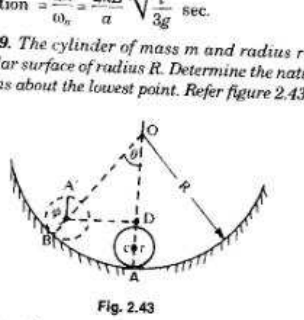


Fig. 2.43

Solution. When the cylinder is in the lowest position, the point A' coincides with A, then

$$\text{Arc } AB = \text{Arc } A'B$$
$$R\theta = r\,\phi$$

So $\quad \phi = \dfrac{R\,\theta}{r}$

The kinetic energy of the cylinder is due to translation and rotation as well.

The distance $OC = R - r$

So translational velocity of the cylinder centre = $(R-r)\dot\theta$

Rotational velocity of cylinder = $(\dot\phi - \dot\theta)$

$$\text{P.E.} = mgh = mg\,.\,CD = mg(OC - OD)$$
$$= mg[(R-r) - (R-r)\cos\theta] \qquad (\because\ OD = (R-r)\cos\theta)$$
$$= mg(R-r)(1-\cos\theta)$$

K.E. due to translation $= \dfrac{1}{2}m(R-r)^2\dot\theta^2$

K.E. due to rotation $\quad = \dfrac{1}{2}I(\dot\phi - \dot\theta)^2$

$$= \frac{1}{2}\cdot\frac{1}{2}mr^2\left(\frac{R\dot\theta}{r} - \dot\theta\right)^2 = \frac{1}{4}mr^2\left(\frac{R}{r} - 1\right)^2\dot\theta^2$$

The total energy of the system is written as

$$mg(R-r)(1-\cos\theta) + \frac{1}{2}m(R-r)^2\dot\theta^2 + \frac{1}{4}mr^2\left(\frac{R}{r}-1\right)^2\dot\theta^2$$

$$= \frac{3}{4}m(R-r)^2\dot\theta^2 + mg(R-r)(1-\cos\theta) = \text{constant}$$

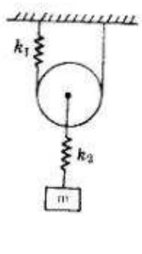
Differentiating the above equation with respect to time, we get

$$2\cdot\frac{3}{4}m(R-r)^2\,\dot\theta\,\ddot\theta + mg(R-r)\sin\theta\,\dot\theta = 0 \qquad (\sin\theta = \theta \text{ when } \theta \text{ is very small})$$

$$\frac{3}{2}(R-r)\ddot\theta + g\,\theta = 0$$

So
$$\omega_n = \sqrt{\frac{2g}{3(R-r)}} \text{ rad/sec}$$

Example 2.40. *Determine the natural frequency of the mass $m = 15$ kg as shown in figure 2.44, assuming that the cords do not stretch and slide over the pulley rim. Assume that the pulley has no mass.*



Given, $k_1 = 8 \times 10^3$ N/m

$k_2 = 6 \times 10^3$ N/m

Solution. At any instant the force in spring k_2 is twice to that of k_1 as mass m is acting at the centre of pulley. If we find equivalent stiffness of the system, the natural frequency can be determined.

Let us say P is the force in spring k_2 and so $P/2$ will be in k_1.

Displacement of mass m is given as

$$x_2 = \frac{P}{k_2}$$

So extension in spring k_1 is $x_1 = \frac{P}{2k_1}$

The mass moves half of x_1 *i.e.* $x_1/2$

Thus the displacement of mass m due to k_1 is given as

$$\frac{x_1}{2} = x_3 = \frac{P}{4k_1}$$

Total movement of mass m is given as

$$x = \frac{x_1}{2} + x_2$$

$$= \frac{P}{4\,k_1} + \frac{P}{k_2}$$

The equivalent stiffness of the system.

$$k_e = \frac{P}{x} = \frac{P}{\frac{P}{4k_1} + \frac{P}{k_2}} = \frac{1}{\frac{1}{4k_1} + \frac{1}{k_2}}$$

$$= \frac{4k_1k_2}{k_2 + 4k_1}$$

The natural frequency of mass m is given as

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_e}{m}} = \frac{1}{2\pi}\sqrt{\frac{4k_1k_2}{(k_2 + 4k_1)m}}$$

$$= \frac{1}{\pi}\sqrt{\frac{k_1k_2}{(k_2 + 4k_1)m}}$$

$$= \frac{1}{\pi}\sqrt{\frac{8 \times 10^3 \times 6 \times 10^3}{(6 \times 10^3 + 4 \times 8 \times 10^3)15}} = 2.9 \text{ Hz}$$

EXAMPLE 2.41. *A homogeneous soild cylinder of length L, cross ectional area A and specific gravity S (S < 1.0) is floating in water with 's axis vertical. Neglecting any accompanying motion of water, deter- ine the differential equation of motion and the period of oscillation of e cylinder if it is depressed slightly and then released.*

(A.M.I.E., 94)

SOLUTION. Refer figure 2.45.

The volume of the cylinder = $A.L$

Mass of cylinder = $\frac{A.L.S.}{g}$

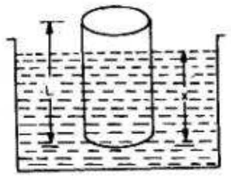


Fig. 2.45.

Let x be the depth up to which the cylinder is submerged in water. Then weight of water displaced = $A \cdot x$ and this provides the restoring force. When the cylinder is depressed slightly, the mass will be subjected to acceleration $\ddot{x}$, then equation of motion

$$m\ddot{x} + kx = 0$$

$$\frac{SAL}{g}\ddot{x} + Ax = 0$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{A.g}{SAL}} = \sqrt{\frac{g}{S.L}} \text{ rad sec}$$

Time period, $= 2\pi\sqrt{\frac{S.L}{g}}$ sec.

EXAMPLE 2.42. *A steel shaft of length L and diameter d is used as a torsion spring for the wheel of a light automobile as shown in figure 2.46. Mass of the wheel and tyre assembly is m and its radius of gyration about its axle is r. Determine the natural frequency of the system with the wheel locked to the arm. How will the natural frequency change if the wheel is not locked to the arm and free to rotate about its axle ?*

Solution. The moment of inertia of wheel and tyre about shaft axis is given as

$$I = mr^2 + ma^2 = m\,(r^2 + a^2)$$

...(when the wheel is not locked to arm)

and $\quad I = ma^2 \quad$ ...(when the wheel is locked to arm)

Torsional stiffness of shaft is given by

$$= \frac{G.J}{L} = \frac{G}{L} \ldots$$

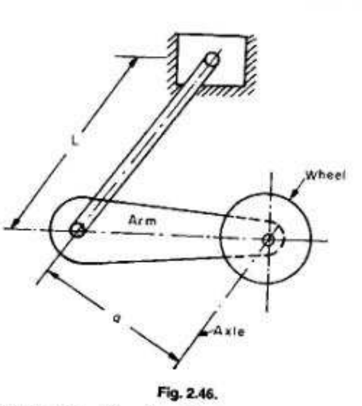


Fig. 2.46.

We know that equation of motion is given by

$$I \cdot \ddot{\theta} + K_t \cdot \theta = 0 \quad \text{or} \quad \ddot{\theta} + \frac{K_t}{I}\theta = 0$$

and the natural frequency ω is given by

$$\omega = \sqrt{\frac{K_t}{I}}$$

Natural frequency when the wheel is not locked to arm

$$\omega_1 = \sqrt{\frac{G}{L}\frac{\pi}{32}d^4/m\,(r^2 + a^2)} = \sqrt{\frac{G\,\pi\,d^4}{32\,L\,m\,(a^2 + r^2)}}$$

Natural frequency when the wheel is locked to arm

$$\omega_2 = \sqrt{\frac{G}{L}\left(\frac{\pi}{32}d^4\right)/ma^2} = \sqrt{\frac{G\,.\,\pi\,.\,d^4}{32\,L\,ma^2}}$$

EXAMPLE 2.43. *A governor is shown in figure 2.47 schematically. The two links which carry the balls of mass m each are connected by a spring of stiffness k and has a natural length of 2e. Find out the expression for the inclination of the links with vertical when the governor rotates at a speed ω.*

SOLUTION. Since the spring expands about the spindle on both sides of it, so it is assumed that the spring is fixed at the centre of the coil. Thus the spring will be equivalent to two springs. Hence, equivalent stiffness of spring will be 2 k. Various forces acting on the arms and balls are shown in figure 2.48 (b).

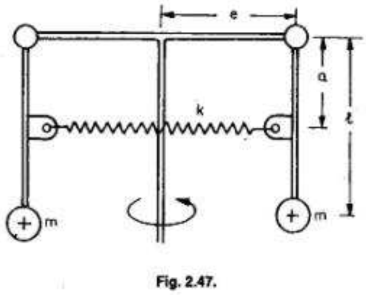


Fig. 2.47.

When the governor rotates with angular speed ω, let the arms be inclined to the vertical at an angle θ. For equilibrium, taking moment of all forces about 0, we get

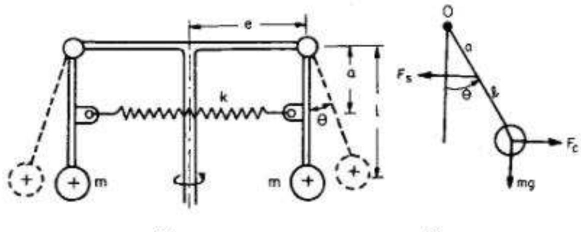
$$F_C \cdot l\cos\theta = m \cdot g \cdot l\sin\theta + Fs \cdot a\cos\theta \qquad ...(i)$$

where F_C = Centrifugal force = $m\omega^2 r$

and $\quad F_S$ = Spring force = stiffness × deflection

$$= 2\,k \times 2a\sin\theta$$

$$= 4ak\sin\theta$$



(a) (b)

Fig. 2.48.

Substituting the values in eqn. (*i*), we get

$$m\omega^2 r \cdot l\cos\theta = m \cdot g \cdot l\sin\theta + 4a^2 k\sin\theta\cos\theta$$

For small angles and neglecting obliquity effects,

$$\sin\theta \simeq \theta, \quad \cos\theta = 1$$

Equation (*i*) can be written as

$$m\omega^2 rl = m\,.\,g\,.\,l\,.\,\theta + 4a^2 k\,.\,\theta$$

$$m\omega^2 r\,.\,l = (mgl + 4a^2 k)\,\theta \qquad ...(ii)$$

Radius of rotation r can be written as

$$r = e + l\,.\,\sin\theta$$

$$= e + l\,.\,\theta \qquad (\text{since } \sin\theta \simeq \theta)$$

Equation (*ii*) can be written as

$$m\omega^2 (e + l\,.\,\theta)\,.\,l = (mgl + 4a^2 k)\,\theta$$

$$e\,m\omega^2 l + m\omega^2 l^2\,.\,\theta = (mgl + 4a^2 k)\,\theta$$

or

$$\theta = \frac{m\omega^2 el}{(mgl + 4a^2 k - m\omega^2 l^2)}$$

Problems

1. A car having a mass of 1500 kg deflects its spring 3 cm under its own load. Find the natural frequency of car in vertical direction.
2. The natural frequency of a spring-mass system is 15 Hz. An extra 3 kg mass is coupled to its mass and natural frequency reduces by 3 Hz. Find the mass and stiffness of the system.
3. Find the natural frequency of the system shown in figure 2.1 P. With and without spring at the mid span of the elastic beam.

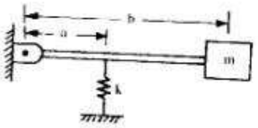


Fig. 2.1 P.

4. A spring mass system has a period 0.25 sec. What will be the new period if the spring constant is increased by 50% ?
5. A spring mass system has a natural frequency of 12 Hz. When the spring constant is reduced by 800 N/m, the frequency is changed by 50%. Determine the mass and spring constant of the original system.
6. Determine the natural frequency of vibration of the system shown in figure 2.2 P. Assume the bar AB as weightless and rigid.
7. A torsion pendulum has to have a natural frequency of 6 Hz. What length of a steel wire of diameter 2 mm should be used for this pendulum. The inertia of the mass fixed at the free end is $0.0098\ \text{kg-m}^2$. Take $G = 0.83 \times 10^{11}\ \text{N/m}^2$

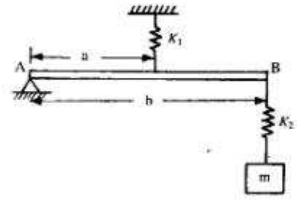
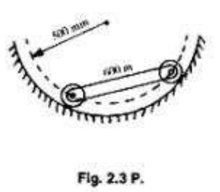


Fig. 2.2 P.

8. A bar 600 mm long rolls on wheels of negligible weight on a circular path with a radius of 500 mm as shown in figure 2.3 P. Determine the natural frequency of oscillation for the bar if it moves in the vertical plane when it is displaced slightly from its equilibrium position.



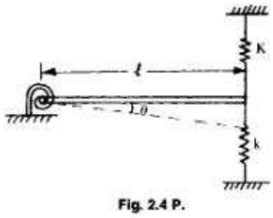
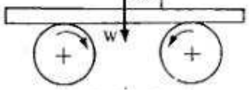


Fig. 2.3 P. Fig. 2.4 P.

9. The uniform stiff rod is restrained to move vertically by both linear and torsional springs as shown in figure 2.4 P. Calculate the frequency of the vertical oscillation of the rod.
10. Using Rayleigh's method find the natural frequency of a cantilever beam due to its weight only.
11. To measure the radius of curvature of a surface a sphere of radius 6 cm is placed on it. For small displacements along the curve the sphere executes 50 vibrations in 70 seconds. Calculate the radius of curvature.
12. For determining the coefficient of dry friction, the device shown in figure 2.5 P is used. A bar rests on two equal disks rotating with equal speeds in opposite directions. If the bar is displaced from the position of equi-



librium and released, it performs harmonic oscillations by moving back and forth about its axis. Determine an expression for the coefficient of dry friction in terms of the natural frequency.

13. A weightless steel cantilever, 450 mm long, 20 mm wide and 10 mm deep, supports a mass of 25 kg at its free end. The deflection of free end is further constrained by a spring 5 N/mm stiffers. Determine the natural frequency of the system.

$$E = 2.11 \times 10^{11}\ \text{N/m}^2 \qquad (\text{P.U., 99})$$

14. Figure 2.6 P shows a rectangular block of mass M resting on the top of a semi-cylindrical surface. If the block is slightly tipped at one end find the frequency of oscillations.

$$\left[\textbf{Ans. } n = \frac{1}{2\pi}\sqrt{\frac{12\,(r - d/2)\,g}{(l^2 + 4d^2)}}\right]$$

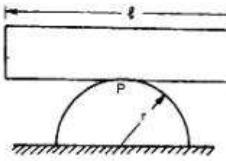


Fig. 2.6 P.

15. A round rod of diameter 10 mm. is bent into a right angle and it is used to support a weight $W = 10$ kg as shown in figure 2.7 P. Calculate the natural frequency of the system; neglect the weight of the rod.

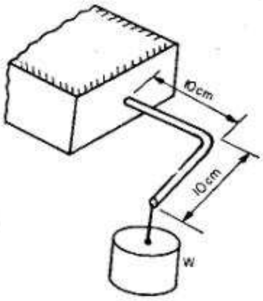


Fig. 2.7 P.

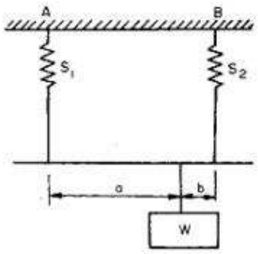
Given $E = 2.11 \times 10^6\ \text{kg/cm}^2$

$G = 0.843\ \text{kg/cm}^2$

16. A steel shaft 6 cm diameter and 50 cm long fixed at one end carries a flywheel of weight 1000 kgf and radius of gyration 30 cm at its free end. Find the frequency of free longitudinal transverse, and torsional vibrations.

$$E = 2 \times 10^6\ \text{kgf/cm}^2,\ C = 3.8 \times 10^6\ \text{kgf/cm}^2. \qquad (\text{K.U.})$$

17. A hollow shaft of 15 cm external diameter, and 10 cm internal diameter 150 cm long has one of its ends fixed. The other end carries a disc of 600 kgf weight. Determine the frequency of longitudinal and transverse vibrations. Assume $E = 2 \times 10^6\ \text{kgf/cm}^2$. (K.U.)
18. A shaft supported freely at the ends has a mass of 100 kg placed 25 cm from one end. Find the frequency of the natural transverse vibration if the length of the shaft is 75 cm, $E = 200\ \text{GN/m}^2$ and shaft diameter is 4 cm. (K.U.)
19. A load W is vertically suspended on two springs of constants S_1 and S_2 as shown in figure 2.8 P. Determine the resultant spring constant and the frequency of the load. (K.U.)



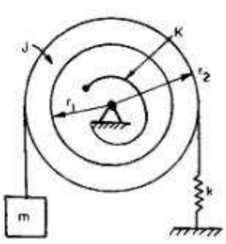


Fig. 2.8 P. Fig. 2.9 P.

20. An integral pulley shown in the above figure 2.9 P is restrained in its movement about its own axis by a torsional spring of stiffness K and a linear spring of stiffness k. A load of mass m is hung from the smaller pulley by means of an inextensible string. Using D'Alembert's principle derive the equation of motion for small oscillations and determine the natural frequency. Take J as mass moment of inertia of the pulley about its axis. (U.P.S.C., 92)
21. The propeller of a ship, of weight 10^5 N and polar mass moment of inertia $10{,}000\ \text{kg-m}^2$, is connected to the engine through a hollow stepped steel propeller shaft, as shown in figure 2.10 P. Assuming that water provides a viscous damping ratio of 0.1, determine the torsional vibratory response of the propeller when the engine induces a harmonic angular displacement of 0.05 sin 314t rad at the base (point A) of the propeller shaft. Modulus of rigidity for steel = $8 \times 10^6\ \text{N/cm}^2$

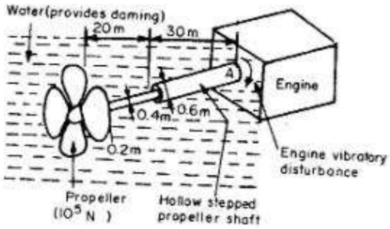
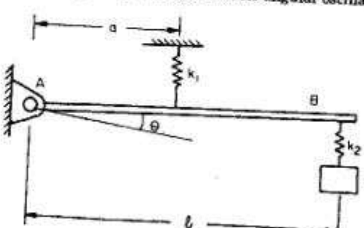


Fig. 2.10 P.

22. A rigid massless bar of length l is hinged a its left end and carries a spring K_2 with mass m at its right end. The bar is also supported by a spring K_1 at a distance a from the left hinge, as shown in figure 2.11 P. Determine the natural frequency of the bar in Hz for angular oscillation.



Take $K_1 = 1000$ N/m
$K_2 = 2000$ N/m
$l = 3$ m

Fig. 2.11 P.

23. The uniform slender rod AB of length 1 and weight w is attached to a torsional spring CD that has a torsional stiffness K_t. Coil spring of stiffness K are attached to the ends of the rod AB as shown in figure 2.12 P. Determine the undamped natural frequency in Hz for small angular oscillation of the rod in terms of system parameters.

(Roorkee Uni. 1999-2000)

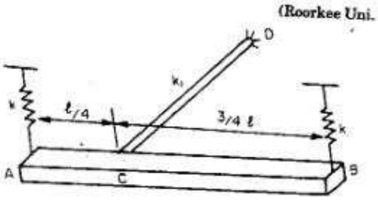


Fig. 2.12 P.

23. (a) A weight W is attached to the end of a small cantilever beam of length l by a cable of stiffness k shown in figure 2.13 P. If $W = 50$ N, $EI = 1000$ N-m^2 and $k = 2000$ N/m. Determine the length l of the beam that will make the system's undamped natural frequency $f_n = 2$ Hz. Consider the mass of both cable and beam as negligible compared with the mass of the weight W.

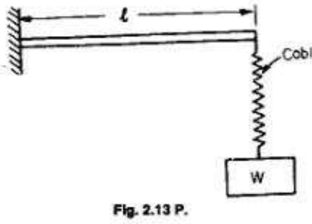


Fig. 2.13 P.

(b) Calculate the natural frequency in sideway, (in X-direction) for the frame of figure 2.14 P and also the period of vibration.

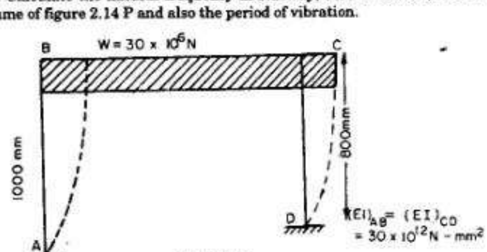
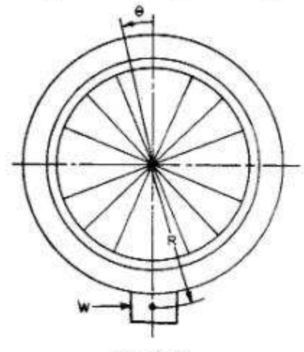


Fig. 2.14 P.

(c) A bicycle wheel and tire are supported so that they are free to rotate about their centroidal axis through the hub of the wheel. A small weight W is taped to the tire as shown in figure 2.15 P at a distance R from the axis of rotation. When this weight is displaced slightly from the vertical axis shown, the wheel is observed to oscillate 3 cycles every 10 seconds. If $R = \phi.28$ m and $W = 3.34$ N, determine the centroidal mass moment of Inertia I of the wheel and tire.

(Roorkee Uni., 94-95)



3

Free Damped Vibration

3.1. INTRODUCTION

Damping is the resistance offered by a body to the motion of a vibratory system. The resistance may be applied by a liquid or solid internally or externally. For example, the motion of the car wheels in water is resisted by the water itself, the wheels of car are put to some resistance/friction because of road surface while moving on it. Because of this resistance vibrations die out over a few cycles of motion.

Mass, stiffness and damping are the main characteristics of a vibratory system. Out of these three the first two are the inherent properties of the system. If the value of damping is small in mechanical systems, it will be having negligible influence on the natural frequency of the system. The vibratory system has some energy which is dissipated during the motion. At the start of the vibratory motion the amplitude of vibration is maximum which goes on decreasing and finally is lost completely with the passage of time. The rate of decreasing the amplitude depends upon the amount of damping.

The main advantage of providing damping in mechanical systems is just to control the amplitude of vibration so that the failure occurring because of resonance may be avoided.

In this chapter various types of damping is discussed. Differential equations of motion for a single degree of freedom system are derived and in the end some problems are solved.

3.2. TYPES OF DAMPING

There are mainly four types of damping used in mechanical systems :

(a) Viscous damping

(b) Coulomb damping

(c) Structural damping

(d) Non-linear, Slip or interfacial damping.

(a) Viscous damping

When the system is allowed to vibrate in a viscous medium, the damping is called as viscous. Viscosity is the property of a fluid by virtue of which it offers resistance to the motion of one layer over the adjacent one.

It can be explained by figure 3.1 where two plates are separated by fluid film of thickness t. The upper plate is allowed to move parallel

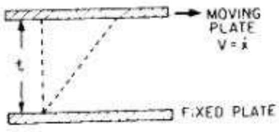


Fig. 3.1. Viscosity.

to the fixed plate with a velocity $\dot{x}$. The net force F required for maintaining the velocity $\dot{x}$ of the plate is expressed as

$$F = \frac{\mu A}{t}\dot{x} \qquad ...(3.2.1)$$

where A = area of the plate

t = thickness of the fluid film

μ = coefficient of absolute viscosity of the film

The force F can also be written as

$$F = c\dot{x}$$

So $$c = \mu A/t \qquad ...(3.2.2)$$

where c is viscous damping coefficient. The main components of a viscous damper are cylinder, piston and viscous fluid.

There is clearance between the cylinder walls and the piston. More the clearance, more will be the velocity of the piston in the viscous fluid and it will offer less value of viscous damping coefficient. The basic system is shown in figure 3.2. The damping force is opposite to the direction of velocity.

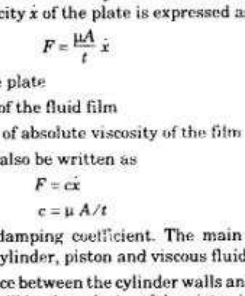


Fig. 3.2.

The damping resistance depends on the pressure difference on the both sides of the piston in the viscous medium. Figure 3.3 shows the example of free vibrations with viscous damping. The equation of motion for the system can be written as

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad ...(3.2.3)$$

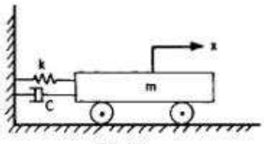


Fig. 3.3.

The differential equations used in the analysis are simple and can be solved easily. This type of damping is widely used in engineering.

Energy Dissipation in Viscous Damping

For a vibratory body some amount of energy is dissipated because of damping. This energy dissipation can be per cycle. For a viscously damped system the force F is expressed as

$$F = c\dot{x} = c\frac{dx}{dt}$$

where $\dot{x} = \dfrac{dx}{dt}$

Work done $dW = F \cdot dx = \left(c\dfrac{dx}{dt}\right) \cdot dx$

The rate of change of work per cycle *i.e.*

$$\text{Energy dissipated } \Delta E = \int_0^{2\pi/\omega}\left(F \cdot \frac{dx}{dt}\right)dt = \int_0^{2\pi/\omega} c\left(\frac{dx}{dt}\,\frac{dx}{dt}\right)dt$$

$$\Delta E = \int_0^{2\pi/\omega} c\left(\frac{dx}{dt}\right)^2 \cdot dt \qquad \text{...(3.2.4)}$$

Let us assume the simple harmonic motion of the type

$$x = A \sin \omega t$$

$$\left(\frac{dx}{dt}\right)^2 = \omega^2 A^2 \cos^2 \omega t$$

The equation (3.2.4) can be written as

$$\Delta E = \int_0^{2\pi/\omega} c\omega^2 A^2 \left(\frac{1 + \cos 2\omega t}{2}\right)dt$$

$$\Delta E = \pi c \omega A^2 \qquad \text{...(3.2.5)}$$

From the above equation it is clear that the energy dissipation per cycle is proportional to the square of the amplitude of motion.

The total energy of a vibrating system can be either maximum of its potential or kinetic energy. The maximum kinetic energy of the system can be written as

$$E = (K.E.)_{max} = \frac{1}{2} m\dot{x}^2_{max}$$

$$= \frac{1}{2} m\omega^2 A^2 \qquad \text{...(3.2.6)}$$

We can find the ratio of ΔE to E.

This ratio is known as specific damping capacity of the system.

Thus specific damping ratio

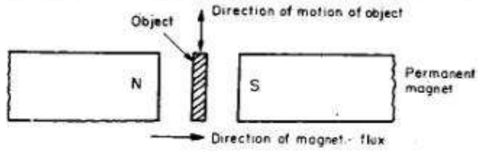
$$\beta = \frac{\Delta E}{E} = \frac{\pi c \omega A^2}{\frac{1}{2} m\omega^2 A^2}$$

$$= 2\left(\frac{c}{2m}\right)\left(\frac{2\pi}{\omega}\right) = \frac{2c\pi}{m\omega} \qquad \text{...(3.2.7)}$$

which is a constant quantity.

This relation is very useful in the design of vibratory instruments. The damping materials are rated by their damping capacity β.

Eddy Current Damping. This type of damping is based on the principle of generation of eddy current which provides the damping. If a non-ferrous conducting object (such as a plate, rod, etc.) is moved in a direction perpendicular to the lines of magnetic flux which is produced by a permanent magnet, then as the object moves, current is induced in the object. The current is proportional to the velocity of the object assuming that the magnetic flux and the dimensions of the body remain constant. This current is the induced eddy current and sets up a magnetic field so as to oppose the original magnetic field that has induced it. This provides a resistance to the motion of the object in the magnetic field. The resisting or damping force produced by this flux field from eddy currents is also proportional to the velocity. This is a mechanical damping of viscous type. This type of damping is used in vibrometers and in some vibration control systems. The magnetic field can be set up by using a permanent magnet or a coil wound on a magnetic core with current passing through it. Refer figure 3.3 (a).



(*b*) **Coulomb Damping**

When one body is allowed to slide over the other, the surface of one body offers some resistance to the movement of the other body on it. This resisting force is called force of friction. Thus force of friction arises only because of relative movement between the two surfaces. Some amount of energy is wasted in overcoming this friction as the surfaces are dry.

So it is sometimes known as dry friction. The general expression for coulomb damping is

$$F = \mu R_N \qquad \text{...(3.2.8)}$$

where μ is the coefficient of friction and R_N is the normal reaction. Friction force F is proportional to the normal reaction R_N on the mating surface. The system is shown in figure 3.4.

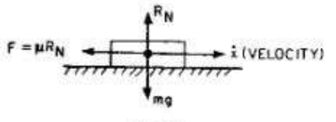
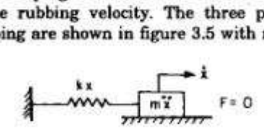
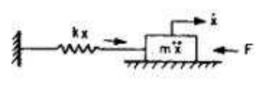


Fig. 3.4

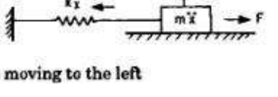
The friction force acts in a direction opposite to the direction of velocity. The damping resistance is almost constant and does not depend on the rubbing velocity. The three possible conditions of coulomb damping are shown in figure 3.5 with mathematical expressions.



(*a*) Equilibrium position $\qquad m\ddot{x} + kx = 0$



(*b*) Mass is moving to the right $\qquad m\ddot{x} + kx + F = 0$



(*c*) Mass is moving to the left $\qquad m\ddot{x} + kx - F = 0$

Let us consider the leftward movement of the body the equation for which can be written as

$$m\ddot{x} + kx = F \qquad \text{...(3.2.9)}$$

The solution of the above equation can be written as

$$x = B\cos\sqrt{\frac{k}{m}}\,t + D\sin\sqrt{\frac{k}{m}}\,t + \frac{F}{k} \qquad \text{...(3.2.10)}$$

where $\omega = \sqrt{k/m}$.

Let us assume the motion characteristics of the system as

$$x = x_0 \quad \text{at } t = 0$$
$$\dot{x} = 0 \quad \text{at } t = 0$$

we get, $\quad B = \left(x_0 - \dfrac{F}{k}\right), D = 0$

So equation (3.2.10) can be written as

$$x = (x_0 - F/k)\cos\sqrt{\frac{k}{m}}\,t + \frac{F}{k} \qquad \text{...(3.2.11)}$$

This solution holds good for half the cycle. When $t = \pi/\omega$, half the cycle is complete. So displacement for half the cycle can be obtained from the above equation.

$$x = (x_0 - F/k)\cos\pi + F/k$$
$$= -(x_0 - F/k) + F/k$$
$$= -\left(x_0 - \frac{2F}{k}\right) \qquad \text{...(3.2.12)}$$

This is the amplitude for left extreme position of the body. It is clear that the initial displacement x_0 is reduced by $2F/k$. In the next half cycle when the body moves to the right the initial displacement will be reduced by $2F/k$. So in one complete cycle the amplitude reduces by $4F/k$. The amplitude decay for coulomb damping is shown in figure 3.6. The natural frequency of the system remains unchanged in coulomb damping.

(*c*) **Structural damping**

It is the inherent characteristics of the material and the resistance is offered by the elastic properties from within the body. There is intermolecular friction in the structure which opposes its movement. The magnitude of this damping is very small as compared to other dampings. Experiments show that for elastic materials for loading and unloading conditions a loop is formed on stress-strain curve. This loop is called hysteresis loop. Refer figure 3.6 (a). The area of this loop is the amount of energy dissipated in one cycle during vibrations. This type

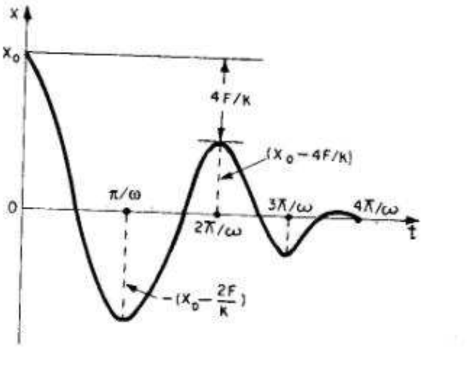


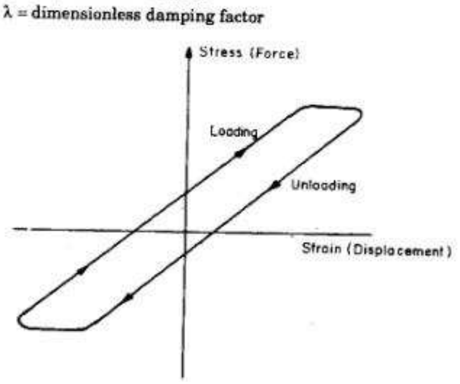
Fig. 3.5. Amplitude decay in coulomb damping

of damping is sometimes known as hysteresis damping. It has been found that it is not the function of frequency, but is approximately proportional to the square of amplitude of vibration. It is proportional to the stiffness of the system. The energy loss per cycle is expressed as

$$E = \pi k \lambda A^2 \qquad \text{...(3.2.13)}$$

where A = amplitude of vibration

λ = dimensionless damping factor



In the above expression λ is related to the property of the material and $k\lambda$ is representing the shape, size and property of the material.

Equation (3.2.13) can be written as

$$E = \beta A^2 \qquad \text{...(3.2.14)}$$

where $\beta = \pi k \lambda$

If this energy is treated equal to the energy dissipation by viscous damping

$$\pi c \omega A^2 = \beta A^2$$

$$c = \frac{\beta}{\pi \omega}$$

The damping force can be written as

$$F = c\dot{x}$$

$$= \frac{\beta \dot{x}}{\pi \omega} = \frac{\beta \omega A}{\pi \omega} = \frac{\beta A}{\pi} \qquad \text{...(3.2.15)}$$

The amplitude decay is found to be exponential in nature.

When force (F) is plotted against displacement (x), then a close loop as shown in figure 3.6 (b) is formed. The area of the loop denotes the energy dissipated by the damper in one cycle of motion which is given by eqn. (3.2.13).

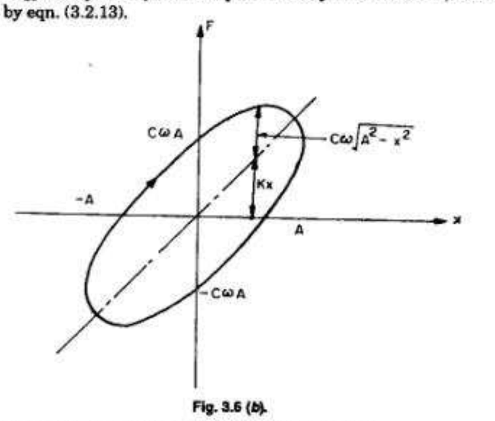


Fig. 3.6 (b).

(d) **Non-Linear, Slip or Interfacial Damping**

The machine elements are connected through various types of

which causes dissipation of vibrational energy when the interface of machine elements or parts in contact are under fluctuating loads. The amount of damping depends on surface roughness of contacting parts, contact pressure and the amplitude of vibration. The energy dissipated per cycle depends upon the coefficient of friction, the pressure at the contacting parts and amplitudes. There is an optimum value of pressure for which the energy dissipated is maximum. This value is different for different amplitudes. Larger the energy dissipation, larger is the effective damping in tbe system. Refer figure 3.6 (c).

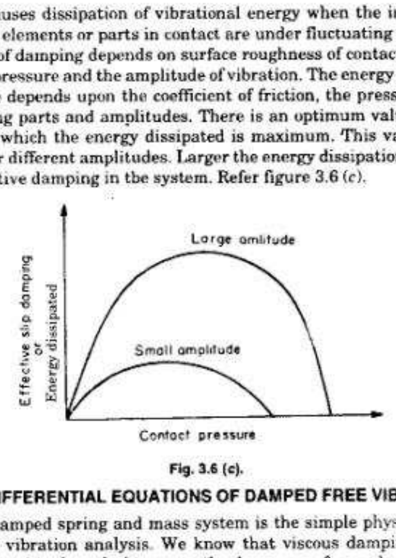


Fig. 3.6 (c).

3.3. DIFFERENTIAL EQUATIONS OF DAMPED FREE VIBRATION

A damped spring and mass system is the simple physical model used for vibration analysis. We know that viscous damping force is proportional to the velocity across the damper, so for analysis purpose we are using viscous damping here. Consider a mass m attached from one end of the spring k, the other of which is fixed. A damper is also provided as shown in figure 3.7.

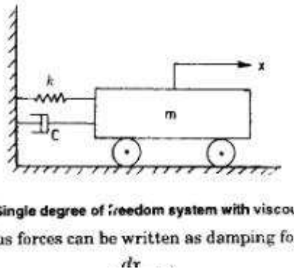


Fig. 3.7. Single degree of freedom system with viscous damping.

The various forces can be written as damping force

dx

accelerating force $\frac{md^2x}{dt^2} = m\ddot{x}$

and spring force kx

Thus the equation of motion can be written as

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad \text{...(3.3.1)}$$

This is called the characteristics equation of the system. This is the differential equation of second order in x.

Assuming a solution of the form

$x = e^{ut}$ where u is a constant to be determined.

$$\text{Then} \quad \left.\begin{aligned} \dot{x} &= ue^{ut} \\ \ddot{x} &= u^2 e^{ut} \end{aligned}\right\} \qquad \text{...(3.3.2)}$$

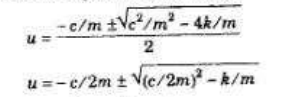


Fig. 3.8. Free body diagram.

Substituting the values of $\dot{x}$ and $\ddot{x}$ in equation (3.3.1) from the above equation, we get

$$mu^2e^{ut} + cue^{ut} + ke^{ut} = 0$$

or

$$u^2 + \frac{cu}{m} + \frac{k}{m} = 0 \qquad \text{...(3.3.3)}$$

Solving the above equation for u, we get

$$u = \frac{-c/m \pm \sqrt{c^2/m^2 - 4k/m}}{2}$$

$$u = -c/2m \pm \sqrt{(c/2m)^2 - k/m} \qquad \text{...(3.3.4)}$$

The two roots can be written as

$$\left.\begin{aligned} u_1 &= -c/2m + \sqrt{(c/2m)^2 - k/m} \\ u_2 &= -c/2m - \sqrt{(c/2m)^2 - k/m} \end{aligned}\right\} \qquad \text{...(3.3.5)}$$

and

Now the solution of equation (3.3.1) can be written as

$$x = A_1 e^{u_1 t} + A_2 e^{u_2 t} \qquad \text{...(3.3.6)}$$

where A_1 and A_2 are two arbitrary constants and u_1 and u_2 its two roots.

This equation can be written as

$$x = A_1 e^{\{-c/2m + \sqrt{(c/2m)^2 - k/m}\}t} + A_2 e^{\{-c/2m - \sqrt{(c/2m)^2 - k/m}\}t} \qquad \text{...(3.3.7)}$$

Critical Damping Constant and Damping Ratio

The critical damping c_c is defined as the value of damping coefficient c for which the mathematical term $(c/2m)^2 - \frac{k}{m}$ in equation (3.3.7) is equal to zero *i.e.*

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} = 0$$

or
$$\frac{c_c}{2m} = \sqrt{\frac{k}{m}}$$

or
$$c_c = 2m\sqrt{k/m} = 2m\omega \qquad ...(3.3.8)$$

The ratio of c to c_c is termed as damping ratio. It is indicated by the symbol ε. Mathematically, it can be written as

$$\varepsilon = \frac{c}{c_c} \qquad ...(3.3.9)$$

Let us consider the term $\frac{c}{2m}$ of equation (3.3.7)

$$\frac{c}{2m} = \left(\frac{c}{c_c}\right)\frac{c_c}{2m} = \varepsilon\,\omega \qquad ...(3.3.10)$$

So equation (3.3.7) can be written with the help of equation (3.3.10) as

$$x = A_1 e^{(-\varepsilon + \sqrt{\varepsilon^2 - 1})\omega t} + A_2 e^{(-\varepsilon - \sqrt{\varepsilon^2 - 1})\omega t} \qquad ...(3.3.11)$$

The nature of the system depends upon the value of damping. Depending upon the value of damping ratio ε, the damped systems are put into three categories which are as follows :

(A) Over-damped system

When the value of damping ratio ε in equation (3.3.11) is more than one *i.e.* $\varepsilon > 1$, the system is known as over-damped one. This motion is called aperiodic. When $t = 0$ the displacement is the sum of A_1 and A_2 *i.e.*

$$x = A_1 + A_2$$

The values of A_1 and A_2 are negative. The value of displacement x goes on decreasing with time. The characteristics of this type of motion are shown in figure 3.9. The system is non-vibratory in nature. The system comes to equilibrium in an exponential manner. Once the system is disturbed, it will take infinite time to come back to equilibrium position.

The values of constants A_1 and A_2 can be determined from initial

x_0 and $\dot{x}_0$ respectively, then we can have two equations and two unknowns.

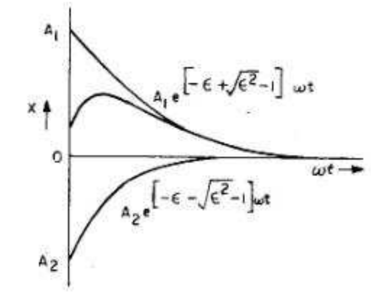


Fig. 3.9. Displacement-time curve for overdamped system.

Putting the conditions to equation (3.3.11), we get

$$x_0 = A_1 + A_2$$

and
$$\dot{x}_0 = \left(\frac{dx}{dt}\right)_{t=0} = A_1[-\varepsilon + \sqrt{\varepsilon^2 - 1}]\omega + A_2[-\varepsilon - \sqrt{\varepsilon^2 - 1}]\omega \qquad ...(3.3.12)$$

So from equation (3.3.12) the values of A_1 and A_2 can be determined. The displacement time plots for overdamping and critical damping showing the variation of displacement (x) with the angle turned ($\omega \cdot t$) for different values of $\xi \geq 1$ can be seen from figure 3.9 (a).

(B) Critically Damped System

The system is said to be critically damped when $\varepsilon = 1$ *i.e.*

$$\frac{c}{2m} = \sqrt{\frac{k}{m}}$$

The two roots of equation (3.3.6) u_1 and u_2 are equal to each other,

$$u_1 = u_2 = -\varepsilon\,\omega = -\omega$$

So the approximate solution of equation (3.3.3) may be written as

$$x = A_1 e^{-\omega t} + A_2 t e^{-\omega t} = (A_1 + A_2 t)e^{-\omega t} \qquad ...(3.3.13)$$

In the above equation A_1 and A_2 are arbitrary constants whose values can be determined from initial conditions as in case of over-

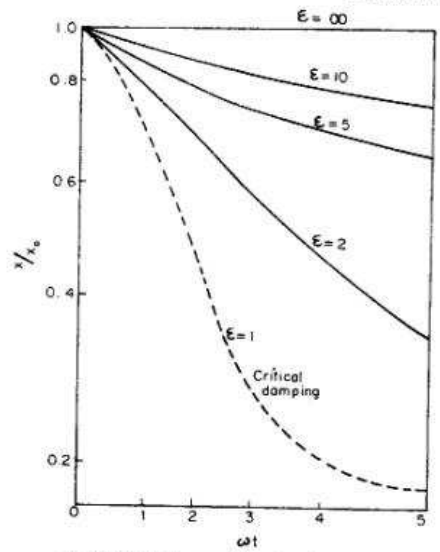


Fig. 3.9 (a). Displacement-time plots for over-damping and critical damping ($\varepsilon \geq 1$).

Applying initial conditions to equation (3.3.12), we get

$$x_0 = A_1$$

$$\dot{x}_0 = \left(\frac{dx}{dt}\right)_{t=0} = A_1(-\omega) + A_2 \qquad ...(3.3.14)$$

$$\dot{x}_0 = -\omega A_1 + A_2$$

$$\dot{x}_0 = -\omega x_0 + A_2$$

$$A_2 = \dot{x}_0 + \omega x_0$$

So the solution of equation (3.3.13) can be written as

$$x = x_0 e^{-\omega t} + (\dot{x}_0 + \omega x_0)\,t e^{-\omega t}$$

$$= [x_0 + (\dot{x} + \omega x_0)t]\,e^{-\omega t} \qquad ...(3.3.15)$$

We observe in equation (3.3.15) that the value of x decreases as t increases and finally becomes zero as t tends to infinity. This is also an aperiodic motion.

Making use of this property (critical damping) the electrical instruments are designed where the displaced body is brought to equilibrium in minimum possible time. Displacement-time curve is shown in figure 3.10. Refer figure 3.9 (a) also.

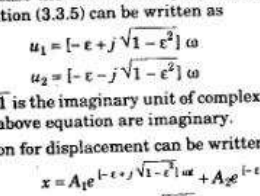


Fig. 3.10. Displacement-time curve for critically damped system.

(C) Under-damped System

In this case the value of damping ratio ε is less than unity. The roots of equation (3.3.5) can be written as

$$u_1 = [-\varepsilon + j\sqrt{1 - \varepsilon^2}]\,\omega$$
$$u_2 = [-\varepsilon - j\sqrt{1 - \varepsilon^2}]\,\omega \qquad ...(3.3.16)$$

where $j = \sqrt{-1}$ is the imaginary unit of complex roots. It is clear that the roots of the above equation are imaginary.

Equation for displacement can be written as

$$x = A_1 e^{[-\varepsilon + j\sqrt{1-\varepsilon^2}]\omega t} + A_2 e^{[-\varepsilon - j\sqrt{1-\varepsilon^2}]\omega t}$$

$$= e^{-\varepsilon\omega t}\,[A_1 e^{j\sqrt{1-\varepsilon^2}\,\omega t} + A_2 e^{-j\sqrt{1-\varepsilon^2}\,\omega t}]$$

We know that $e^{jx} = \cos x + j\sin x$.

$$= e^{-\varepsilon\omega t}[A_1\cos\sqrt{1-\varepsilon^2}\,\omega t + A_1 j\sin\sqrt{1-\varepsilon^2}\,\omega t + A_2\cos\sqrt{1-\varepsilon^2}\,\omega t - A_2 j\sin\sqrt{1-\varepsilon^2}\,\omega t$$

$$= e^{-\varepsilon\omega t}[(A_1 + A_2)\cos\sqrt{1-\varepsilon^2}\,\omega t + (A_1 - A_2)\,j\sin\sqrt{1-\varepsilon^2}\,\omega t]$$

$$= e^{-\varepsilon\omega t}\,[C_1\cos\sqrt{1-\varepsilon^2}\,\omega t + C_2\sin\sqrt{1-\varepsilon^2}\,\omega t]$$

where $C_1 = A_1 + A_2$

and $C_2 = (A_1 - A_2)\,j$

$$x = C_3\,e^{-\varepsilon\omega t}\sin(\sqrt{1-\varepsilon^2}\,\omega t + \phi_1)$$

or
$$= C_4\,e^{-\varepsilon\omega t}\cos(\sqrt{1-\varepsilon^2}\,\omega t + \phi_2) \qquad ...(3.3.17)$$

where $C_1, C_2, C_3, C_4, \phi_1$ and ϕ_2 are arbitrary constants which can be determined from the initial conditions. The amplitude decreases exponentially with time as can be seen from equation (3.3.17).

The term $\sqrt{1-\varepsilon^2}\,\omega = \omega_d$ ...(3.3.18)

is called the frequency of damped vibration. The variation of damped natural frequency ω_d with the damping coefficient ξ can be seen from figure 3.10 (a). The displacement-time plot for an under damped system

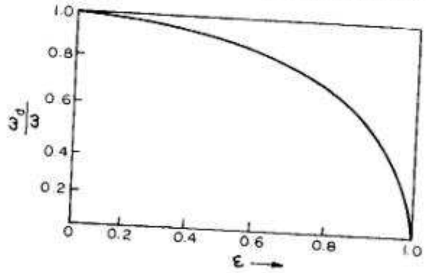


Fig. 3.10 (a). Variation of Damped Natural Frequency in Underdamped System.

showing the variation of displacement (x) with the angle turned ($\omega \cdot t$) for different values of $\xi < 1$ can be seen from figure 3.10 (b).

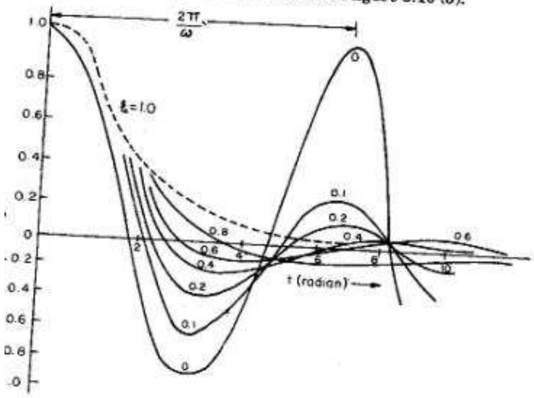


Fig. 3.10 (b). Displacement

It is clear that ω_d is always less than undamped natural frequency ω. We obtain oscillatory motion in this case. Theoretically the system will never come to rest although the amplitude of vibration may be very-very small. The displacement-time curve for underdamped system is shown in figure 3.11. Time period of the motion is given as

$$\frac{2\pi}{\omega\sqrt{1-\varepsilon^2}} \quad \text{or} \quad \frac{2\pi}{\omega_d} = t_d$$

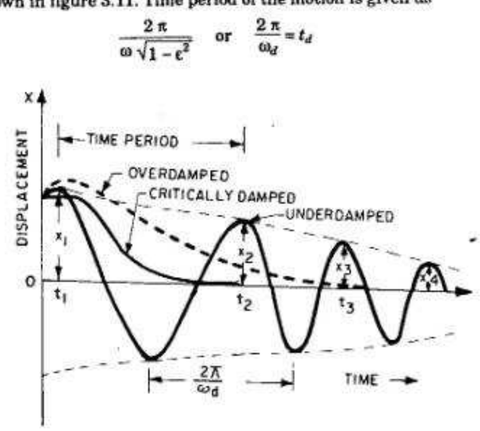


Fig. 3.11.

All the three types of damping responses ($x - t$) are presented in this figure.

Use of Critical Damping

From figure 3.11, it can be seen that for a periodic motion, a critically damped system has the least amount of damping out of the three systems i.e. overdamped, underdamped and critically damped. This means that the vibrating body which has been displaced from its mean position would come to the state of rest in the smallest possible time without executing oscillations about the mean position i.e. the body will come to mean position without overshooting. (Overshooting means that the oscillating body after oscillating on one side of the mean position, does not cross to the other side of the mean position).

This feature of critical damping is used for practical applications in large guns so that after firing, the guns return to their original position in the minimum time without vibrating and are thus ready for next firing without delay. If the damping provided is an overdamped one, then considerable delay will be caused.

3.4. LOGARITHMIC DECREMENT

It is defined as the natural logarithm of the ratio of any two successive amplitudes on the same side of the mean line. Let us refer figure 3.11 for two successive amplitudes x_1 and x_2.

As per the definition, logarithmic decrement δ is given as

$$\delta = \ln \frac{x_1}{x_2} \qquad ...(3.4.1)$$

We rewrite equation (3.3.17) for amplitude as

$$x = C_4 e^{-\varepsilon\omega t} \cos(\sqrt{1-\varepsilon^2}\,\omega t + \phi_2)$$

Let t_1 and t_2 denote the times corresponding to two successive amplitudes. We can find the ratio of amplitudes x_1 and x_2 as

$$\frac{x_1}{x_2} = \frac{C_4 e^{-\varepsilon\omega t_1} \cos(\sqrt{1-\varepsilon^2}\,\omega t_1 + \phi_2)}{C_4 e^{-\varepsilon\omega t_2} \cos(\sqrt{1-\varepsilon^2}\,\omega t_2 + \phi_2)}$$

$$= e^{-\varepsilon\omega(t_1 - t_2)} \frac{\cos(\sqrt{1-\varepsilon^2}\,\omega t_1 + \phi_2)}{\cos(\sqrt{1-\varepsilon^2}\,\omega t_2 + \phi_2)} \qquad ...(3.4.2)$$

Let us assume $t_2 = t_1 + t_d$

where $t_d = \frac{2\pi}{\omega_d}$ is the period of damped vibration. The term

$$\frac{\cos(\omega_d t_1 + \phi_2)}{\cos[\omega_d(t_1 + t_d) + \phi_2]} \qquad [\text{as } \sqrt{1-\varepsilon^2}\,\omega = \omega_d]$$

or

$$\frac{\cos(\omega_d t_1 + \phi_2)}{\cos\left\{\omega_d\left(t_1 + \frac{2\pi}{\omega_d}\right) + \phi_2\right\}} = \frac{\cos(\omega_d t_1 + \phi_2)}{\cos[(\omega_d t_1 + 2\pi) + \phi_2]} \qquad ...(3.4.3)$$

Again considering equation (3.4.2) and using equation (3.4.3) in it, we have

$$\frac{x_1}{x_2} = e^{-\varepsilon\omega(t_1 - t_1 - t_d)} = e^{\varepsilon\omega t_d} = e^{\frac{\varepsilon\omega 2\pi}{\omega_d}}$$

$$\frac{x_1}{x_2} = e^{\frac{\varepsilon\omega 2\pi}{\sqrt{1-\varepsilon^2}\,\omega}} = e^{2\pi\varepsilon/\sqrt{1-\varepsilon^2}}$$

or

$$\delta = \log_e \frac{x_1}{x_2} = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}} \qquad ...(3.4.4)$$

when the value of ε is very small, above equation can be written as

$$\delta \approx 2\pi\varepsilon \qquad ...(3.4.5)$$

Figure 3.12 shows the variation trend of the logarithmic decrement δ with ε as expressed by equations (3.4.4) and (3.4.5).

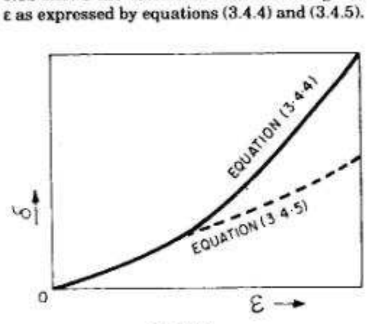


Fig. 3.12.

If the system executes n cycles, the logarithmic decrement δ can be written as

$$\delta = \frac{1}{n} \log_e \frac{x_1}{x_{n+1}} \qquad ...(3.4.6)$$

where x_1 = amplitude at the starting position

x_{n+1} = amplitude after n cycles

It can be proved mathematically as

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \dots \frac{x_{n-1}}{x_n} = e^{\delta}$$

$$\left(\frac{x_1}{x_2}\right)\left(\frac{x_2}{x_3}\right)\left(\frac{x_3}{x_4}\right)\dots\left(\frac{x_n}{x_{n+1}}\right) = (e^{\delta})^n$$

$$n\,\delta = \ln \frac{x_1}{x_{2+1}}$$

So

$$\delta = \frac{1}{n} \ln \frac{x_1}{x_{2+1}} \qquad ...(3.4.7)$$

3.4.1 Vibrational Energy and Logarithmic Decrement

Let the amplitude at any given instant be x_1 and that after one cycle be x_2.

$\therefore$ Logarithmic decrement $\delta = \ln \frac{x_1}{x_2}$

or

$$\frac{x_1}{x_2} = e^{\delta}$$

or $$\frac{x_2}{x_1} = e^{-\delta}$$

Expanding $e^{-\delta}$, we get

$$e^{-\delta} = 1 - \delta + \frac{\delta^2}{2!} - \frac{\delta^3}{3!} + ...$$

Let E_1 be the vibrational energy at amplitude x_1

$$\therefore \quad E_1 = \frac{1}{2} kx_1^2$$

Similarly, let E_2 be the vibrational energy at amplitude x_2, so

$$E_2 = \frac{1}{2} kx_2^2$$

Now $$\frac{E_1 - E_2}{E_1} = 1 - \frac{E_2}{E_1} = 1 - \frac{x_2^2}{x_1^2}$$

$$= 1 - e^{-2\delta} \qquad \text{(substituting the value of } \frac{x_2}{x_1})$$

$$= 1 - \left[1 - 2\delta + \frac{(2\delta)^2}{2!} - \frac{(2\delta)^3}{3!} + ... \right]$$

$$= 2\delta - \frac{(2\delta)^2}{2!} + \frac{(2\delta)^2}{3!} - ....$$

For small δ, higher powers of δ may be neglected, so

$$\frac{E_1 - E_2}{E_1} = 2\delta$$

r $$\delta = \frac{\Delta E}{2E_1}$$

'here $\Delta E = E_1 - E_2$ = energy dissipated in one cycle

OLVED EXAMPLES

***Example 3.1.** A damping force having magnitude 2 cos $(2\pi t - \pi/4)$ N, ives 5 cos $2\pi t$ m displacement. Calculate*

(a) the energy dissipated during first 5 seconds, and

(b) the energy dissipated during the first 3/4 sec.

Solution. The displacement leads the force by $\pi/4 = 45°$ angle. here will be energy dissipation. We know that force and displacement re given as $F = F_0 \cos(\omega t - \phi)$, $x = A \cos \omega t$

Time period of force **F** and displacement x is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi} = 1 \text{ sec}$$

(a) In 5 seconds there will be 5 complete cycles.

$$\text{Energy dissipated} = n\left(\pi F_0 A \sin\frac{\pi}{4}\right)$$

$$= 5(\pi \times 2 \times 5 \times \sin 45°) \text{ N} - \text{m}$$

$$= 111.01 \text{ N} - \text{m}$$

(b) During 3/4 sec only a part of cycle is complete. To find the energy during this period, we have expression as

$$\text{Energy dissipation} = \int_0^t F \frac{dx}{dt} dt$$

$$= \int_0^{3/4} \left[2\cos\left(2\pi t - \frac{\pi}{4}\right) \frac{d}{dt} 5\cos 2\pi t \right] dt$$

$$= \int_0^{3/4} \left[2\cos\left(2\pi t - \frac{\pi}{4}\right)(-5 \times 2\pi) \sin 2\pi t \right] dt$$

$$= \int_0^{3/4} \left[(-20\pi)\cos\left(2\pi t - \frac{\pi}{4}\right) \sin 2\pi t \right] dt$$

$$= \int_0^{3/4} \left[(-20\pi)\left(\cos 2\pi t \,.\, \cos\frac{\pi}{4} + \sin 2\pi t \sin\frac{\pi}{4}\right) (\sin 2\pi t) \right] dt$$

$$= \int_0^{3/4} (-44.406)\left(\frac{1}{2}\sin 4\pi t + \frac{2}{2}\sin^2 2\pi t\right) dt$$

$$= -40.62 \text{ N-m}$$

***Example 3.2.** Find the equation of motion for the system shown in figure 3.7 when (a) $\varepsilon = 1.0$, (b) $\varepsilon = 0.3$ and (c) $\varepsilon = 2.0$, if the mass m is displaced by a distance of 3 cm and released.*

Solution. (a) When $\varepsilon = 1.0$ the system is critically damped.

$$x = (A_1 + A_2 t)e^{-\omega t}$$

The displacement is 3 cm at $t = 0$, so putting this initial condition, we get

$$3 = A_1$$

$$\left(\frac{dx}{dt}\right)_{t=0} = \dot{x} = -\omega A_1 + A_2 = 0$$

$$= -3\omega + A_2$$

$$A_2 = 3\omega$$

Therefore, the displacement x is given as

$$x = (3 + 3\omega t)e^{-\omega t}$$

or $$x = 3(1 + \omega t)e^{-\omega t}$$

(b) When $\varepsilon = 0.3$, the system is underdamped.

The displacement is given as

$$x = A_1 e^{-\varepsilon\omega t} \sin(\sqrt{1 - \varepsilon^2}\, \omega t + \phi)$$

$$= A_1 e^{-.3\omega t} \sin(\sqrt{1 - .09}\, \omega t + \phi)$$

$$= A_1 e^{-.3\omega t} \sin(0.953\, \omega t + \phi)$$

To find A_1 and ϕ, we put the boundary conditions as

$$x|_{t=0}\; 3 = A_1 \sin\phi$$

$$\dot{x}|_{t=0} = 0 = -.3 \sin\phi + .953 \cos\phi$$

$$\tan\phi = \frac{.953}{.3} = 3.176$$

$$\phi = 72° \, 31'$$

$$\therefore \quad A_1 = 3/\sin\phi = 3.145$$

The equation of motion can be written as

$$x = 3.145\, e^{-0.3\omega t} \sin(.953\, \omega t + 72° \, 31')$$

(c) The system is overdamped for $\varepsilon = 2.0$

The equation for displacement is given as

$$x = A_1 e^{[-\varepsilon + \sqrt{\varepsilon^2 - 1}]\omega t} + A_2 e^{[-\varepsilon - \sqrt{\varepsilon^2 - 1}]\omega t}$$

$$= A_1 e^{(-2 + \sqrt{3})\omega t} + A_2 e^{(-2 - \sqrt{3})\omega t}$$

Applying the initial conditions to the above equation to find A_1 and A_2

$$x|_{t=0} = 3 = A_1 + A_2$$

$$\dot{x}|_{t=0} = 0 = (-2 + \sqrt{3})\,\omega A_1 + (-2 - \sqrt{3})\omega A_2$$

$$-.2679 A_1 - 3.732 A_2 = 0$$

$$A_1 + 13.93 A_2 = 0$$

Solving it, we get

$$A_2 = -0.232$$

$$A_1 = 3.232$$

***Example 3.3.** A gun barrel having mass 560 kg is designed with the following data :*

Initial recoil velcity 36 m/sec ; Recoil distance on firing 1.5 m

Calculate

(a) spring constant. (b) damping coefficient, and

(c) time required for the barrel to return to a position 0.12 m from its initial position. (P.U., 99)

Solution. (a) The kinetic energy of the barrel must be equal to the strain energy of the spring, so

$$\frac{1}{2} m\dot{x}^2 = \frac{1}{2} kx^2$$

$$\frac{1}{2} \times 560 \times 36 \times 36 = \frac{1}{2} k(1.5)^2$$

$$k = 322.56 \times 10^3 \text{ N/m}$$

(b) The critical damping is given by

$$C_c = 2\sqrt{km} = 2\sqrt{322560 \times 560} = 26880 \text{ Ns/m}$$

(c) Natural frequency ω can be determined as

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{322560}{560}} = 24 \text{ rad/sec}$$

$$\text{Time period } \frac{2\pi}{\omega} = \frac{2\pi}{24} = 0.26166 \text{ sec}$$

The time taken by the gun barrel is only $\frac{1}{4}$ of the total cycle time because recoil takes place only during the quarter of cycle. where T = cycle time

$$\text{Time for recoil} = \frac{1}{4} \cdot \text{Time period} = \frac{1}{4} \times 0.26166 = 0.0654 \text{ sec}$$

For critical damping the displacement is given as

$$x = (A_1 + A_2 t)\, e^{-\omega t}$$

Initial conditions are given as

$$\left.\begin{aligned} x &= 1.5 \text{ m} \\ \dot{x} &= 0 \end{aligned}\right\} \text{ at } t = 0$$

So $$1.5 = A_1$$

$$\dot{x} = -\omega A_1 + A_2$$

$$0 = -1.5\,\omega + A_2$$

$$A_2 = 1.5\,\omega = 1.5 \times 24 = 36$$

Here $x = .12$ m

$$.12 = (1.5 + 36t)e^{-24t}$$

$$= 1.5(1 + 24t)\,e^{-24t}$$

$$.08 = (1 + 24t)e^{-24t}$$

Solving this equation for 't' by hit and trial method :-

t	$(1+24\,t)$	e^{-24t}	$x = (1 + 24\,t)\,e^{-24t}$
0.10	3.4	0.0907	0.308
0.15	4.6	0.02732	0.125
0.20	5.8	0.00823	0.047
0.18	5.32	0.0133	0.0707
0.17	5.08	0.0169	0.085

$\therefore$ t can be approximately taken as 0.17 seconds.

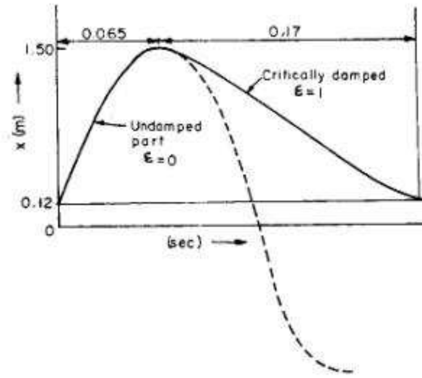


Fig. 3.12 (a).

Solving it for t, we have

$$t = 0.17 \text{ sec}$$

Total time $= 0.17 + 0.0654 = 0.2354$ sec

***Example 3.4.** A damper offers resistance 0.05 N at constant velocity .04 m/sec. The damper is used with $k = 9$ N/m. Determine the damping*

parameter and frequency of the system ...

Solution. Damping force $F = C\dot{x}$

$$\dot{x} = .04 \text{ m/sec}, \quad F = .05 \text{ N}$$

$$C = \frac{F}{\dot{x}} = \frac{.05}{.04} = 1.25 \text{ N-sec/m}$$

$$C_c = 2\sqrt{km} = 2\sqrt{9 \times .1} = 1.897 \text{ N-sec/m}$$

$$\varepsilon = \frac{C}{C_c} = \frac{1.25}{1.897} = 0.658$$

The system is under-damped. The frequency of damped vibration

$$\omega_d = \sqrt{1-\varepsilon^2}\;\omega = \sqrt{[1-(.658)^2] \times \frac{9}{0.10}}$$

$$= 7.14 \text{ rad/sec}$$

***Example 3.5.** A vibrating system is defined by the following parameters :*

$$m = 3 \text{ kg},\ k = 100 \text{ N/m},\ C = 3 \text{ N-sec/m}$$

Determine (a) the damping factor, (b) the natural frequency of damped vibration, (c) logarithmic decrement, (d) the ratio of two consecutive amplitudes and (e) the number of cycles after which the original amplitude is reduced to 20 percent.

Solution. Critical damping is determined as

$$C_c = 2\sqrt{km} = 2\sqrt{100 \times 3} = 34.64 \text{ N-sec/m}$$

(*a*) $$\varepsilon = \frac{C}{C_c} = \frac{3}{34.64} = 0.086$$

(*b*) $$\omega_d = \omega\sqrt{1-\varepsilon^2} = \sqrt{\frac{k}{m}(1-.086^2)}$$

$$= \sqrt{\frac{100}{3}(1-.086^2)} = 5.75 \text{ rad/sec}$$

$$f_d = \frac{\omega_d}{2\pi} = \frac{5.75}{2\pi} = .92 \text{ Hz}$$

(*c*) $$\delta = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}} = \frac{2\pi \times .086}{\sqrt{1-.086^2}} = 0.542$$

(*d*) The ratio between two consecutive amplitudes say x_1/x_2

$$\delta = \ln\frac{x_1}{x_2}$$

or $$e^{\delta} = \frac{x_1}{x_2}$$

$$\frac{x_1}{x_2} = e^{.542} = 1.72$$

(*e*) $$\delta = \frac{1}{n}\ln\frac{x_1}{x_n}$$

$$n = \frac{1}{\delta}\ln\frac{x_1}{x_n} = \frac{1}{.542}\ln\left(\frac{x_1}{x_1/5}\right) = \frac{1}{.542}\ln 5$$

$$n = 2.96 \text{ cycles}$$

***Example 3.6.** A vibratory system in a vehicle is to be designed with the following parameters :*

$$k = 100 \text{ N/m},\ C = 2 \text{ N-sec/m},\ m = 1 \text{ kg}$$

Calculate the decrease of amplitude from its starting value after complete oscillations and (b) the frequency of oscillation.

Solution. (*a*) $$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{1}} = 10 \text{ rad/sec}$$

$$C_c = 2\sqrt{km} = 2\sqrt{100 \times 1} = 20 \text{ N-sec/m}$$

$$\varepsilon = \frac{C}{C_c} = \frac{2}{20} = 0.10$$

$$\delta = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}} = \frac{2\pi \times 0.10}{\sqrt{1-0.1^2}} = 0.63$$

We know that, $$\delta = \frac{1}{n}\ln\frac{x_1}{x_{n+1}}$$

$$n = 3$$

$$.63 \times 3 = \ln\frac{x_1}{x_{n+1}}$$

$$\frac{x_1}{x_{n+1}} = e^{1.89} = 6.62$$

(*b*) $$\omega_d = \omega\sqrt{1-\varepsilon^2}$$

$$= 10\sqrt{1-.1^2} = 9.95 \text{ rad/sec}$$

***Example 3.7.** A thin plate of area A and weight W is attached to the end of a spring and allowed to oscillate in a viscous fluid as shown in Figure 3.13. If f_1 is the frequency of oscillation of the system in air and f_2 in the liquid, show that*

$$\alpha = \frac{2\pi W}{gA}\sqrt{f_1^2 - f_2^2}$$

where the damping force on the plate is $F_d = \alpha 2Av$, v being the velocity.

Solution. $$F_d = \alpha\, 2\, Av$$

We know that damping force is generally given by the relation

$$F_d = c\dot{x}$$

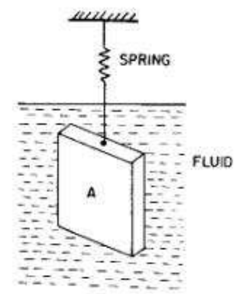


Fig. 3.13.

So comparison gives $c = 2A\alpha$

$$f_1 = \frac{\omega}{2\pi} \quad \text{where } \omega = \sqrt{k/m}$$

$$f_2 = \frac{\omega}{2\pi}\sqrt{1-\varepsilon^2}$$

$$\frac{f_1}{f_2} = \frac{1}{\sqrt{1-\varepsilon^2}}$$

or $$1-\varepsilon^2 = \left(\frac{f_2}{f_1}\right)^2$$

$$\varepsilon^2 = (f_1^2 - f_2^2)/f_1^2 \qquad \left[\varepsilon = \frac{C}{C_c} = \frac{2\alpha A}{2\sqrt{km}}\right]$$

$$\left(\frac{2\,\alpha A}{2\sqrt{km}}\right)^2 = \frac{(f_1^2 - f_2^2)}{f_1^2}$$

$$\alpha^2 = \left(\frac{\sqrt{km}}{A}\right)^2 \frac{(f_1^2 - f_2^2)}{f_1^2}$$

$$\alpha = \frac{\sqrt{km}}{Af_1}\sqrt{f_1^2 - f_2^2}$$

$$= \frac{1}{A}\sqrt{\frac{k}{m}}\,\frac{m}{f_1}\sqrt{f_1^2 - f_2^2} = \frac{m}{A}\frac{\omega}{f_1}\sqrt{f_1^2 - f_2^2}$$

$$= \frac{W}{gA}\frac{2\pi f_1}{f_1}\sqrt{f_1^2 - f_2^2}$$

$$= \frac{2\pi W}{gA}\sqrt{f_1^2 - f_2^2}$$

EXAMPLE 3.8. *Derive equation of motion for the system shown in figure 3.14.*

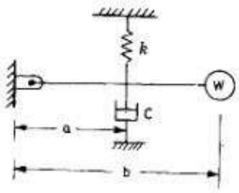


Fig. 3.14.

If m = 1.5 kg, k = 4900 Ns/m, a = 6 cm and b = 14 cm, determine the value of C for which the system is critically damped. (P.U., 91)

SOLUTION. Let us consider the downward angular displacement θ of the weight W.

The displacement of spring $= a\theta$

So the spring force $= ka\theta$

damping force $= C\dot{x} = Ca\dot{\theta}$

The equation of motion can be written as

$$I\ddot{\theta} + a(ka\theta) + a(Ca\dot{\theta}) = 0$$

$$\frac{W}{g} b^2\ddot{\theta} + Ca^2\dot{\theta} + ka^2\theta = 0$$

or

$$\ddot{\theta} + \frac{Ca^2 g}{Wb^2}\dot{\theta} + \frac{ka^2 g}{Wb^2}\theta = 0$$

The solution of the above equation

$$u_{1,2} = -\frac{Ca^2 g}{2Wb^2} \pm \sqrt{\left(\frac{Ca^2 g}{2Wb^2}\right)^2 - \frac{ka^2 g}{Wb^2}}$$

The system is critically damped when the radical is zero

$$\frac{C_c a^2 g}{2Wb^2} = \frac{a}{b}\sqrt{\frac{kg}{W}}$$

$$C_c = \frac{2b}{a}\sqrt{\frac{Wk}{g}} = \frac{2b}{a}\sqrt{km}$$

$$= \frac{2 \times 14}{6}\sqrt{4900 \times 1.5} = 400 \text{ N-sec/m}$$

EXAMPLE 3.9. *The torsional pendulum with a disc of moment of inertia J = 0.05 kg–m² immersed in a viscous fluid is shown in figure 3.15. During vibrations of pendulum, the observed amplitudes on the same side of the neutral axis for successive cycles are found to decay 50% of the initial value. Determine*

(*a*) *logarithmic decrement*

(*b*) *damping torque per unit velocity*

(*c*) *the periodic time of vibration*

(*d*) *the frequency when the disc is removed from the fluid.*

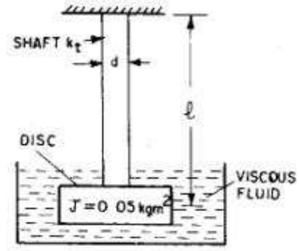


Fig. 3.15. Torsional Pendulum.

Assume $G = 4.5 \times 10^{10}$ N/m² for the material of shaft

d = 0.10 m, l = 0.50 m, M.I. of disc = 0.05 kg–m². (P.U., 98)

SOLUTION. (*a*) Say initial displacement is θ, it remains 50% so $\delta = \log_e(\theta/\theta/2) = \log_e 2$

$$\delta = 0.693$$

(*b*)
$$\delta = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}}$$

So
$$(1-\varepsilon^2)\,\delta^2 = 4\pi^2\varepsilon^2$$

$$\delta^2 = (4\pi^2 + \delta^2)\varepsilon^2$$

$$\varepsilon = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.693}{\sqrt{4\pi^2 + (.693)^2}} = 0.109$$

We know that $\varepsilon = C/C_c$

or $C = \varepsilon C_c$

But $C_c = 2\sqrt{k_t J}$ (in torsion)

$$C = \varepsilon\, 2\sqrt{k_t J}$$

$$J = .05 \text{ kg–m}^2$$

$$k_t = \frac{GI}{l} \quad \left[k_t = \frac{T}{\theta} = \frac{GI}{l}\right]$$

$$= \frac{4.5 \times 10^{10} \times \pi/32 \times (.1)^4}{.50} \quad \left(I = \frac{\pi}{32}d^4\right)$$

$$k_t = 8.83125 \times 10^5 \text{ Nm/rad}$$

So
$$C = \varepsilon \times 2\sqrt{8.83125 \times 10^5 \times .05}$$

$$= 0.109 \times 2 \times 2.1013 \times 10^2 = 45.809 \text{ Nm/rad}$$

This is damping torque per unit velocity.

(*c*) Periodic time of oscillation $= \dfrac{2\pi}{\omega_d}$

$$= \frac{2\pi}{\omega\sqrt{1-\varepsilon^2}} = \frac{2\pi}{\sqrt{\frac{k_t}{J}}\,[1-(.109)^2]}$$

$$= \frac{2\pi}{\sqrt{\frac{8.83125 \times 10^5}{.05}}\,(.9881)} = 1.503 \times 10^{-3} \text{ sec}$$

(*d*) When the disc is removed from viscous fluid, the natural frequency is given

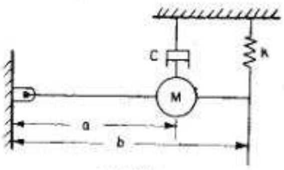
$$f = \frac{\omega}{2\pi}$$

But
$$\omega = \sqrt{\frac{k_t}{J}} = \sqrt{\frac{8.83125 \times 10^5}{.05}}$$

$$= 4202.67 \text{ rad/sec}$$

So
$$f = \frac{4202.67}{2\pi} = 669.2 \text{ Hz}$$

EXAMPLE 3.10. *Determine suitable expression for equation of motion of the damped vibratory system shown in figure 3.16. Find the critical damping coefficient when a = 0.10 m, b = 0.13 m, k = 4900 N/m and M = 1.5 kg.* (P.U. 78)



SOLUTION. Let us assume that the mass *M* is displaced by an angle θ, then

Spring force $= k(b\theta)$

Damping force $= C(a\,\dot{\theta})$

Equation of motion can be written as

$$I\ddot{\theta} + kb(b\theta) + Ca(a\dot{\theta}) = 0$$

$$Ma^2\ddot{\theta} + Ca^2\dot{\theta} + kb^2\theta = 0 \qquad [\because I = Ma^2]$$

or
$$\ddot{\theta} + \frac{Ca^2\dot{\theta}}{Ma^2} + \frac{kb^2\theta}{Ma^2} = 0$$

$$\ddot{\theta} + \frac{C\dot{\theta}}{M} + \frac{k}{M}\frac{b^2}{a^2}\theta = 0$$

The roots of the equation are

$$u_{1,2} = \frac{-C}{2M} \pm \sqrt{\left(\frac{C}{2M}\right)^2 - \frac{k}{M}\frac{b^2}{a^2}}$$

The critical damping coefficient is given as

$$\left(\frac{C_c}{2M}\right)^2 = \frac{k}{M}\frac{b^2}{a^2}$$

or
$$C_c = \frac{2b}{a}\sqrt{kM}$$

$$= 2 \times \frac{.13}{.10}\sqrt{4900 \times 1.5} = 222.90 \text{ N-sec/m}$$

EXAMPLE 3.11. *A mass of 10 kg is kept on two slabs of isolators placed one over the other. One of the isolators is of rubber having a stiffness of 3 kN/m and damping coefficient of 100 N-sec/m while the other isolator is of felt with stiffness of 12 kN/m and damping coefficient of 300 N.sec/m. If the system is set in motion in vertical direction, determine the damped and undamped natural frequencies of the system.* (M.D.U., 94)

SOLUTION. The isolators are connected in series. We can find k_e and C_e as equivalence for both.

$$\frac{1}{k_e} = \frac{1}{3000} + \frac{1}{12000}$$

$$k_e = 2400 \text{ N/m}$$

$$\frac{1}{C_e} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{100} + \frac{1}{300} = \frac{4}{300}$$

$$C_e = 75 \text{ N-sec/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2400}{10}} = 15.49 \text{ rad/sec}$$

$$f_n = \omega_n/2\pi = 2.47 \text{ Hz}$$

$$\varepsilon = \frac{C_c}{2\sqrt{k_e m}} = \frac{75}{2 \times \sqrt{2400 \times 10}} = 0.24$$

Damped natural frequency

$$\omega_d = \sqrt{1-\varepsilon^2}\ \ \omega_n = \sqrt{1-(.24)^2}\ (15.49)$$

$$= 15.03 \text{ rad/sec}$$

Example 3.12. *A horizontal spring mass system with coulomb damping has a mass of 5.0 kg attached to a spring of stiffness 980 N/m. If the coefficient of friction is 0.025, calculate*

(a) *the frequency of free oscillations,*

(b) *the number of cycles corresponding to 50% reduction in amplitude if the initial amplitude is 5.0 cm, and*

(c) *the time taken to achieve this 50% reduction.*

(M.D.U., 94 ; P.U., 94)

Solution. For a horizontal system the force is given as

$$F = \mu \text{ mg} = .025 \times 5 \times 9.81 = 1.226 \text{ N}$$

(a) The natural frequency

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{980}{5}} = 14 \text{ rad/sec}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{14}{2\pi} = 2.23 \text{ Hz}$$

(b) Amplitude after 50% reduction is half of the initial amplitude i.e., 0.025 m

Reduction in amplitude per cycle

$$= \frac{4F}{k} = \frac{4 \times 1.226}{980}$$

$$= 5 \times 10^{-3} \text{ m}$$

Cycles completed in 50% reduction

$$= \frac{.025}{.005} = 5 \text{ cycles.}$$

(c) Time taken to achieve 50% reduction

$$= \text{No. of cycles} \cdot \frac{2\pi}{\omega_n} \qquad \left(T = \frac{2\pi}{\omega_n}\right)$$

$$= 5 \times \frac{2\pi}{14} = 2.42 \text{ sec}$$

Example 3.13. *For the system shown in figure 3.17, the characteristic of the dashpot is such that when a constant force of 49 N is applied to the piston its velocity is found to be constant at 0.12 m/sec.*

(a) *Determine the value of C*

(b) *Would you expect the complete system to be periodic or aperiodic ?*

(P.U., 89)

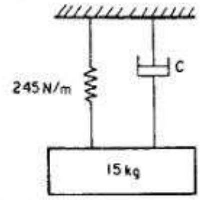


Fig. 3.17

Solution. (a) The viscous force is given by

$$F = C\dot{x}$$

$$C = \frac{F}{\dot{x}} = \frac{49}{.12} = 408.33 \text{ N-sec/m}$$

$$(b)\ \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{245}{15}} = 4.04 \text{ rad/sec}$$

The equation of motion can be written as

$$m\ddot{x} + C\dot{x} + kx = 0$$

$$\varepsilon = \frac{C}{2m\omega_n} = \frac{408.33}{2 \times 15 \times 4.04} = 3.36$$

Since ε is more than unity, so the system is overdamped. The motion will be aperiodic in nature.

Example 3.14. *A body of 5 kg is supported on a spring of stiffness 200 N/m and has dashpot connected to it which produces a resistance of 0.002 N at a velocity of 1 cm/sec. In what ratio will the amplitude of vibration be reduced after 5 cycles.* (P.U., 92 ; M.D.U., 91)

Solution. $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{200}{5}} = 6.32$ rad/sec

$$C = F/\dot{x} = .002/.01 = 0.2 \text{ N-sec/m}$$

Critical damping $C_c = 2m\omega_n = 2 \times 5 \times 6.32$

$$= 63.2 \text{ N-sec/m}$$

Damping ratio $\varepsilon = \frac{C}{C_c} = \frac{.2}{63.2} = 3.16 \times 10^{-3}$

Logarithmic decrement δ can be written as

$$\delta = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}} = \frac{2\pi \times 3.16 \times 10^{-3}}{\sqrt{1-(3.16 \times 10^{-3})^2}}$$

$$\delta = 0.0198$$

We know that, $\delta = \frac{1}{n} \ln \frac{x_1}{x_{n+1}}$

$$\delta \cdot n = \ln \frac{x_1}{x_{n+1}}$$

where n no. of cycles = 5

$$.0198 \times 5 = \ln \frac{x_1}{x_6} = 0.0990$$

$$\frac{x_1}{x_6} = 1.104.$$

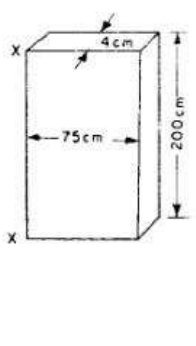
Example 3.15. *A door 200 cm high, 75 cm wide and 4 cm thick and weighing 35 kg is fitted with an automobile door closer. The door opens against a spring with a modulus of 1 kg-cm/radian. If the door is opened 90° and released, how long will it take the door to be within 1° of closing ? Assume the return spring of the door to be critically damped.*

(P.U., 93)

Solution. For a critically damped system, the equation of motion can be written as

$$x = (A_1 + A_2 t)e^{-\omega t}$$

given that at $\begin{cases} t = 0 \\ x = \pi/2, \text{ so } \dot{x} = 0 \end{cases}$



we get $\pi/2 = A_1,\ \frac{\omega\pi}{2} = A_2$

Substituting these values in the above equation, we get

$$x = \left(\frac{\pi}{2} + \frac{\omega\pi}{2}t\right)e^{-\omega t} = \pi/2(1+\omega t)e^{-\omega t}$$

Applying the second boundary condition, we get

$$\frac{\pi}{180} = \frac{\pi}{2}(1+\omega t)e^{-\omega t} \qquad (1° = \pi/180)$$

$$e^{\omega t} = 90(1+\omega t)$$

Solving the above equation, we find

$$\omega t = 6.516$$

Here moment of inertia of the door about the xx-axis can be written as

$$I_{xx} = \frac{1}{2}m(a^2 + b^2) + m(a/2)^2 = \frac{1}{2}ma^2 + \frac{1}{2}mb^2 + \frac{ma^2}{4}$$

$$= \frac{3}{4}ma^2 + \frac{1}{2}mb^2$$

Given $a = 75$ cm, $b = 4$ cm, $m = 35$ kg.

Substituting the values, we find

$$I_{xx} = 147936 \text{ kg-cm}^2$$

The frequency of vibration is given by

$$\omega_n = \sqrt{\frac{k_t}{I_{xx}}}$$

units $\left|\begin{array}{l} k_t = 1 \text{ kg-cm/rad} = 981 \text{ kg cm}^2/\text{rad sec}^2 \\ I_{xx} = 147936 \text{kg-cm}^2 \end{array}\right.$

$$\omega_n = \sqrt{\frac{981}{147936}} = 0.08 \text{ rad/sec}$$

We know that $\omega_n \cdot t = 6.516$

So $t = \frac{6.516}{0.08} = 81.45$ sec

Example 3.16. *The damped vibration record of a spring-mass-dashpot system shows the following data :*

Amplitude on second cycle = *1.2 cm*

Amplitude on third cycle = *1.05 cm*

Spring constant, $k = 8$ *kg/cm*

Weight on the spring, $W = 2$ *kg*

Determine the damping constant, assuming the viscous damping.

(P.U., 85)

SOLUTION. Frequency $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{2/981}} = 62.64$ rad/sec

$$\delta = \ln \frac{x_2}{x_3}, \qquad \text{where } x_2 = 1.2 \text{ cm}, \; x_3 = 1.05 \text{ cm}$$

$$\delta = \ln \frac{1.2}{1.05} = 0.1335$$

$$\delta = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}} \qquad \text{(logarithmic decrement relation)}$$

$$\frac{0.1335}{2\pi} = \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} = 0.0212$$

$$\varepsilon^2 = (1-\varepsilon^2)\, 4.5 \times 10^{-4}$$

$$\varepsilon^2(1 + 4.5 \times 10^{-4}) - 4.5 \times 10^{-4} = 0$$

$$\varepsilon^2(1.00045) = 4.5 \times 10^{-4}$$

$$\varepsilon = 0.0212$$

$$\varepsilon = \frac{C}{C_c}$$

So, $C = \varepsilon \,.\, C_c = \varepsilon(2m\,\omega_n)$

$$= .0212 \times \left(2 \times \frac{2 \times 62.64}{9.81}\right) = 0.541 \text{ kg-sec/m}$$

***EXAMPLE 3.17.** A body of mass $M = 1$ kg, lies on a dry horizontal plane and is connected by spring to a rigid support. The body is displaced from the unstressed position by an amount equal to 0.255 m with the tension in the spring at this displacement equal to 5 Kg, and then released with zero velocity. How long will the body vibrate and at what distance from the unstressed position will it stop if the coefficient of friction is 0.25 ?* (P.U., 91)

SOLUTION. $m = 1$ kg

$\mu = 0.25$

Spring force $= 5 \times 9.81 = 49.05$ N

$kx = 49.05$ N

$$k = \frac{49.05}{x} = \frac{49.05}{0.255} = 192.35 \text{ N/m}$$

for horizontal system

$$F = \mu\, mg = .25 \times 1 \times 9.80 = 2.45 \text{ N}$$

Natural frequency $\omega_n = \sqrt{k/m} = \sqrt{\frac{192.35}{1}}$

$$\omega_n = 13.86 \text{ rad/sec}$$

Total displacement of body = 0.255 m

Reduction in amplitude/cycle

$$\frac{4F}{k} = \frac{4 \times 2.452}{192.35} = 0.0509 \text{ m}$$

No. of cycles performed $= \frac{.255}{.0509} = 5$

Time taken = No. of cycles $\times \frac{2\pi}{\omega_n}$

$$= 5 \times \frac{2\pi}{13.86} = 2.27 \text{ sec}$$

***EXAMPLE 3.18.** A vertical spring mass system has a mass of 0.5 kg and on initial deflection of 0.2 cm. Find the spring stiffness and the natural frequency of the system.* (P.U., 85)

SOLUTION. $m = 0.5$ kg

Initial deflection = 0.2 cm

Spring force $= mg = .5 \times 9.8$

$= 4.9$ N

Spring constant $= F/x$

$$= \frac{4.9}{.2 \times 10^{-2}} = 2450 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2450}{0.5}} = 70 \text{ rad/sec}$$

$$f_n = \frac{70}{2\pi} = 11.14 \text{ Hz}$$

***EXAMPLE 3.19.** Free vibration records of 1 tonne machine mounted on an isolator is shown in figure 3.19. Identify the type of isolator and its characteristics i.e. the spring.* (P.U., 94)

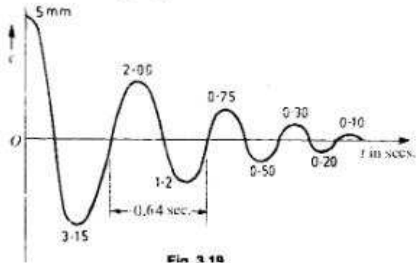


Fig. 3.19

SOLUTION. $m = 1$ tonne $= 1000$ kg

logarithmic decrement for n cycles is given by

Here $n = 4$ cycles

$$\delta = \frac{1}{n} \log_e \frac{x_1}{x_{n+1}}$$

$$= \frac{1}{4} \log_e \frac{5}{0.10} = 0.978$$

We have equation

$$\delta = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}}$$

$$0.978 = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}}$$

$$\frac{\varepsilon}{\sqrt{1-\varepsilon^2}} = 0.155$$

$$\varepsilon^2 = 0.023$$

So $\varepsilon = 0.15$

Damped frequency $\omega_d = \frac{2\pi}{T}$

$$\omega_d = \frac{2\pi}{0.64} = 9.8 \text{ rad/sec}$$

$$\frac{\omega_d}{\omega_n} = \sqrt{1-\varepsilon^2}$$

So $\omega_n = \frac{\omega_d}{\sqrt{1-\varepsilon^2}} = \frac{9.8}{\sqrt{1 - .15 \times .15}}$

$$\omega_n = 9.9 \text{ rad/sec}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\omega_n^2 = \frac{k}{m}$$

So $k = m\,\omega_n^2$

$$= 1000 \times (9.9) = 98010 \text{ N/m}$$

Critical damping

$$C_c = 2m.\omega_n = 2 \times 1000 \times 9.9$$

$$= 19800 \text{ N-m/rad}$$

$$\varepsilon = \frac{C}{C_c}$$

So $C = \varepsilon.C_c = 0.15 \times 19800 = 2970$ N – m/rad

***EXAMPLE 3.20.** Damped vibration of spring-mass-dashpot system gives the following information :*

Amplitude of second cycle = 1.2 cm

Amplitude of third cycle = 1.05 cm

Spring constant = 8 kg/cm

Weight of spring W = 2 kg

Determine the damping constant, assume it to be viscous. (P.U., 88)

SOLUTION. We can find logarithmic decrement by the relation

$$\delta = \ln \frac{x_2}{x_3} = \frac{1.2}{1.05} = 0.133$$

Also $\delta = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}}$

So $\frac{\varepsilon^2}{(1-\varepsilon^2)} = \frac{\delta^2}{4\pi^2} = \frac{(.133)^2}{4\pi^2}$

which gives, $\varepsilon = 0.02$

$$\varepsilon = \varepsilon\, C_c = 0.02 \times 2\sqrt{km}$$

$$= 0.02 \times 2\sqrt{\frac{8 \times 2}{981}} = 0.016 \frac{\text{kg . cm}}{\text{rad}}$$

***EXAMPLE 3.21.** The single pendulum is pivoted at point O as shown in figure 3.20.*

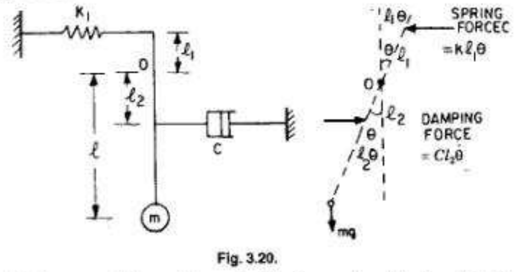


Fig. 3.20.

If the mass of the rod is negligible for small oscillation, find the damped natural frequency of the pendulum. (P.U., M.E. Civil, 94)

SOLUTION. The equation of motion can be written as

$$ml^2\ddot{\theta} = -k_1 l_1^2\theta - Cl_2^2\dot{\theta} - mgl\theta$$

$$ml^2\ddot{\theta} + Cl_2^2\dot{\theta} + (k_1l_1^2 + mgl)\theta = 0$$

$$\ddot{\theta} + \frac{Cl_2^2\dot{\theta}}{ml^2} + \frac{(k_1l_1^2 + mgl)}{ml^2}\theta = 0$$

The general equation for such a system is

$$\ddot{\theta} + 2\varepsilon\,\omega_n\,\dot{\theta} + \omega_n^2\,\theta = 0$$

Let us compare this equation with general form, we have

$$2\varepsilon\omega_n = \frac{Cl_2^2}{ml^2} \quad ...(1)$$

and $$\omega_n^2 = \frac{(k_1l_1^2 + mgl)}{ml^2} \quad ...(2)$$

So $$\omega_n = \frac{Cl_2^2}{ml^2 \times 2\varepsilon} \quad ...(3) \text{ from equation (1)}$$

or $$\omega_n^2 = \left(\frac{Cl_2^2}{2\varepsilon ml^2}\right)^2 \quad ...(4)$$

From equation (4) and (2), we get

$$\left(\frac{Cl_2^2}{2\varepsilon ml^2}\right)^2 = \left(\frac{k_1l_1^2 + mgl}{ml^2}\right)$$

$$(1 - \varepsilon^2) = 1 - \frac{c^2l_2^4}{4ml^2(k_1l_1^2 + mgl)} \quad ...(5)$$

We know that damped frequency ω_d is given by the expression

$$\omega_d^2 = (1 - \varepsilon^2)\omega_n^2$$

Using equation (5) in the above expression, we get

$$\omega_d^2 = \left(1 - \frac{C^2l_2^4}{4ml^2(k_1l_1^2 + mgl)}\right)\left(\frac{k_1l_1^2 + mgl}{ml^2}\right)$$

(Putting ω_n^2 from equation (2))

$$= \left(\frac{4ml^2(k_1l_1^2 + mgl) - C^2l_2^4}{4ml^2\,(k_1l_1^2 + mgl}\right)\left(\frac{k_1l_1^2 + mgl}{ml^2}\right)$$

So $$\omega_d = \sqrt{\frac{k_1l_1^2 + mgl}{ml^2} - \left(\frac{Cl_2}{2ml^2}\right)^2}$$

EXAMPLE 3.22. *A gun barrel of mass 600 kg. has a recoil spring of tiffness 294000 N/meter. If the barrel recoils 1.3 m on firing, deter-nine,*

(a) the initial recoil velocity of the barrel

(b) the critical damping coefficient of the dashpot which is engaged at the end of the recoil stroke.

(c) the time required for the barrel to return to a position 5 cm from the initial position. (P.U., 88)

SOLUTION. The kinetic energy of the barrel must be equal to the potential energy of the spring, so

$$\frac{1}{2}m\dot{x}^2 = \frac{1}{2}kx^2$$

$$\dot{x} = \sqrt{\frac{kx^2}{m}} = \sqrt{\frac{294000 \times 1.3 \times 1.3}{600}}$$

$$\dot{x} = 28.77 \text{ m/sec}$$

(b) The critical damping is given by

$$C_c = 2\sqrt{km}$$
$$= 2\sqrt{294000 \times 600}$$
$$= 26563 \text{ N-sec/m}$$

(c) The natural frequency ω_n is given as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{294000}{600}} = 22.1 \text{ rad/sec}$$

Time period $$= \frac{2\pi}{\omega_n} = \frac{2\pi}{22.1} = 0.283 \text{ sec}$$

$$\text{Time for recoil} = \frac{1}{4}\cdot\text{Time period} = \frac{1}{4} \times 0.383 = 0.071 \text{ sec}$$

For critical damping the displacement is given as

$$x = (A_1 + A_2t)e^{-\omega_n t}$$

The initial conditions are given as

$$\left.\begin{array}{l} x = 1.3 \\ \dot{x} = 0 \end{array}\right\} \text{ at } t = 0$$

$$1.3 = A_1$$
$$0 = A_2 - A_1\omega_n$$
$$A_2 = 28.7$$

So the equation of motion can be written as

$$x = (1.3 + 28.7t)e^{-22.1t}$$

We have to find the value of t when we are given $x = 0.05$ m

Solving it $t = 0.22$ sec

Solving the above equation for t when $x = 0.05$ m by hit and trial method :

t	$1.3 + 28.7t$	$e^{-22.1t}$	x
0.10	4.17	0.11	0.460
0.20	7.05	0.012	0.085
0.21	7.33	0.0096	0.071
0.22	7.63	0.0077	0.049

$\therefore$ t can be approximately taken as 0.22 seconds.

Total time = 0.22 + 0.071 = 0.29 sec

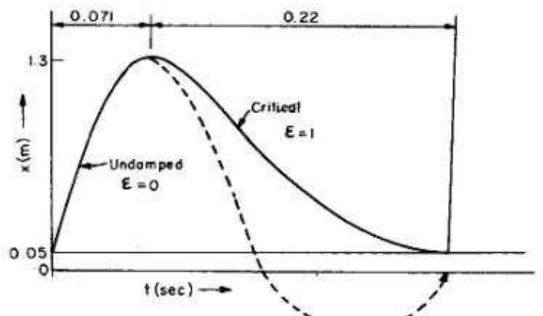


Fig. 3.20 (a).

EXAMPLE 3.23. *A mass of 1 kg is attached to a spring having a stiffness of 3920 N/m. The mass slides on a horizontal surface, the coefficient of friction between mass and surface being 0.1. Determine the frequency of vibrations of the system and the amplitude after one cycle if the initial amplitude is 0.25 cm.* (P.U., 90)

SOLUTION. $m = 1$ kg, $\mu = 0.1$

$$k = 3920 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3920}{1}} = 62.6 \text{ rad/sec}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{62.6}{2\pi} = 9.97 \text{ Hz}$$

Force $$= \mu\,mg = .1 \times 1 \times 9.8 = .98 \text{ N}$$

Reduction in amplitude/cycle

$$= \frac{4F}{k} = \frac{4 \times .98}{3920} = .001 \text{ m}$$

Finally the amplitude

$$= .0025 - .001 = .0015\text{m}$$

EXAMPLE 3.24. *Determine the power required to vibrate a spring mass system with an amplitude of 15 cm and at a frequency of 100 Hz. The system has a damping factor 0.05 and a damped natural frequency of 22 Hz as found out from the vibration record. The mass of the system is 0.5 kg.* (P.U., 94)

SOLUTION. Energy dissipated/cycle

$$= \pi C\omega X^2 \text{(N-m)}$$
$$\omega = 2\pi \times 100 \text{ rad/sec} = 200\pi \text{ rad/sec}$$
$$X = 0.15 \text{ m (Amplitude)}$$
$$\varepsilon = .05$$

when ε is very small, we can take

$$C = C_c\varepsilon$$
$$C = 2m\omega_n\varepsilon$$
$$= 2 \times .5 \times 44\pi \times .05 \qquad \omega_n = 2\pi \times 22$$
$$= 6.90 \text{ N-sec/m} \qquad = 44\pi \text{ rad/sec}$$

Energy dissipated per cycle

$$= \pi \times 6.90 \times 200\pi \times .15 \times .15$$
$$= 306.5 \text{ N-m}$$

EXAMPLE 3.25. *A torsional pendulum when immersed in oil indicates its natural frequency as 200 Hz. But when it was put to vibration in vacuum having no damping, its natural frequency was observed as 250 Hz. Find the value of damping factor of the oil.*

SOLUTION. The expression for torsional vibrations in vacuum ($\varepsilon = 0$) is

$$I\ddot{\theta} + k\theta = 0$$
$$\ddot{\theta} + \frac{k}{I}\theta = 0$$
$$\omega_n = \sqrt{\frac{k}{I}} \text{ rad/sec}$$
$$f_n = \frac{\omega_n}{2\pi} = 250$$
$$\omega_n = 2\pi \times 250 = 500\pi \text{ rad/sec}$$

For vibration when immersed in oil

$$f_d = 200 \text{ Hz}$$

So $\omega_d = 200 \times 2\pi = 400\pi$ rad/sec

$$\omega_d = \sqrt{1-\varepsilon^2}\ \omega_n$$

$$400\ \pi = \sqrt{1-\varepsilon^2}\ 500\ \pi$$

$$.8 = \sqrt{1-\varepsilon^2}$$

or $.64 = 1 - \varepsilon^2$

$$\varepsilon^2 = 1 - .64 = .36$$

$$\varepsilon = 0.6$$

EXAMPLE 3.26. *The disc of a torsional pendulum has a moment of inertia of 600 kg-cm² and is immersed in a viscous fluid. The brass shaft attached to it is of 10 cm diameter and 40 cm long. When the pendulum is vibrating, the observed amplitudes on the same side of the rest position for successive cycles are 9°, 6° and 4°. Determine*

(a) logarithmic decrement

(b) damping torque at unit velocity, and

(c) the periodic time of vibration.

Assume for the brass shaft,

$$G = 4.4 \times 10^{10}\ N/m^2. \qquad \text{(M.D.U., 93)}$$

SOLUTION. (*a*) The logarithmic decrement is given by

$$\delta = \ln\frac{\theta_1}{\theta_2} = \ln\frac{9}{6} = \ln 1.5$$

$$\delta = 0.405$$

(*b*) $\delta = \dfrac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}}$

or $\delta^2(1-\varepsilon^2) = 4\pi^2\varepsilon^2$

$$(.405)^2\,(1-\varepsilon)^2 = 4\pi^2\varepsilon^2$$

$$.164\,(1-\varepsilon^2) = 39.43\varepsilon^2$$

$$\varepsilon = 0.0645$$

We know that, $\varepsilon = C/C_c$

and $C_c = 2\sqrt{k_t I}$

$$I = 600 \text{ kg-cm}^2 = 0.06 \text{ kg-m}^2$$

$$k_t = \frac{GI_P}{l} = \frac{G}{l}\frac{\pi}{32}d^4$$

$$= \frac{4.4\times10^{10}}{0.4}\times\frac{\pi}{32}\times(.1)^4$$

$$C = \varepsilon C_c = 0.0645 \times 2\sqrt{k_t I}$$

$$= 0.0645 \times 2\sqrt{1.08\times10^6\times0.06}$$

$$= 32.83 \text{ N-m/rad}$$

(*c*) $T = \dfrac{2\pi}{\omega_d} = \dfrac{2\pi}{\omega_n\sqrt{1-\varepsilon^2}}$ = Periodic time

$$\omega_n = \sqrt{\frac{k_t}{I}} = \sqrt{\frac{1.08\times10^6}{.06}} = 4242.6 \text{ rad/sec}$$

$$T = \frac{2\pi}{4242.6\sqrt{1-(.0645)^2}}$$

$$= 1.48\times10^{-3} \text{ sec.}$$

EXAMPLE 3.27. *The system shown in fig 3.21 is displaced from its static equilibrium position to the right a distance of 0.01 m. An impulsive force acts towards the left on the mass at the instant of its release to give it an initial velocity V_0 in that direction. If the system has the following parameters :*

$$k = 15700\ N/m,\ c = 1570\ N\text{-}sec/m,\ m = 9.8\ kg$$

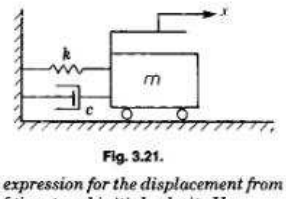


Fig. 3.21.

(a) Derive the expression for the displacement from the equilibrium position in terms of time t and initial velocity V_0.

(b) What value of V_0 would be required to make the mass pass the position of the static equilibrium 1/100 sec after it is applied ? (M.D.U., 95)

SOLUTION. (*a*) The equation of motion for the system shown in figure 3.21 can be written as

$$m\ddot{x} + c\dot{x} + kx = 0$$

The natural frequency of the system can be determined

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{15700}{9.8}} = 40 \text{ rad/sec}$$

The damping factor $\varepsilon = \dfrac{c}{2m\omega_n} = \dfrac{1570}{2\times9.8\times40} = 2.0$

The value of ε is more than one, so it is an overdamped system, its equation (3.3.11) can be written as :

$$x = A_1 e^{[-\varepsilon+\sqrt{\varepsilon^2-1}]\omega_n t} + A_2 e^{[-\varepsilon-\sqrt{\varepsilon^2-1}]\omega_n t}$$

Substituting the values of ε and ω_n in the above equation, we get

$$x = A_1 e^{[-2+\sqrt{2^2-1}]40t} + A_2 e^{[-2-\sqrt{2^2-1}]40t}$$

$$x = A_1 e^{[-2+\sqrt{3}]40t} + A_2 e^{[-2-\sqrt{3}]40t}$$

$$= A_1 e^{-10.71t} + A_2 e^{-149.3t}$$

The initial condition are given as

(*i*) $x = 0.01$ at $t = 0$

(*ii*) $\dot{x} = -V_0$ at $t = 0$

$$\dot{x} = \frac{dx}{dt} = -10.71 A_1 e^{-10.71} - 149.3 A_2 e^{-149.3t}$$

Applying the conditions, we get

$$.01 = A_1 + A_2$$

$$-V_0 = -10.71 A_1 - 149.3 A_2$$

So $A_1 = -7.2\times10^{-3} V_0 + .0107$

$$A_2 = 7.2\times10^{-3} V_0 - .0007$$

Now the equation of displacement can be written by substituting the values of A_1 and A_2 in above equation, so

$$x = (-7.2\times10^{-3} V_0 + .0107)e^{-10.71t} + (7.2\times10^{-3} V_0 - .0007)e^{-149.3t}$$

This is the general expression.

(*b*) Static equilibrium means $x = 0$ at $t = 1/100$ sec. Substituting the values $x = 0$ and $t = 1/100$ in the general expression, we get

$$0 = (-7.2\times10^{-3} V_0 + .0107)(0.8984) + (7.2\times10^{-3} V_0 - .0007)\times.2246$$

Solving it $V_0 = 2.01$ m/sec

EXAMPLE 3.28. *The successive amplitudes of vibrations of vibratory system as obtained under free vibration are 0.69, 0.32, 0.19, 0.099 units respectively. Determine the damping ratio of the system.* (P.U., 95)

SOLUTION. Logarithmic decrement δ is given as

$$\delta = \frac{1}{3}\log_e\left(\frac{0.69}{0.099}\right) = 0.647$$

But we know the relation for damping ratio

$$\delta = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}}$$

$$\delta^2 = \frac{4\pi^2\varepsilon^2}{1-\varepsilon^2}$$

$$(.647)^2 = \frac{4\pi^2\varepsilon^2}{1-\varepsilon^2} \Rightarrow .4186 - .4186\,\varepsilon^2 = 39.438\,\varepsilon^2$$

$$39.8566\,\varepsilon^2 = .4186$$

$$\varepsilon = 0.102.$$

Note. When the individual successive amplitude ratios are not same, we take the logarithmic decrement for the first and the last amplitude i.e. over the entire range of available data. In this case there are three cycles ($n = 3$).

EXAMPLE 3.29. *A shock absorber is to be designed so that its overshoot is 10% of the initial displacement when released. Determine the damping factor. If the damping factor is reduced to one half this value, what will be the overshoot ?* (A.M.I.E., 1993)

SOLUTION. Let us assume that the damper of shock absorber is of viscous type, logarithmic decrement δ is given by

$$\delta = \log_e\left(\frac{x_1}{x_2}\right) = \log_e(10) = 2.30258$$

Also we know that, $\delta = \dfrac{2\pi\xi}{\sqrt{1-\xi^2}}$ ...(*i*)

If ξ is very small $\delta = 2\pi\xi$ or $\xi = \dfrac{\delta}{2\pi} = \dfrac{2.30258}{2\pi} = 0.366$

Exact value of ξ can be determined with the help of equation (*i*)

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \quad\text{or}\quad \delta^2 = \frac{4\pi^2\xi^2}{1-\xi^2}$$

or $(2.30258)^2 = \dfrac{4\pi^2\xi^2}{1-\xi^2}$

or $\xi = 0.344$

Again, damping factor reduced to half

$$\xi' = \frac{\xi}{2} = \frac{0.344}{2} = 0.172$$

Then $\delta' = \dfrac{2\pi\xi'}{\sqrt{1-\xi'^2}} = \dfrac{2\pi\times0.172}{\sqrt{1-(.172)^2}}$

$$\delta' = 1.095$$

Now overshoot, $\delta' = \log_e\left(\dfrac{x_1}{x'_2}\right)$

$$1.095 = \log_e(x_1/x_2'),\ \ln 1.095 = \frac{x_1}{x'_2}$$

or $x_2' = \dfrac{x_1}{[illegible]} = x_1\times0.3345 = 33.45\%$

Problems

1. Prove that for the damped spring-mass system, the peak amplitude occurs at a frequency ratio given by the expression.

$$\left(\frac{\omega}{\omega_n}\right)_p = \sqrt{1-2\varepsilon^2}$$

2. For a system having viscous damping, plot a curve for the number of cycles elapsed for the amplitude to decay to 50% of the initial value, against the damping factor.

3 A body of mass 1500 kg is suspended on a leaf spring. The system was set into vibration and the frequency of vibration was measured as 0.982 Hz. The successive amplitudes were measured to be 4.8 cm, 4.1 cm, 3.4 cm, 2.7 cm. Determine the spring stiffness and the coulomb damping.

4. Show that for viscous damping, the loss factor is independent of the amplitude and proportional to the frequency.

5. A transformer of 3000 kg mass is mounted on an isolator embodying a coil spring and a coulomb damper. Design the isolator, so that the natural frequency of the system is 4.5 Hz and an initial displacement of 5 mm given to the transformer dies out completely in three cycles.

6. The mass of a vibrating system weighs 20 N and is made to vibrate in a viscous medium. Determine the damping ratio and damping coefficient. When a harmonic exciting force of 30 N results in a resonant amplitude of 15 mm with a period of 0.2 sec.

7. Derive the differential equations of motion for the system shown in figure 3.1 P

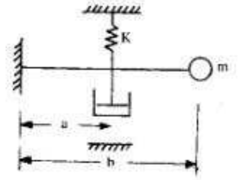


Fig. 3.1 P

8. A gun barrel weighing W kg has a recoil spring whose stiffness is k kg/m. If $W = 450$ kg, $k = 36000$ kg/cm and the barrel recoils 1 m on firing determine.

(a) The initial recoil velocity of the barrel

(b) The critical damping coefficient of a dashpot which is engaged at the end of the recoil stroke.

(c) The time required for the barrel to return to a position 5 cm from its initial position.

9. Show that, for small amounts of damping, the damping ratio ε can be expressed as

$$\varepsilon = \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1}$$

where ω_2 and ω_1 denote the frequencies on either side of the resonance where the amplitude is $\left(\frac{1}{\sqrt{2}}\right)$ of its maximum value.

10. A mass of 3 kg is supposed on an isolator having a spring constant of 3000 N/m and viscous damping. If the amplitude of free vibration of the mass falls to one half its original value in 2 sec., determine the damping coefficient of the isolator.

11 A slab door, 2 m high, 0.75 m wide, 40 mm thick and with a mass of 36 kg, is fitted with an automatic door closer. The door opens against a torsion spring with a modulus of 10 N-m/radian. Determine the necessary damping to critically dampen the return swing of the door. If the door is opened 90° and released, how long will it take until the door is within 1° of closing ?

12. In the figure 3.2 P, A is frictionless hinge about which the massless-rigid bar AB of length l rotates. Find the natural frequency of the system for small deflection of the mass m.

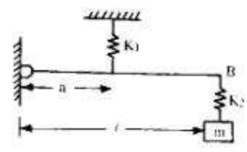


Fig. 3.2 P

13. A mass of 4.5 kg hangs from a spring and makes damped vibrations. The time of 50 complete oscillations is found to be 20 sec and the ratio of first downward displacement to the sixth is found to be 2.25. Find the stiffness of the spring in kN/m and the damping force in N/m/sec. (K.U.)

14. The mass of a machine is 100 kg. Its vibrations are damped by a viscous dashpot which diminishes amplitude of vibrations from 4 cm to 1 cm in three complete oscillations. If the machine is mounted on four springs each of stiffness 250 Newton per cm, and (i) resistance of the dashpot at unit velocity and (ii) the periodic time of the damped vibrations.

(I.A.S., 92)

15. A machine has a mass of 200 kg. It is placed on two different isolators and the corresponding free vibrating records are shown in figures 3.3 (a) P and

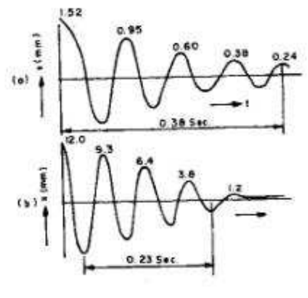


Fig. 3.3 P.

3.3 (b) P. Determine in each case the type of damping and its characteristics. Also determine the undamped natural frequencies.

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4

Forced Vibration

4.1. INTRODUCTION

There are vibrations the amplitude of which is maintained almost constant due to the application of external forces. We can take the case of ringing electric bell, it rings so long as the electric supply is there. But as soon as the electric supply is off, the functioning of the bell is stopped. Sound is caused because of vibration. Ringing of bell means the amplitude of vibration is maintained almost constant throughout the operation. This type of vibration which occurs under the influence of external force, is called forced vibration. Machine tools during operations have this type of vibrations. In free vibrations the oscillations die out in course of time due to energy dissipation by damping. The time taken depends upon the amount of damping present in the system.

The external force keeps the system vibrating. This force is called external excitation. The excitation may be periodic, impulsive or random in nature. Again the periodic force may be harmonic and non-harmonic. Vibrations because of impulsive forces are called transient. Earthquake is because of random forces.

In this chapter, we discuss the application of harmonic excitation, its effects on the system and vibration measuring instruments. Solved numerical problems are presented in the end.

4.2 SOURCES OF EXCITATION

The external excitation to a system can be easily detected. This excitation is in the form of motion and so produced by one dynamic system to another. Both such systems are connected together rigidly and form one dynamic system having several degrees of freedom.

Another excitation is internal and occurs due to unbalance in the system. There are various reasons of unbalance in the system few of which are listed here.

(a) **Thermal effects.** Different members of the machine are made of metals having different coefficient of thermal expansion. The heating of such members during machining operation give rise to unbalance due to variations in the rate of heat transfer and expansion on the interfaces of the adjacent members.

(b) **Resonance.** During resonance of a unit or a part large amplitudes are produced which give rise to the unbalance.

(c) **Loose or defective mating part.** During the operation, the loose or defective part may lead to resonance which is responsible for unbalance. Similarly, coupling misalignment or defective assembly or installation may be the cause of unbalance.

(d) **Bent shaft.** The bent shaft is prone to failure due to unbalance caused at the critical speeds.

(e) **Bearing or journal defects.** These defects such as loose parts, improper clearance, surface roughness, out of roundness and excessive and play of shaft give rise to unbalance in the system.

(f) **Variation in turning moment of the engine.** We have studied the turning moment diagrams of two stroke and four stroke engines. The torque variation is periodic. There is uneven power supply during the strokes which is responsible for unbalance e.g. punching operation in a punching press.

(g) **Mass of rotating parts not distributed uniformly.** It causes eccentricity of the mass and ultimately the unbalance or centrifugal force are produced in the system.

(h) **Magnetic effects.** The various stray magnetic fields in the vicinity of the machine elements and the various currents flowing through them, are the reasons of unbalance.

4.3. EQUATIONS OF MOTION WITH HARMONIC FORCE

Consider a spring mass system with viscous damping, while a harmonic force of frequency ω and amplitude F acts on it, as shown in figure 4.1.

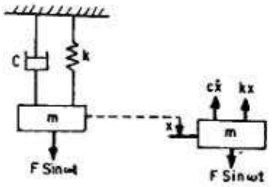


Fig. 4.1. Force vibration with viscous damping.

The mass m is displaced from its equilibrium position by a distance x in the downward direction. The mass m is put to three forces spring force kx, damping force $c\dot{x}$ and harmonic excitation $F \sin \omega t$. The direction of these forces is shown in figure 4.1.

The equation of motion can be written as

$$m\ddot{x} + c\dot{x} + kx = F \sin \omega t \qquad ...(4.3.1)$$

This is second order linear differential equation with constant coefficient. The general solution of the above equation is of the form

$$x = x_c + x_p \qquad ...(4.3.2)$$

where x_c = complementary solution

x_p = particular solution

x_c is the solution of the homogeneous equation

$m\ddot{x} + c\dot{x} + kx = 0$ which we have already discussed in the previous chapter in detail. A solution $x_p = A \sin (\omega t - \phi)$ ....(4.3.3)

can be assumed where A is the amplitude of vibration and ϕ is the phase of the displacement with respect to harmonic force.

$$\dot{x}_p = \omega A \cos (\omega t - \phi)$$
$$= \omega A \sin (\omega t - \phi + \pi/2)$$
$$\ddot{x}_p = \omega^2 A \sin (\omega t - \phi + \pi) \qquad ...(4.3.4)$$

Substituting equation (4.2.4) in equation (4.2.1), we get

$$m\omega^2 A \sin (\omega t - \phi + \pi) + c\omega A \sin (\omega t - \phi + \pi/2) + kA \sin (\omega t - \phi) - F \sin \omega t = 0 \qquad ...(4.3.5)$$

However, x_c vanishes because of damping with time. Equation (4.3.5) represents four forces namely inertia force, damping force, spring force and harmonic force. By the application of these forces, the system is supposed to be in equilibrium. The vector diagram coming out from this equation (4.3.5) is drawn in figure 4.2. It is shown in the figure that spring force is perpendicular to damping force and damping force is perpendicular to inertia force.

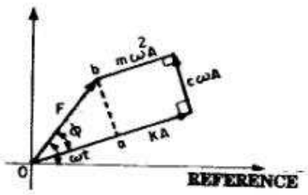


Fig. 4.2. Vector representation of forced vibration with damping.

From figure 4.2, let us consider triangle oab

$$F^2 = (kA - m\omega^2 A)^2 + (c\omega A)^2$$

$$A = \frac{F}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$= \frac{F/k}{\sqrt{\left(\frac{k}{k} - \frac{m\omega^2}{k}\right)^2 + \left(\frac{c\omega}{k}\right)^2}}$$

$$= \frac{F/k}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{c\omega}{k}\right)^2}} \qquad ...(4.3.6) \quad (\text{Putting } \omega_n^2 = m/k)$$

and $\tan \phi = \frac{ab}{oa} = \frac{c\omega A}{kA - m\omega^2 A} = \frac{c\omega}{k - m\omega^2}$

$$= \frac{c\omega/k}{1 - \frac{\omega^2}{\omega_n^2}} \qquad ...(4.3.7)$$

We know that $\frac{c\omega}{k} = \frac{c}{c_c}\left(\frac{c_c}{2m}\right)\left(\frac{2m}{k}\right) . \omega$

$$= \varepsilon\, \omega_n \frac{2}{\omega_n^2} . \omega = 2\,\varepsilon \frac{\omega}{\omega_n}$$

Putting this value in equation (4.3.7), we get

$$\tan \phi = \frac{2\varepsilon\, \omega/\omega_n}{1 - (\omega/\omega_n)^2} \qquad ...(4.3.8)$$

Again equation (4.3.6) can be written as

$$A = \frac{F/k}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [(2\varepsilon\omega/\omega_n)^2]}} \qquad ...(4.3.9)$$

Let us assume that $F/k = X_s$ where X_s is called zero frequency deflection. The above equation can be put as

$$A = \frac{X_s}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\varepsilon\omega/\omega_n)^2]}}$$

or

$$\frac{A}{X_s} = \frac{1}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [(2\varepsilon\omega/\omega_n)^2]}} \qquad ...(4.3.10)$$

The non-dimensional quantity A/X_s is known as magnification factor or amplitude ratio.

So the particular solution of equation (4.3.3) can be written as

$$x_p = X_s \frac{\sin (\omega t - \phi)}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [(2\varepsilon\omega/\omega_n)^2]}} \qquad ...(4.3.11)$$

The complete solution can be written as

$$x = x_c + x_p$$

The value of x_c can be taken from equation (3.3.17)

So x can be written as

$$x = A_1 e^{-\varepsilon\omega_n t} \cos (\sqrt{1 - \varepsilon^2}\, \omega_n t + \phi_1)$$
$$+ \frac{X_s \sin (\omega t - \phi)}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [(2\varepsilon\, \omega/\omega_n)^2]}} \qquad ...(4.3.12)$$

The values of constant A_1 and ϕ_1 can be determined from initial conditions.

The frequency at which maximum amplitude occurs can be obtained by differentiating equation (4.3.10) with respect to (ω/ω_n) and equating the differential to zero.

$$\frac{d(A/X_s)}{d(\omega/\omega_n)} = \frac{2[1 - (\omega/\omega_n)^2]\,[-2(\omega/\omega_n)] + 2[2\varepsilon\, \omega/\omega_n]\,[2\varepsilon]}{2\,\{[1 - (\omega/\omega_n)^2]^2 + [2\varepsilon\, \omega/\omega_n]^2\}^{3/2}}$$
$$= 0$$

which leads to $\frac{\omega_{max}}{\omega_n} = \sqrt{1 - 2\varepsilon^2}$ ...(4.3.13)

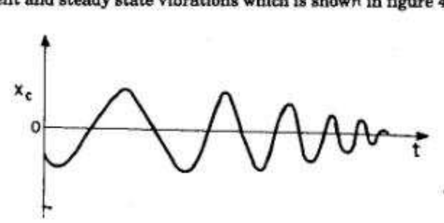
where ω_{max} is the frequency corresponding to the maximum amplitude. At resonance $\omega = \omega_n$, putting this value in equation (4.3.10)

$$\frac{A}{X_s} = \frac{1}{2\varepsilon} \qquad ...(4.3.14)$$

4.3.1. Total Response

The first part of equation (4.3.12) vanishes with time while the second part remains into existence. The amplitude remains constant due to second part and it is called steady vibration. The vibration because of first part is called transient and it occurs at the damped natural frequency of the system.

The complete solution of equation (4.3.12) is the superposition of transient and steady state vibrations which is shown in figure 4.3.



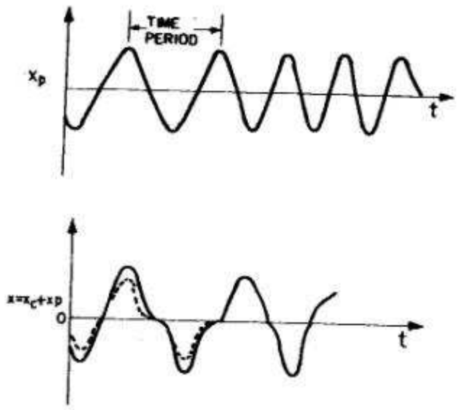


Fig. 4.3.

4.3.2. Characteristic Curves

The ratio ω/ω_n is called the frequency ratio where ω is the frequency of excitation. Similarly, A/X_s is known as magnification factor or amplitude ratio.

A curve between frequency ratio and magnification factor is known as frequency response curve. Similarly a curve between phase angle and frequency ratio is known as phase-frequency response curve. Both curves which are drawn with the help of equations (4.3.10) and (4.3.8) as shown in figure 4.4.

The following points are noted from these equations and figures :

1. At zero frequency magnification is unity and damping does not have any effect on it.
2. Damping reduces the magnification factor for all values of frequency.
3. The maximum value of amplitude occurs a little towards left of resonant frequency.
4. At resonant frequency the phase angle is 90°.
5. The phase angle increases for decreasing value of damping above resonance.
6. The amplitude of vibration is infinite at resonant frequency

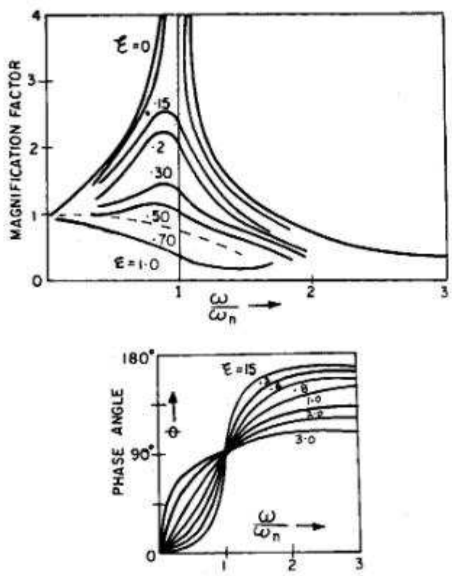


Fig. 4.4.

7. The amplitude ratio is below unity for all values of damping which are more than 0.70.
8. The variation in phase angle is because of damping. Without damping it is either 180° or 0.

4.3.3. Variation of Frequency Ratio ω/ω_n

There are three possibilities of ω variation *i.e.*, $\omega < \omega_n$, $\omega = \omega_n$ and $\omega > \omega_n$. This variation of ω will affect the magnitude of various forces acting on the system. The three cases are discussed here.

(*a*) $\omega/\omega_n << 1$

We know that inertia and damping forces are given by the expressions $m\omega^2 A$ and $c\omega A$ respectively. It is clear that both the terms are affected by ω. When the value of ω is very small, inertia and damping forces are reduced considerably

angle is reduced and F is balanced by spring force kA which are almost equal and opposite in magnitude. See figure 4.5 (*a*).

(*b*) $\omega/\omega_n = 1$

When the frequency of excitation ω increases and becomes equal to the natural frequency ω_n, resonance occurs. The phase angle becomes 90°. Inertia force and spring force are found to be equal and opposite. Excitation force balances the damping force *i.e.* $c\omega A = F$. Thus giving the amplitude of vibration at resonance

$$A = F/c\omega \qquad \text{...(4.3.3.1.)}$$

See figure 4.5(*b*).

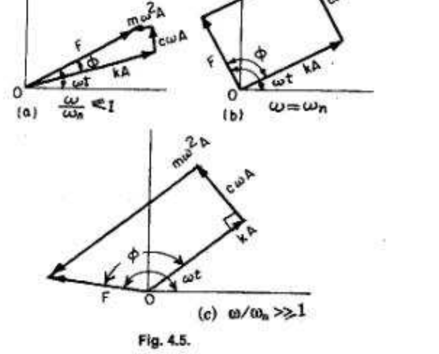


Fig. 4.5.

(*c*) $\omega/\omega_n >> 1$

At very high frequencies ω, inertia force increases very rapidly and its magnitude is very large. Damping and spring force are small in magnitude. When the value of ω/ω_n is very high, the phase angle ϕ is very close to 180°. See figure 4.5 (*c*).

4.4. RESPONSE OF A ROTATING AND RECIPROCATING UNBALANCE SYSTEM

A machine having rotor as one of its components is called a rotating machine, like turbines and I.C. engines. The problem of un-

coincide with the axis of rotation. The distance between axis of rotation and the centre of gravity is called eccentricity e and the mass acting at a distance e from the axis of rotation is known as eccentric mass m_0. A rotating machine supported on springs is shown in figure 4.6. The mass of the machine including the eccentric mass is m and the angular speed of rotor as ω rad/sec.

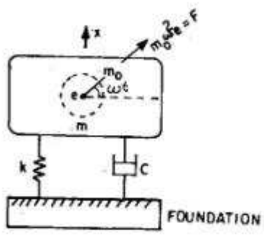


Fig. 4.6. Rotating unbalance.

k and c being the spring stiffness and damping coefficient respectively. The system may be assumed to have single degree of freedom as it can move in one direction only. Let the eccentric mass m_0 make an angle ωt with the reference axis. At any moment vertical displacement is given as $x + e \sin \omega t$. The centrifugal force of eccentric mass will be $m_0\,\omega^2 e$. This force can be resolved into two components, *i.e.*, vertical and horizontal. Vertical component is of much significance which acts as equivalent excitation and can be expressed as $m_0\, e\, \omega^2 \sin \omega t$.

The differential equation of motion can be written as

$$(m - m_0)\frac{d^2x}{dt^2} + m_0\frac{d^2}{dt^2}(x + e\sin\omega t) + kx + c\frac{dx}{dt} = 0$$

$$(m - m_0)\,\ddot{x} + m_0\,\ddot{x} - m_0\omega^2 e\sin\omega t + kx + c\dot{x} = 0$$

or

$$m\ddot{x} + c\dot{x} + kx = m_0\omega^2 e\sin\omega t \qquad \text{...(4.4.1)}$$

This equation represents damped forced vibration. Comparing this equation with equation (4.3.1), we see that $m_0\omega^2 e = F$ and rest of the things are same. $m_0 e\omega^2$ acts as the excitation force on the system.

To find the amplitude A and phase angle ϕ, above equation can be solved like equation (4.3.1).

$$A = \frac{m_0 e\omega^2/k}{\sqrt{[\text{illegible}]}} \cdot$$

$$\frac{A}{\frac{m_0 e}{m}} = \frac{(\omega/\omega_n)^2}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [(2\varepsilon\,\omega/\omega_n)^2]}} \qquad \left(\because \omega_n^2 = \frac{k}{m}\right) \qquad ...(4.4.2)$$

At resonance $\omega = \omega_n$

$$\frac{A}{\frac{m_0 e}{m}} = \frac{1}{2\varepsilon} \qquad ...(4.4.3)$$

The phase angle can be written as

$$\tan\phi = \frac{2\varepsilon(\omega/\omega_n)}{1-(\omega/\omega_n)^2} \qquad ...(4.4.4)$$

The complete solution in this case can be written as

$$x = x_c + x_p$$
$$= A_1 e^{-\varepsilon\omega_n t}\cos\left(\sqrt{1-\varepsilon^2}\,\omega_n t + \phi_2\right)$$
$$+ \frac{m_0 e\,\omega^2/k}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\varepsilon\frac{\omega}{\omega_n}\right]^2}} \qquad ...(4.4.5)$$

The variation of $A \dfrac{m_0 e}{m}$ with frequency ratio for various values of damping factor is shown in figure 4.7 while the curve between the phase angle and frequency ratio is the same as shown in figure 4.4. The following points are concluded from this equation and figure :

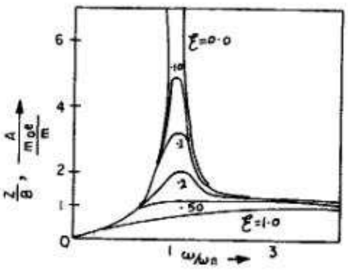


Fig. 4.7. Amplitude frequency response curve.

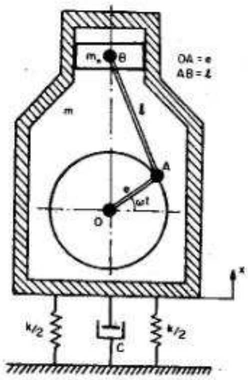
1. Damping factor plays very important role during resonance

2. When the value of ω is very small as compared to ω_n, it is known as low speed system. For a low speed system the value of $\left(\frac{A}{\frac{m_0 e}{m}}\right) \longrightarrow 0$.
3. Similarly, for a high speed system ω is very high, then $\left(\frac{A}{\frac{m_0 e}{m}}\right) \longrightarrow 1$.
4. At very high speed the effect of damping seems to be negligible.
5. A comparison between figures 4.4 and 4.7 shows that peak amplitude occurs to the right of resonance in figure 4.7 and on the contrary the peak amplitude falls to the left of resonance in figure 4.4.
6. At resonance $\omega = \omega_n$ and so

$$\frac{A}{\frac{m_0 e}{m}} = \frac{1}{2\varepsilon}$$

$$A_{resonance} = \frac{m_0 e}{2m\varepsilon}$$

Considering a reciprocating engine as shown in figure 4.7 (a), let the equivalent mass of the reciprocating part be m_0 and the total mass



of the engine including the reciprocating part is m. The crank length and the connecting rod length are e and l respectively.

Inertia force due to reciprocating mass m_0 is approximately equal to $F = m_0 e\omega^2\left[\sin\omega t + \left(\frac{e}{l}\right)\sin 2\omega t\right]$. If e is very small as compared to l, second harmonic of the above equation i.e. $\frac{e}{l}\sin 2\omega t$ part can be neglected and the exciting force F becomes equal to $m_0 e\,\omega^2 \sin\omega t$ which is the same as that for rotating unbalance. Hence, for small value of e, the previous analysis of rotating unbalance is applicable to this also.

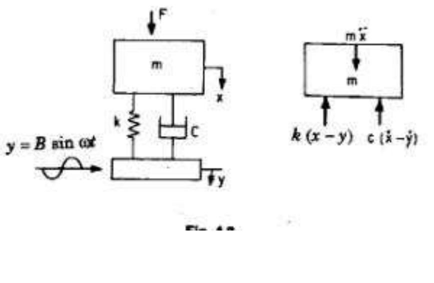
4.5. SUPPORT MOTION

In case of locomotives or vehicles the wheels act as base or support for the system. The wheels can move vertically up and down on the road surface during the motion of the vehicle.

At the same time there is relative motion between the wheels and the chassis. So chassis is having motion relative to the wheels and the wheels are having motion relative to the road surface. The amplitude of vibration in case of support motion depends on the speed of vehicle and nature of road surface. The vibration measuring instruments are designed on the support motion approach. Such systems are supposed to have single degree of freedom for the simplicity of mathematical expression. In a vibratory system where the support is put to excitation absolute and relative motion become important from subject point of view.

4.5.1. Absolute Motion

Absolute motion of a mass means its motion with respect to the coordinate system attached to the earth. As shown in figure 4.8, the absolute displacement of support is $y = B\sin\omega t$ and the absolute displacement of the mass m from its equilibrium position is x. The displacement of mass m relative to the support is z. The net elongation of the spring is $(x-y)$ and the relative motion between the two ends of the damper is $(\dot{x}-\dot{y})$. Then $z = x - y$ and $\dot{z} = \dot{x} - \dot{y}$.



The equation of motion can be written as

$$m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$$

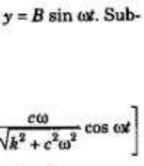
or $$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \qquad ...(4.5.1.1)$$

The support is subjected to harmonic vibration, $y = B\sin\omega t$. Substituting this value of y in equation (4.5.1.1), we get

$$m\ddot{x} + c\dot{x} + kx = cB\omega\cos\omega t + kB\sin\omega t$$
$$= B[k\sin\omega t + c\omega\cos\omega t]$$
$$= B\sqrt{k^2+c^2\omega^2}\left[\frac{k}{\sqrt{k^2+c^2\omega^2}}\sin\omega t + \frac{c\omega}{\sqrt{k^2+c^2\omega^2}}\cos\omega t\right]$$
$$= B\sqrt{k^2+c^2\omega^2}\,[\cos\alpha\sin\omega t + \sin\alpha\cos\omega t]$$
$$m\ddot{x} + c\dot{x} + kx = B\sqrt{k^2+c^2\omega^2}\,\sin(\omega t+\alpha) \qquad ...(4.5.1.2)$$

and $$\tan\alpha = \frac{c\omega}{k}$$

where $$\alpha = \tan^{-1}\frac{c\omega}{k} = \tan^{-1}(2\varepsilon\,\omega/\omega_n) \qquad ...(4.5.1.3)$$



Steady state solution can be written as

$$x = A\sin(\omega t + \alpha - \phi) \qquad ...(4.5.1.4)$$

Comparing equation (4.5.1.2) with equation (4.3.1), we see that

$$F = B\sqrt{k^2+c^2\omega^2}.$$

Let us go back to equation (4.3.9) *i.e.*

$$A = \frac{F/k}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [(2\varepsilon\,\omega/\omega_n)^2]}}$$

Steady state amplitude can be written from equation (4.5.1.2) as

$$A = \frac{B\sqrt{k^2+c^2\omega^2}/k}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\varepsilon\,\omega/\omega_n]^2}}$$

$$\frac{A}{B} = \frac{\sqrt{1+(2\varepsilon\,\omega/\omega_n)^2}}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [(2\varepsilon\,\omega/\omega_n)^2]}} \qquad ...(4.5.1.5)$$

$$\frac{F}{k} = \text{static deflection}\left(\because \frac{c\omega}{k} = \frac{C}{C_c}\left(\frac{C_c}{2m}\right)\left(\frac{2m}{k}\right)\omega = \varepsilon\,\omega_n\frac{2}{\omega_n^2}\omega = 2\varepsilon\frac{\omega}{\omega_n}\right)$$

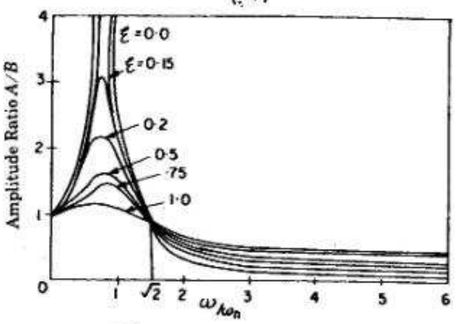
The ratio $\frac{A}{B}$ is called the displacement transmissibility which is the ratio of the amplitude of the body to the amplitude of the support.

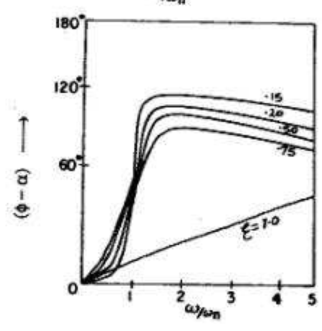
We know that $\tan\alpha = \frac{c\omega}{k}$

or $$\alpha = \tan^{-1}\frac{c\omega}{k} = \tan^{-1} 2\varepsilon\,\omega/\omega_n$$

and $$\tan\phi = \frac{2\varepsilon\,\omega/\omega_n}{1-(\omega/\omega_n)^2}$$

$$\phi = \tan^{-1}\frac{2\varepsilon\,\omega/\omega_n}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$





So phase $(\phi - \alpha)$ can be written as

$$\phi - \alpha = \tan^{-1}\left[\frac{2\varepsilon\,\omega/\omega_n}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] - \tan^{-1}\left(2\varepsilon\frac{\omega}{\omega_n}\right) \quad ...(4.5.1.6)$$

With the help of equations (4.5.1.5) and (4.5.1.6) steady state amplitude and phase are shown in figure 4.9. We conclude the following points from this figure :

1. At frequency ratio $\omega/\omega_n = \sqrt{2}$ for all values of damping the amplitude ratio (A/B) is unity.
2. The amplitude ratio is less than one for all values of damping when $\omega/\omega_n > \sqrt{2}$.
3. For high values of frequency ratio $(\omega/\omega_n >> \sqrt{2})$, the amplitude ratio is quite small which means that the support displacement is almost negligible.
4. For high values of frequency ratio the damping does not have any important role so far as the amplitude ratio is concerned.
5. The phase angle is not 90° at resonance as in case of force excitation (see figure 4.4).

4.5.2. Relative Motion

In section (4.5.1), we have assumed that z is the relative displacement of the mass with respect to the support. So it can be written as

$$z = x - y$$

or $$\dot{z} = \dot{x} - \dot{y}$$
$$\ddot{z} = \ddot{x} - \ddot{y} \quad ...(4.5.2.1)$$

Substituting the above relations in equation (4.5.1.1), we get

$$m(\ddot{z} + \ddot{y}) + c\dot{z} + kz = 0$$
$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y}$$

The sinusoidal support excitation is assumed in the system *i.e.* $y = B\sin\omega t$. Substituting the value of y in the above equation, we have

$$m\ddot{z} + c\dot{z} + kz = m\omega^2 B\sin\omega t \quad ...(4.5.2.2)$$

This equation is known as relative equation of motion and it is of the same form as equation (4.4.1) *i.e.*

$$m\ddot{x} + c\dot{x} + kx = m_0\,\omega^2 e\sin\omega t$$

Let us assume that the steady state relative amplitude Z lags the excitation by angle α, so

$$z = Z\sin(\omega t - \alpha)$$

$\frac{Z}{B}$ ratio can be written as

$$\frac{Z}{B} = \frac{(\omega/\omega_n)^2}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\varepsilon\,\omega/\omega_n]^2}} \quad ...(4.5.2.3)$$

and $$\alpha = \tan^{-1}\left[\frac{2\varepsilon\,\omega/\omega_n}{1-(\omega/\omega_n)^2}\right] \quad ...(4.5.2.4)$$

The characteristics of this equation are shown in figures 4.7 and 4.4 for frequency.

The following conclusions are drawn from these figures and equation :

1. When $(\omega/\omega_n) > 3$, the amplitude ratio $\left(\frac{Z}{B}\right)$ is almost unity. It means that the relative amplitude Z and the support amplitude B are equal. This is the principle on the basis of which vibration measuring instruments are designed which will be discussed later on in this chapter.
2. For $\left(\frac{Z}{B}\right)$ ratio being unity, the mass m will be having no displacement.
3. Damping does not have any effect on $\frac{Z}{B}$ ratio for high values of ω/ω_n *i.e.* for $\omega/\omega_n > 3$.

4.6. VIBRATION ISOLATION

The high speed engines and machines when mounted on foundations and supports cause vibrations of excessive amplitude because of unbalanced forces set up during their working. These are the disturbing forces which damage the foundation on which the machines are mounted. So the vibrations transmitted to the foundation should be eliminated or reduced considerably by using some devices such as springs, dampers, etc., between the foundation and the machine. These devices isolate the vibrations by absorbing some disturbing energy themselves and allow only a fraction of it to pass through them to the foundation. Thus the amplitude of vibration is minimised and the adjoining structure or foundation is not put to heavy disturbances. The isolation is expressed in terms of force or motion. Lesser the amount of force or motion transmitted to the foundation greater is said to be the isolation. So machines are mounted on isolation mounts. There are two basic requirements for an isolator : Firstly, there should be no rigid connection between the unit (Machine, engine or vibrating body, etc.) and the base otherwise the undesired vibrations will be completely transmitted from the unit to the base. It may damage the supporting structure.

Secondly, it should be ensured that the isolators remain together in case the damping material (rubber, cork, felt, pad, etc.) fails. It should be just to keep the machine or unit in the safe position with respect to the support.

The materials normally used for vibration isolation are rubber, felt cork, metallic springs, etc. These are put between the foundation and the vibrating body.

Rubber as an isolator is quite useful for shear loading. Its sound transmissibility is very low. Its properties are influenced by heat, gasoline, oil, etc. So it cannot be used at high temperature in the presence of gasoline and oil. It is preferred for light loads and high frequency oscillations.

The damping factor of felt is high so it is specially used for low frequency ratios. Its pads are put between the machine and the support. It is preferred to use many pads at a time rather than a single large pad of the same size.

Cork is suitable for compressive loads. It is not perfectly elastic. At high loads it becomes more flexible.

Metal springs used are of two types namely helical spring and leaf spring. The spring has high sound transmissibility which can be reduced by covering it with pads of felt, cork or rubber. The spring can be used in all working conditions as it is not affected by air, water, oil or usual temperature variation. They are useful for high frequency ratios.

4.7. TRANSMISSIBILITY

Let us consider figure 4.8 for analysis. The spring and damping forces are kx and $c\dot{x}$ respectively.

The amplitude of vibration is A, so the maximum value of these forces will be kA and $c\omega A$ respectively. The forces are perpendicular to each other, because of phase difference being 90° so their resultant F_T is given as

$$F_T = \sqrt{(kA)^2 + (c\omega A)^2} = A\sqrt{(k^2 + c^2\omega^2)} \quad ...(4.7.1)$$

In the above equation F_T is the force transmitted to the foundation. The disturbing force is F. The ratio of F_T to F is called transmissibility. So it can be expressed mathematically, as

$$\text{T.R.} = \frac{F_T}{F} \quad ...(4.7.2)$$

From equation (4.2.9), we have

$$F = Ak\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\varepsilon\,\omega/\omega_n]^2} \quad ...(4.7.3)$$

Substituting the values of F and F_T in equation (4.6.2), we have

$$\text{T.R.} = \frac{A\sqrt{k^2 + c^2\omega^2}}{Ak\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\varepsilon\,\omega/\omega_n]^2}}$$

$$= \frac{\sqrt{1 + (c^2/k^2)\omega^2}}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\,\varepsilon\omega/\omega_n]^2}}$$

$$\text{T.R.} = \frac{F_T}{F} = \frac{\sqrt{1 + (2\varepsilon\,\omega/\omega_n)^2}}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\varepsilon\,\omega/\omega_n]^2}} \qquad \text{...(4.7.4)}$$

The above equation is the same as equation (4.4.1.5), so

$$\frac{F_T}{F} = \frac{A}{B} \qquad \text{...(4.7.5)}$$

If $\frac{F_T}{F} = 0.25$, then the system is said to have 25% transmissibility.

The following informations are available from equation (4.7.4) and figure 4.9 :

1. The area of vibration isolation starts when transmissibility is less than unity and $\omega/\omega_n > \sqrt{2}$. Thus when the frequency ω of exciting force is given, the isolation mounts can be designed such that $\omega/\omega_n > \sqrt{2}$. If ω is small the value of ω_n should be small and the value of natural frequency $\left(\omega_n = \sqrt{\frac{k}{m}}\right)$ results from high value of mass or low value of stiffness. Thus if the disturbing frequency ω is low the isolation can be accomplished by adding certain amount of mass to the system.
2. Damping is important specially at resonance otherwise TR will be excessively large.
3. In the vibration isolation region $\omega/\omega_n > \sqrt{2}$, so equation (4.6.4) in absence of damping can be written as
$$T.R. = \frac{1}{(\omega/\omega_n)^2 - 1}$$
4. Unity value of T.R. occurs at two points where ω/ω_n is zero and $\sqrt{2}$ for all values of damping.
5. Under ideal operating conditions, T.R. must be zero. Hence, ω/ω_n should be as large as possible. Thus, the natural frequency of the system should be small *i.e.* weak spring with fairly heavy mass.

4.8 FORCED VIBRATIONS WITH COULOMB DAMPING

(Refer section 3.2 also)

For a single degree of freedom system with coulomb damping or dry friction damping, subjected to a harmonic force $F \sin \omega t$, the equation of motion is given by

$$M\ddot{x} + kx \pm \mu R_N = F \sin \omega t \qquad \text{...(4.8.1)}$$

An exact solution is available due to early work of J.P. Den Hartog for small damping force so that the motion is continuous. For high value of dry damping force the motion does not remain continuous. If the dry friction force is small compared to harmonic force, an approximate solution is required for this case.

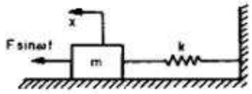


Fig. 4.10.

The sign of frictional force (μR_N) is positive when the mass moves from left to right and negative when the mass moves from right to left. By finding an equivalent viscous damping ratio, we find an approximate solution of eqn. (4.8.1). For viscous damping ratio, we equate the energy dissipated due to dry friction to the energy dissipated by an equivalent viscous damper during a full cycle of motion. The energy dissipated by friction force μR_N in a quarter cycle is $\mu R_N A$. So in a full cycle, the energy dissipated by dry friction damping is given by

$$\Delta E = 4\,\mu\,R_N\,A = 4F_f A \qquad \text{...(4.8.2)}$$

where F_f = Frictional force

A = Amplitude of motion

If equivalent viscous damping is denoted by Ce, the energy dissipated during a full cycle is given by eqn. (3.2.5).

$$\Delta E = \pi\,C_e\,\omega A^2 \qquad \text{...(4.8.3)}$$

Equating the above two equations, we obtain an equivalent viscous damping factor which is measured in terms of dry friction

$$\pi\,C_e\,\omega A^2 = 4\,F_f A$$

$$C_e = \frac{4\,F_f}{\pi\,\omega A} = \frac{4\,\mu\,R_N}{\pi\,\omega A} \qquad \text{...(4.8.4)}$$

The steady state amplitude for the system having viscous damping is given by eqn. (4.3.6).

$$A = F/k\Big/\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{C\omega}{k}\right)^2}$$

Substituting the equivalent damping coefficient C_e in place of C in eqn. 4.3.6, we get

$$A = \frac{F/k}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{4\,F_f}{\pi\,Ak}\right)^2}} \qquad \text{...(4.8.5)}$$

This equation contains A within the radical. Squaring and algebraically solving the above equation for A, we get

$$\frac{A}{F/k} = \sqrt{\frac{1-\left(\frac{4F_f}{\pi F}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2}} \qquad \text{...(4.8.6)}$$

$$= \sqrt{\frac{1-\left(\frac{4F_f}{\pi F}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}} \qquad \text{...(4.8.7)}$$

If $\frac{F}{F_f} > \frac{4}{\pi}$ the numerator of the fraction under the radical is nega-
ive, the radical is imaginary, and the approximate solution can not be
sed. However, the exact solution was derived by Den Hartog.

To avoid imaginary values of A so that A is always real, we need
) have

$$1-\left(\frac{4F_f}{\pi F}\right)^2 > 0 \qquad \text{...(4.8.8)}$$

$$\frac{4F_f}{\pi F} < 1 \text{ or } \frac{F}{F_f} > \frac{4}{\pi} \qquad [\text{where } F_f = \mu\,R_N]$$

$$\frac{F_f}{F} < \frac{\pi}{4} \qquad \text{...(4.8.9)}$$

The phase angle ϕ can be found using eqn. (4.3.7) by substituting
, in place of C.

$$\phi = \tan^{-1}\left[\frac{C_e\,\omega/k}{1-\omega^2/\omega_n^2}\right] = \tan^{-1}\left[\frac{\frac{4\,F_f}{\pi kA}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] \qquad \text{...(4.8.10)}$$

Substituting the value of A from eqn. (4.8.7) in eqn. (4.8.10), we
tain

$$\phi = \tan^{-1}\left[\frac{\frac{4F_f}{\pi F}}{\left\{1-\left(\frac{4F_f}{\pi F}\right)^2\right\}^{1/2}}\right] \qquad \text{...(4.8.11)}$$

Eqn. (4.8.11) shows that $\tan\phi$ is constant for a given value of F/F_f. ϕ is discontinuous at $\frac{\omega}{\omega_n} = 1$ (resonance) since it takes positive value for $\frac{\omega}{\omega_n} < 1$ and a negative value for $\frac{\omega}{\omega_n} > 1$. Thus Eqn. (4.8.11) can be expressed as

$$\phi = \tan^{-1}\left[\frac{\pm\,4F_f/\pi\,F}{\left\{1-\left(\frac{4F_f}{\pi F}\right)^2\right\}^{1/2}}\right] \qquad \text{...(4.8.12)}$$

Eqn. (4.8.7) shows that friction limits the amplitude of forced vibration for $\frac{\omega}{\omega_n} \neq 1$. At resonance when $\frac{\omega}{\omega_n} = 1$, the amplitude becomes infinite. This is explained below. The energy directed into the system over the cycle when it is excited harmonically at resonance is

$$\Delta E_1' = \int F \sin \omega t\,.\,dx$$

$$= \int_0^T F \sin \omega t\,.\,\frac{dx}{dt}\,.\,dt \qquad \text{...(4.8.13)}$$

From eqn. 4.3.3, $x = A \sin(\omega t - \phi)$

$$\frac{dx}{dt} = \omega A \cos(\omega t - \phi)$$

Substituting $\frac{dx}{dt}$ in (4.8.13), we get

$$\Delta E_1' = \int_0^{2\pi/\omega} F \sin \omega t\,.\,[\omega A \cos(\omega t - \phi)]\,dt \qquad \text{...(4.8.14)}$$

$$\left[\because\ T = \frac{2\pi}{\omega}\right]$$

At resonance $\phi = 90°$, $\therefore$ eqn. (4.8.14) becomes

$$\Delta E_1' = FA\omega\int_0^{2\pi/\omega} \sin^2 \omega t\,dt = \pi\,FA \qquad \text{...(4.8.15)}$$

The energy dissipated from the system is given by eqn. (4.8.2). Since $\pi FA > 4\mu R_N A$ for A to be real value, $\Delta E_1' > \Delta E$ at resonance. Thus more energy is directed into the system per cycle than is dissipated per cycle. This extra energy builds up the amplitude of vibration.

For non-resonant conditions $\left(\frac{\omega}{\omega_n} \neq 1\right)$, the energy input from eqn. (4.8.14) is given by

$$\Delta E_1 = \omega F A \int_0^{2\pi/\omega} \sin \omega t \cos (\omega t - \phi)\, dt = \pi F A \sin \phi$$

(See equation 1.13.1) ...(4.8.16)

Due to presence of $\sin \phi$, the input energy curve is made to coincide with dissipated energy curve, so amplitude is limited. Phase of motion ϕ can be seen to limit the amplitude of motion. Refer figure 4.11.

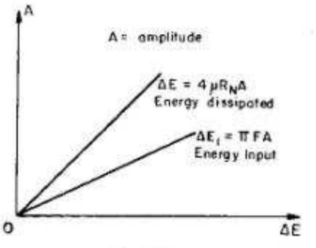


Fig. 4.11.

4.9 FORCED VIBRATION WITH HYSTERESIS OR STRUCTURAL DAMPING

It has been mentioned earlier in section 3.2 that when a material is subjected to cyclic reversal of loading and unloading, energy is dissipated within the material itself.

Considering a single degree of freedom system with hysteresis damping and subjected to a harmonic force $F \sin \omega t$ as shown in figure 4.12. The equation of motion of mass can be derived as

$$m\ddot{x} + \lambda \frac{k}{\omega} \dot{x} + kx = F \sin \omega t \qquad ...(4.9.1)$$

Where $\left(\frac{\lambda k}{\omega}\right)\dot{x}$ denotes the damping force. In contrast to viscous damping, this damping force is a function of forcing frequency 'ω'.

For most structural elements the energy dissipated per cycle is proportional to the square of the amplitude and independent of frequency over a wide range, i.e. from eqn. (3.2.14).

Energy dissipated per cycle $= \beta A^2$ ...(4.9.2)

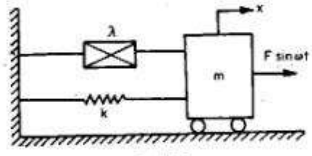


Fig. 4.12.

If C_e is equivalent viscous damping for this case, then equating the energy dissipated per cycle in structural damping to the energy dissipated by an equivalent viscous damper during a full cycle, we get;

$$\pi C_e \omega A^2 = \beta A^2$$

$$C_e = \frac{\beta}{\pi\omega} \qquad ...(4.9.3)$$

From eqn. (3.2.13), we get,

Energy loss per cycle $= \pi k \lambda A^2$ ...(4.9.4)

From eqn. (4.9.2) and (4.9.4), we get

$$\pi k \lambda A^2 = \beta A^2$$

or $\beta = \pi k \lambda$ ...(4.9.5)

or $\lambda = \beta / \pi k$ ...(4.9.6)

The differential equation given by eqn. 4.9.1 can now be written as

$$m\ddot{x} + \left(\frac{\beta}{\pi\omega}\right)\dot{x} + kx = F \sin \omega t \qquad ...(4.9.7)$$

where $\lambda = \frac{\beta}{\pi k}$ is called the structural damping factor or loss factor.

The steady state soln. of eqn. (4.9.1) can be obtained by substituting eqn. (4.3.3) in eqn. (4.9.1). The same results can be obtained by substituting the value of C_e in place of C in eqn. (4.3.6) and eqn. (4.3.7).

Thus the solution of eqn. 4.9.1 is given as :

Steady state amplitude for system having hysteresis damping is given by eqn. 4.3.6 as

$$A = \frac{F/k}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{C_e \omega}{k}\right)^2}} \qquad ...(4.9.9)$$

From eqn. 4.9.3 and eqn. 4.9.5

$$C_e \omega = \frac{\beta}{\pi} = \frac{\pi k \lambda}{\pi} = \lambda k \qquad ...(4.9.10)$$

Substituting eqn. 4.9.10 in eqn. 4.9.9

$$\therefore \quad A = \frac{F/k}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \lambda^2}} \qquad ...(4.9.11)$$

or

$$\frac{A}{F/k} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + [\lambda]^2}} \qquad ...(4.9.12)$$

The phase angle ϕ can be found using eqn. 4.3.7 by substituting C_e in place of C

$$\phi = \tan^{-1}\left[\frac{C_e \omega / k}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right] \qquad ...(4.9.13)$$

From eqn. 4.9.10, we get

$$C_e \omega = \lambda k$$

Substituting value of $C_e \omega$ in eqn. 4.9.13, we get

$$\phi = \tan^{-1}\left[\frac{\lambda}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right] \qquad ...(4.9.14)$$

If the harmonic excitation is assumed to be $Fe^{i\omega t}$, the eqn. 4.9.1 becomes

$$m\ddot{x} + \left(\frac{\lambda k}{\omega}\right)\dot{x} + kx = Fe^{i\omega t} \qquad ...(4.9.15)$$

In this case, response x is also a harmonic function involving the factor $e^{i\omega t}$. Hence, $\dot{x}$ is given by $i\,\omega x$ and eqn. 4.9.15 becomes

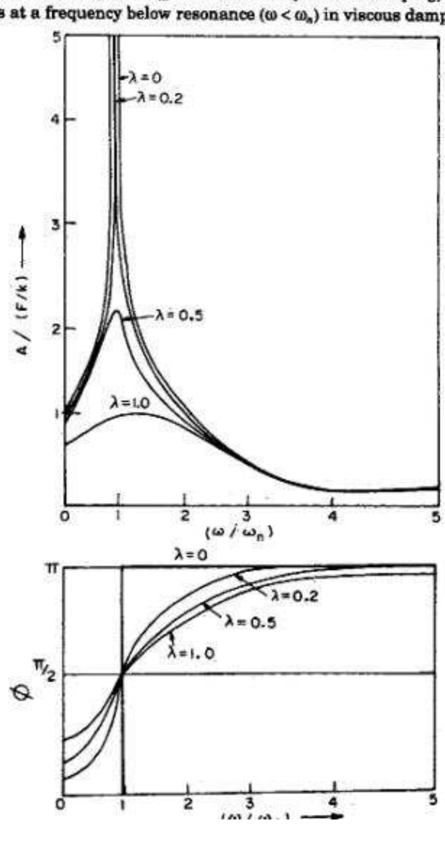
$$m\ddot{x} + k(1 + i\lambda)x = Fe^{i\omega t} \qquad ...(4.9.16)$$

Where $k(1 + i\lambda)$ is called COMPLEX STIFFNESS OR COMPLEX DAMPING. The steady state solution of eqn. (4.9.16) is given by real part of eqn. (4.9.17)

$$x = \frac{Fe^{i\omega t}}{k\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + i\lambda\right]} \qquad ...(4.9.17)$$

Eqn. (4.9.12) and eqn. (4.9.14) are plotted in figure 4.13 for several values of λ. A comparison of figure 4.13 with figure (4.4) for viscous damping reveals the following :

(1) Amplitude ratio $A/(F/k)$ attains its maximum value of $F/k\lambda$ at resonant frequency ($\omega = \omega_n$) in the case of hysteresis damping, while it occurs at a frequency below resonance ($\omega < \omega_n$) in viscous damping.



(2) The phase angle ϕ has a value of $\tan^{-1}(\lambda)$ at $\omega = 0$ in case of hysteresis damping, while it has a value of zero at $\omega = 0$ in viscous damping. This shows that the response can never be in phase with forcing function in case of hysteresis damping.

4.10 FORCED VIBRATIONS WITH COULOMB AND VISCOUS DAMPING

Sometimes there is a compound damping consisting of couloumb damping and viscous damping in parallel arrangement. The equivalent damping coefficient in this case is given by sum of viscous damping coefficient and the equivalent damping coefficient as given by eqn. (4.8.4).

$$\therefore \quad C_e = C + \frac{4F_f}{\pi\omega A} \qquad \ldots(4.10.1)$$

Substituting this value of C_e in place of C in eqn. (4.3.6), we get

$$A = \frac{F/k}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[\left(C + \frac{4F_f}{\pi\omega A}\right)\frac{\omega}{k}\right]^2}} \qquad \ldots(4.10.2)$$

The above eqn. can be written in the following form :

$$\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{C\omega}{k}\right)^2\right\} A^2 + \left[\frac{8F_f C\omega}{\pi k^2}\right] A + \left[\left(\frac{4F_f}{\pi k}\right)^2 - \left(\frac{F}{k}\right)^2\right] = 0 \qquad \ldots(4.10.3)$$

Solving the above quadratic eqn. for A, we get the amplitude of vibration of the system both for Coulomb and viscous damping.

If there is no viscous damping then putting $C = 0$ in above equation, we get the amplitude corresponding to eqn. (4.8.5). If on the other hand there is no coulomb damping then putting $F_f = 0$ in eqn. (4.10.3), the amplitude obtained is same as given by eqn. (4.3.6)

Similarly, from eqn. (4.3.7) we get the phase angle by substituting C_e in place of C.

$$\phi = \tan^{-1}\left[\frac{\left(C + \frac{4F_f}{\pi\,\omega A}\right)\frac{\omega}{k}}{1 - (\omega/\omega_n)^2}\right] \qquad \ldots(4.10.4)$$

4.11. VIBRATION MEASURING INSTRUMENTS

The instruments which are used to measure the displacement, velocity or acceleration of a vibrating body are called vibration measuring instruments. Vibration measuring devices having mass, spring,

dashpot, etc. are known as seismic instruments. The quantities to be measured are displayed on a screen in the form of electric signal which can be readily amplified and recorded. The output of electric signal of the instrument will be proportional to the quantity which is to be measured. The input is reproduced as output very precisely. Two types of seismic transducers known as vibrometer and accelerometer are widely used. A vibrometer or a seismometer is a device to measure the displacement of a vibrating body. Similarly, other device known as an accelerometer is an instrument to measure the acceleration of a vibrating body. Vibrometer is designed with low natural frequency and accelerometer with high natural frequency. So vibrometer is known as low frequency transducer and accelerometer as high frequency transducer.

4.11.1. Vibrometer

Let us consider equation (4.5.2.3) again, *i.e.*

$$\frac{Z}{B} = \frac{(\omega/\omega_n)^2}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\varepsilon\,\omega/\omega_n]^2}}$$

Let us assume $\omega/\omega_n = r$ in the above equation

$$\frac{Z}{B} = \frac{r^2}{\sqrt{[1 - r^2]^2 + [2\varepsilon r]^2}} \qquad \ldots(4.11.1.1)$$

We have plotted the characteristics of this equation in figure 4.7. It is repeated here for convenience in figure 4.14.

When the value of r is very high (more than 3), the above equation can be written as

$$\frac{Z}{B} = \frac{r^2}{\sqrt{[(1 - r^2)]^2}} = 1$$

$$Z = B \qquad \ldots(4.11.1.2)$$

(as $2\varepsilon r$ is very small term, so it is neglected for a wide range of damping factors)

So the relative amplitude Z is showp equal to the amplitude of vibrating body B on the screen. Though Z and B are not in the same phase but B being in single harmonic, will result in the output signal as true reproduction of input quantity. From figure 4.14 it can be seen that for large values of ω/ω_n (r), the ratio $\frac{Z}{B}$ approaches unity for every value of damping. The instrument shown in figure 4.15 works as a vibrometer for very large value of r.

Vibrometer known as low frequency transducer is used to measure the high frequency ω of a vibrating body. Since the ratio r is very high so the natural frequency of the instrument is low. Low natural frequen-

means heavy mass of the body of the instrument which makes its rare application in practice specially in systems which require much sophistication. The frequency range of a vibrometer depends upon several factors such as damping, its natural frequency, etc. It may have natural frequency 1 Hz to 5 Hz.

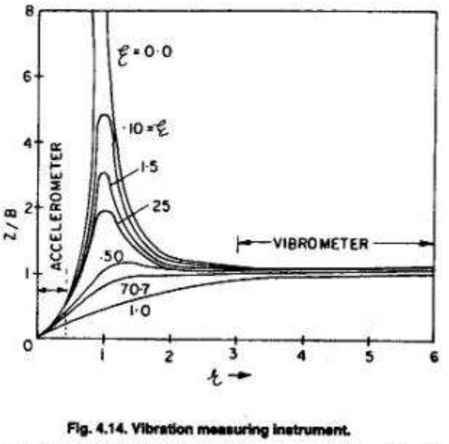
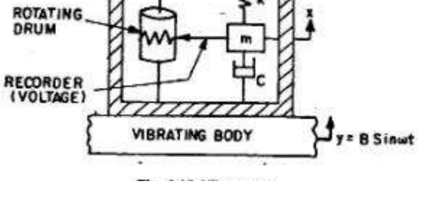


Fig. 4.14. Vibration measuring instrument.

The relative motion (z) between the mass and vibrating body is converted into proportional voltage by assuming the mass as permanent magnet (refer figure 4.15).



4.11.2. Accelerometer

An accelerometer is used to measure the acceleration of a vibrating body. If the natural frequency ω_n of the instrument is very high compared to the frequency ω which is to be measured, the ratio ω/ω_n (r) is very small *i.e.* $\omega/\omega_n << 1$. The range of frequency measurement is shown in figure 4.14. Since the natural frequency of the instrument is high so it is very light in construction. With the help of electronics integration devices it displays velocity and displacement both. Because of its small size and usefulness for determining velocity and displacement besides acceleration, it is very widely used as a vibration measuring device and is termed as high frequency transducer. The voltage signals obtained from an accelerometer are usually very small which can be preamplified to see them bigger in size on oscilloscope. For getting velocity and displacement double integration device may be used and the results are obtained on screen.

Again considering equation (4.5.2.3) and assuming $\omega/\omega_n << 1$, we can also assume that $(\omega/\omega_n) \longrightarrow 0$

$$\frac{Z}{B} = \left(\frac{\omega}{\omega_n}\right)^2 . f \text{ or } Z = \frac{\omega^2 B}{\omega_n^2} f \qquad \ldots(4.11.2.1)$$

where f is a factor which remains constant for the useful range of accelerometer.

$$\text{where} \qquad f = \frac{1}{\sqrt{(1 - r^2)^2 + (2\varepsilon r)^2}} \qquad \ldots(4.11.2.2)$$

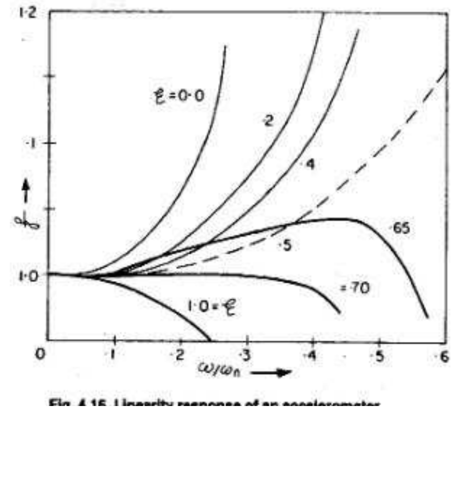


Fig. 4.16 Linearity response of an accelerometer

In this equation $\omega^2 B$ is the acceleration of the vibrating body. It is clearly seen that the acceleration is multiplied by a factor $\frac{1}{(\omega_n)^2}$. To keep the value of factor f equal to 1 for very high range of ω/ω_n ratio, ε should be high in value. The amplitude Z becomes proportional to the acceleration provided the natural frequency remains constant. Thus Z is treated proportional to the amplitude of acceleration to be measured, so it can be written as $Z \alpha$ acceleration. The instrument gives accurate results for very high value of its natural frequency. With the help of equation (4.11.2.2), a figure 4.16 is drawn to show the linear response of the accelerometer. It is seen that for $\varepsilon = 0.70$ there is complete linearity for accelerometer for $\omega/\omega_n \le .25$. Thus the instrument with 100 Hz natural frequency will have a useful frequency range from 0 to 25 Hz at $\varepsilon = 0.70$ and will provide very accurate results. For this purpose electromagnetic type accelerometers are widely used nowadays.

4.12. QUALITY FACTOR AND HALF POWER POINTS

With the help of equation (4.3.10) a curve is drawn for a very small value of damping (say $\varepsilon = 0.05$) as shown in figure 4.17. The curve is drawn between amplitude ratio (A/X_s) and frequency ratio (ω/ω_n).

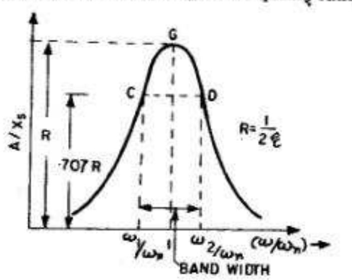


Fig. 4.17.

Equation (4.3.10) at resonance can be as equation (4.3.14) *i.e.*

$$\frac{A}{X_s} = \frac{1}{2\varepsilon} = R \qquad ...(4.12.1)$$

The amplitude ratio at resonance is called as quality factor R. The points where amplitude ratio is $\frac{R}{\sqrt{2}} = .707\ R$ are known as half power points. C and D are the half power points. Assume ω_1 and ω_2 as frequencies corresponding to points C and D respectively.

For a harmonic displacement $A \sin(\omega t + \phi)$, the maximum velocity is $A\omega$ and the energy of vibration of mass m is $\frac{1}{2} m\omega^2 A^2$. Thus the energy of vibration is proportional to the square of amplitude.

The energy absorbed at point G is $\frac{1}{2} m\omega^2 R^2$ and the energy absorbed at points C and D is given as $\frac{1}{2} m\omega^2 \left(\frac{R}{\sqrt{2}}\right)^2 = \frac{1}{2}\left(m\omega^2 \frac{R^2}{2}\right)$ So the energy at points C and D is half of the energy which is at G.

Making use of amplitudes at half power points *i.e.* $\frac{A}{X_s} \cdot \frac{1}{\sqrt{2}} = \frac{R}{\sqrt{2}}$ in equation (4.3.10) both sides, we get

$$\left(\frac{A}{\sqrt{2}\,X_s}\right)^2 = \frac{1}{[1-(\omega/\omega_n)^2]^2 + [2\varepsilon\,\omega/\omega_n]^2}$$

$$\frac{1}{2}\left(\frac{1}{2\varepsilon}\right)^2 = \frac{1}{[1-(\omega/\omega_n)^2]^2 + [2\varepsilon\,\omega/\omega_n]^2}$$

Solving for $(\omega/\omega_n)^2$, we get

$$(\omega/\omega_n)^2 = (1 - 2\varepsilon^2) \pm 2\varepsilon\sqrt{1-\varepsilon^2}$$

Since ε is very small, its higher order powers can be neglected, so

$$\left(\frac{\omega}{\omega_n}\right)^2 = 1 \pm 2\varepsilon$$

$$\left(\frac{\omega_1}{\omega_n}\right)^2 = 1 - 2\varepsilon \quad \text{and} \quad (\omega_2/\omega_n)^2 = 1 + 2\varepsilon$$

$$\frac{\omega_2^2 - \omega_1^2}{\omega_n^2} = 1 + 2\varepsilon - 1 + 2\varepsilon = 4\varepsilon$$

$$\frac{(\omega_1 + \omega_2)}{\omega_n}\,\frac{(\omega_2 - \omega_1)}{\omega_n} = 4\varepsilon$$

At resonance $\quad \frac{\omega_1 + \omega_2}{2} = \omega_n$, so

$$2 \cdot \frac{\omega_2 - \omega_1}{\omega_n} = 4\varepsilon$$

$$\frac{\omega_2 - \omega_1}{\omega_n} = 2\varepsilon$$

The difference of frequency ratios at power points is called band width. Mathematically, it can be written as

$$\frac{1}{R} = \frac{\omega_2 - \omega_1}{\omega_n} = 2\varepsilon = \text{Bandwidth} \qquad ...(4.12.2)$$

4.13. FREQUENCY MEASURING DEVICE

The working of frequency measuring instruments is based on the principle of resonance. At resonance the amplitude of vibration is found to be maximum and then the excitation frequency is equal to the natural frequency of the instrument. Two types of instruments are discussed here.

Fullarton Tachometer. This instrument is known as single reed instrument. It consists of a thin strip carrying small mass attached at one of its free ends. The strip is treated as a cantilever the length of which is changed by means of a screw mechanism as shown in figure 4.18. The strip of the instrument is pressed over the vibrating body to find its natural frequency. We go on changing the length of the strip till amplitude of vibration is maximum. At the instant, the excitation frequency equals the natural frequency of cantilever strip which can be directly seen from the strip itself. The strip has different frequencies for its different lengths.

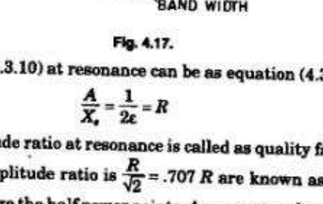


Fig. 4.18. Frequency measuring device.

The natural frequency can be determined with the help of this formula

$$f_n = \frac{1}{2\pi}\sqrt{\frac{3EI}{l^3 m}}\ \text{Hz}$$

where symbols have their usual meanings.

Frahm Tachometer : This is known as multi reed instrument also. It consists of several reeds of known different natural frequencies. There may be a known series of frequencies for the reeds. Small difference in the frequencies of successive reeds will show more accurate results. The instrument is brought in contact with the vibrating body whose frequency is to be measured and one of the reeds will be having maximum amplitude and hence that reed will be showing the frequency of the vibrating body.

The mathematical analysis involved in the calculation of the natural frequency of the vibrating body with the help of a Frahm's Reed Tachometer is discussed below :

Let m be the mass attached to the end of each reed of length l and E be the modulus of elasticity of the reed material.

The static deflection of the reed considering it to be a cantilever fixed at one end is given by

$$x_{st} = \frac{mgl^3}{3EI}$$

where $I = \frac{bd^3}{12}$ = moment of inertia of the reed about the base

We know that $k \cdot x_{st} = m.g$

where k = stiffness of the reed

So natural frequency of the reed $= f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{m \cdot g}{x_{st} \cdot m}} = \frac{1}{2\pi}\sqrt{\frac{g}{x_{st}}}$$

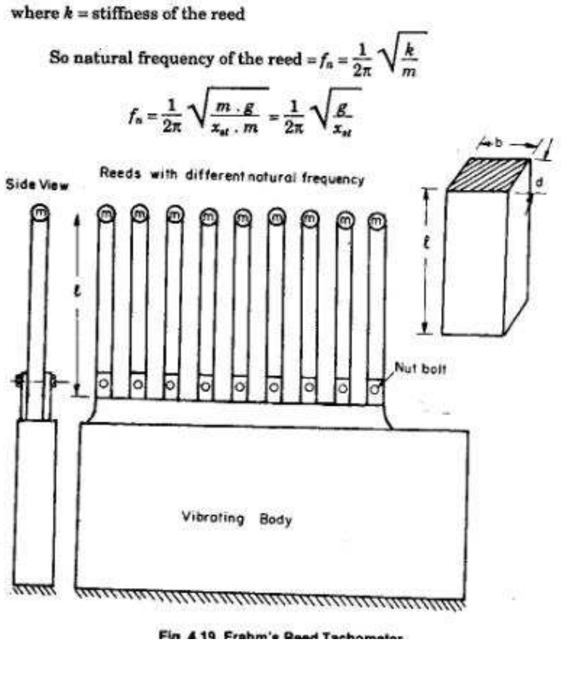


Fig. 4.19. Frahm's Reed Tachometer.

$$= \frac{1}{2\pi}\sqrt{\frac{g \cdot 3EI}{m \cdot g l^3}} = \frac{1}{2\pi}\sqrt{\frac{3EI}{m \cdot l^3}} \text{ Hz}$$

Thus by having different values of mass 'm' or length 'l' of the reed, we can have a series of reeds with definite known frequencies.

The one which has a frequency equal to the natural frequency of the vibrating body, vibrates with a large amplitude. Thus the frequency of the vibrating body can be determined easily by knowing the reed with maximum amplitude.

The accuracy of the instrument depends upon the difference between the value of the natural frequency of the successive reeds.

The instrument will be more accurate if the difference in the value of frequency is smaller. Refer figure 4.19.

4.14. CRITICAL SPEED

We have already discussed that when the natural frequency of the system coincides with the external forcing frequency, it is called resonance. The speeds at which resonance occurs are known as the critical speeds. These speeds are also termed as whirling speeds or whipping.

At these speeds the amplitude of vibration of rotors is excessively large and the large amount of force is transmitted to the foundations or bearings. In the region of critical speeds the system may fail because of violent nature of vibrations in the transverse direction. Therefore, it is very important to find the natural frequency of the shaft to avoid the occurrence of critical speeds which may result in excessive noise and its breakage into pieces.

The critical speed may occur because of eccentric mounting of the rotor, nonuniform distribution of rotor material, bending of shaft ; etc.

Let us consider a shaft rotating horizontally in bearings A and B as shown in figure 4.20.

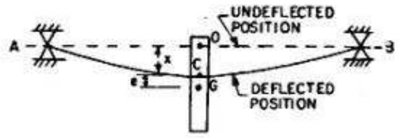


Fig. 4.20.

The shaft being of negligible weight, carries a rotating disc of weight W

Supposing k_t = stiffness of shaft,

e = eccentricity of the disc

= CG (radial distance)

x = lateral deflection of the shaft centre, from O (*i.e.* OC)

ω = uniform angular velocity of shaft

ω_c = critical speed of shaft

C = geometric centre of the disc

G = centre of gravity of the disc which is at a distance e from C

Considering the equilibrium of the disc, two forces are acting on it. The centrifugal force acts radially outwards through G and restoring force acts through C radially inwards. For the equilibrium of the disc, these forces act through the same line and so the points O, C and G must lie on the same straight line. The restoring force is equal to centrifugal force for the disc equilibrium.

Thus, restoring force = Centrifugal force

$$k_t x = \frac{W}{g}\omega^2 (x + e)$$

$$k_t x - \frac{W}{g}\omega^2 x = \frac{W}{g}\omega^2 e$$

$$\frac{x}{e} = \frac{\frac{W}{g}\omega^2}{k_t - \frac{W}{g}\omega^2} = \frac{1}{\frac{k_t g}{W\omega^2} - 1}$$

$$\frac{x}{e} = \frac{1}{\left(\frac{\omega_c}{\omega}\right)^2 - 1} \quad ...(4.14.1)$$

(Assuming $\omega_c^2 = \frac{k_t g}{W} = \frac{g}{\delta}$, where δ is static deflection)

When $\omega_c = \omega$, $\frac{x}{e}$ ratio is infinite. This particular value of ω will be critical speed.

$$\omega_c = \sqrt{\frac{g}{\delta}}$$

and we know that $\omega_c = 2\pi f_c$, so

$$2\pi f_c = \sqrt{\frac{g}{\delta}}$$

$$f_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \text{ Hz} \quad ...(4.14.2)$$

With the help of equation (4.14.1), we can plot a curve between (x/e) versus (ω_c/ω) which is shown in figure 4.21.

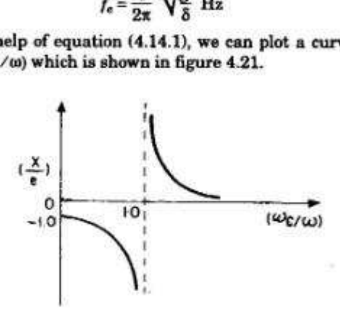


Fig.4.21.

There are three possible combinations of speed *i.e.* $\omega = \omega_c$, $\omega > \omega_c$ and $\omega < \omega_c$.

When $\omega = \omega_c$, the amplitude ratio becomes infinite which causes severe vibration and excessive loads on bearings and the disc will try to fly out. Proper damping is provided to control the disc from flying out.

In case when $\omega < \omega_c$, the amplitude ratio will be positive and C will lie between O and G. When $\omega > \omega_c$, the ratio (x/e) is found to be negative. It means G falls between O and C. Thus at high speeds the disc rotates with heavy side inside. Here heavy side means the centre of gravity. Then the disc rotates about its centre of gravity G and vibrations are minimum. This concept is useful for stabilization of aircrafts at high speeds.

4.15. CRITICAL SPEED WITH DAMPING

When damping is considered in the system, the three points O, C and G do not lie in a straight line as we found in case when damping was not taken into account. In the present case let CG is the eccentricity e, $OC = x$ the deflection of the shaft and $OG = r$ as shown in figure 4.22.

The centrifugal force acts at G and its direction is along OG outwards. The magnitude of this force is equal to $F_G = m\omega^2 r$. The shaft deflects by the amount OC, so, the restoring force $K_t x$ acts at C in the inward direction. This force tries to resist the bending of the shaft. Its direction is towards point O. An equal and opposite force to F_G acts at the geometric centre C of the shaft. This force F_p, is balanced by the damping force $c\omega x$ and restoring force $K_t x$ as clearly shown in figure 4.22.

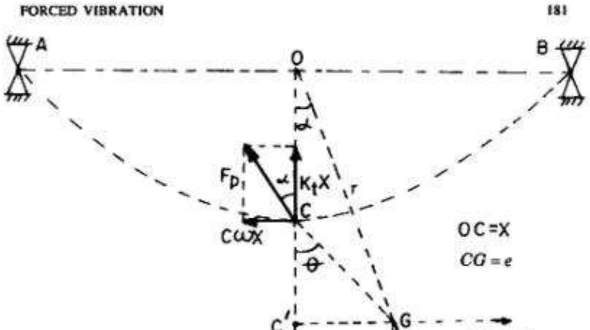


Fig. 4.22.

We can consider the different values of phase angle ϕ.

(*i*) When $\phi = 0$, it is having no damping.

(*ii*) When $\phi < 90°$, it is having damping.

(*iii*) When $\phi = 90°$, the deflection of the shaft is maximum hence, the critical speed.

(*iv*) When $\phi = 180°$, the point G will have a tendency to come closer to point O and there will be least vibrations. This is the case when the speed is more than the critical speed.

From figure, we get

$$GC' = r \sin\alpha = CG \sin\phi = e \sin\phi$$

$$r \sin\alpha = e \sin\phi \quad ...(4.15.1)$$

and

$$OC' = r \cos\alpha$$

But

$$OC + CC' = x + e\cos\phi$$

$$r \cos\alpha = x + e\cos\phi \quad ...(4.15.2)$$

Considering the equilibrium of the system, we have

$$k_t x = F_G \cdot \cos\alpha$$

$$c\omega x = F_G \sin\alpha$$

Rearranging the above equations

$$k_t x - F_G \cos\alpha = 0$$

$$k_t x - m\omega^2 r \cos\alpha = 0 \quad ...(4.15.3)$$

$$c\omega x - F_G \sin\alpha = 0$$

$$c\omega x - m\omega^2 r \sin\alpha = 0 \qquad \text{...(4.15.4)}$$

Substituting the values of $r\cos\alpha$ and $r\sin\alpha$ from equations (4.15.2) and (4.15.1) into equations (4.15.3) and (4.15.4), we get

$$k_t x - m\omega^2 (x + e\cos\phi) = 0$$

$$(k_t - m\omega^2)\, x = m\omega^2 e \cos\phi \qquad \text{...(4.15.5)}$$

$$c\omega x = m\omega^2 e \sin\phi \qquad \text{...(4.15.6)}$$

With the help of above equations (4.15.5) and (4.15.6), we can find the phase angle as

$$\tan\phi = \frac{c\omega x}{(k_t - m\omega^2)\, x} = \frac{c\omega}{k_t - m\omega^2} = \left[\frac{2\varepsilon(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right] \qquad \text{...(4.15.7)}$$

Noting that damping ratio $\varepsilon = \dfrac{C}{2\sqrt{k_t m}} = \dfrac{C}{2\sqrt{\dfrac{k_t}{m}}\, m}$

$$\varepsilon = \frac{C}{2\omega_n m}, \quad 2\varepsilon\, \omega\omega_n = \frac{c\omega}{m}$$

as $$\omega_n = \sqrt{\frac{k_t}{m}}$$

Squaring and adding equations (4.15.5) and (4.15.6), we get

$$[(k_t - m\omega^2)\, x]^2 + [c\omega x]^2 = [m\omega^2 e]^2$$

$$x^2\, \{[k_t - m\omega^2]^2 + [c\omega]^2\} = [m\omega^2]^2\, e^2$$

$$\frac{x^2}{e^2} = \frac{[m\omega^2]^2}{[k_t - m\omega^2]^2 + [c\omega]^2} = \frac{[\omega^2]^2}{\left[\dfrac{k_t}{m} - \omega^2\right]^2 + \left[\dfrac{c\omega}{m}\right]^2}$$

So $$\frac{x}{e} = \frac{\omega^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + \left(\dfrac{c\omega}{m}\right)^2}}$$

$$= \frac{\omega^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\varepsilon\omega\omega_n)^2}}$$

$$= \frac{(\omega/\omega_n)^2}{\sqrt{(1 - \omega/\omega_n)^2 + (2\varepsilon\, \omega\omega_n/\omega_n^2)^2}} \qquad \text{(dividing by } \omega_n^2\text{)}$$

$$\frac{x}{e} = \frac{\beta^2}{\sqrt{(1-\beta^2)^2 + (2\varepsilon\beta)^2}} \qquad \text{...(4.15.8)}$$

SOLVED EXAMPLES

Example 4.1. *A vibrating system having mass 1 kg is suspended by a spring of stiffness 1000 N/m and it is put to harmonic excitation of 10 N. Assuming viscous damping, determine*

(a) *the resonant frequency*

(b) *the phase angle at resonance*

(c) *the amplitude at resonance*

(d) *the frequency corresponding to the peak amplitude and*

(e) *damped frequency*

Take C = 40 N-sec/m

Solution. (a) Frequency at resonance

$$\omega = \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{1}} = 31.62 \text{ rad/sec}$$

Damping factor ε is given as

$$\varepsilon = \frac{c}{2m\omega_n} = \frac{40}{2 \times 1 \times 31.62} = 0.632$$

(b) Phase angle at resonance

$$\tan\phi = \infty$$

So $\phi = \pi/2$

(c) Amplitude at resonance

$$x_{reso} = \frac{F}{c\omega_n} = \frac{10}{40 \times 31.62} = 7.9 \times 10^{-3} \text{ m}$$

(d) The frequency corresponding to the peak amplitude

$$\omega_{peak} = \omega_n\sqrt{1 - 2\varepsilon^2} = 31.62\sqrt{1 - 2(.632)^2}$$

$$= 14.14 \text{ rad/sec}$$

(e) The damped frequency is given as

$$\omega_d = \sqrt{1 - \varepsilon^2}\, \omega_n = \sqrt{1 - (.632)^2}\; 31.62$$

$$= 24.5 \text{ rad/sec}$$

Example 4.2. *The vibrating system as shown in figure 4.6 is displayed for vibrational characteristics. The total mass of the system is 25 kg. At speed of 1000 rpm. the system and the eccentric mass have a phase difference of 90° and the corresponding amplitude is 1.5 cm. The eccentric unbalanced mass of 1 kg has a radius of rotation 4 cm. Determine (a) the natural frequency of the system (b) the damping factor (c) the amplitude at 1500 rpm, and (d) the phase angle at 1500 rpm.*

Solution. Given $m = 25$ kg

$$A = 1.5 \text{ cm}$$

$$m_0 = 1 \text{ kg}$$

$$e = 4 \text{ cm}$$

At phase angle 90°, the condition of resonance will occur *i.e.* $\omega = \omega_n$.

(a) So $\omega_n = 1000/60 = 16.67$ cycles/sec

(b) $$\varepsilon = \frac{m_0 e}{2mA} = \frac{1 \times 4}{2 \times 25 \times 1.5} = 0.053$$

(c) $$A_{1500} = \frac{\dfrac{m_0 e}{m}\, (\omega/\omega_n)^2}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\varepsilon\omega/\omega_n]^2}}$$

$$= \frac{\dfrac{1 \times 4}{25}\left(\dfrac{1500}{1000}\right)^2}{\sqrt{\left[1 - \left(\dfrac{1500}{1000}\right)^2\right]^2 + \left[2 \times .053 \times \dfrac{1500}{1000}\right]^2}}$$

$$A_{1500} = 0.226 \text{ cm}$$

(d) $$\tan\phi_{1500} = \frac{2\varepsilon\, \omega/\omega_n}{1 - (\omega/\omega_n)^2}$$

$$= \frac{2 \times .053 \times 1500/1000}{1 - \left(\dfrac{1500}{1000}\right)^2}$$

$$= -.1272$$

$$\phi = 172°\, 45'.$$

Example 4.3. *An electric motor is supported on a spring and a dashpot. The spring has the stiffness 6400 N/m and the dashpot offers resistance of 500 N at 4.0 m/sec. The unbalanced mass 0.5 kg rotates at 5 cm radius and the total mass of vibratory system is 20 kg. The motor runs at 400 rpm.*

Determine (a) damping factor (b) amplitude of vibration and phase angle (c) resonant speed and resonant amplitude, and (d) forces exerted by the spring and dashpot on the motor.

Solution. Given $k = 6400$ N/m

$$c = \frac{F}{\dot{x}} = \frac{500}{4} = 125 \text{ N . sec/m}$$

$$m_0 = .5 \text{ kg}$$

$$m = 20 \text{ kg}$$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 400}{60} = 41.866 \text{ rad/sec}$$

$$C_c = \text{critical damping} = 2\sqrt{km}$$

$$= 2\sqrt{6400 \times 20} = 715.54 \text{ N-sec/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{6400}{20}} = 17.88 \text{ rad/sec}$$

$$= 170.90 \text{ rpm.}$$

(a) $$\varepsilon = \frac{C}{C_c} = \frac{125}{715.54} = .175$$

(b) $$A = \frac{\dfrac{m_0 e}{m}\, r^2}{\sqrt{[1 - r^2]^2 + [2\varepsilon r]^2}} \qquad \text{(where } r = \omega/\omega_n\text{)}$$

$$= \frac{\dfrac{.5 \times 5}{20}\left(\dfrac{400}{170.90}\right)^2}{\sqrt{1 - \left(\dfrac{400}{170.90}\right)^2 + \left(2 \times .175 \times \dfrac{400}{170.90}\right)^2}}$$

$$A = 0.15 \text{ cm}$$

$$\tan\phi = \frac{2r\varepsilon}{1 - r^2} = \frac{2 \times \dfrac{400}{170.9} \times .175}{1 - \left(\dfrac{400}{170.9}\right)^2} = -.1829$$

$$\phi = 169°\, 41'.$$

(c) The resonant speed is given as

$$\omega_n = 17.88 \text{ rad/sec}$$

$$= 170.90 \text{ rpm.}$$

$$A_{resonant} = \frac{m_0 e}{2\varepsilon m} = \frac{.5 \times 5}{2 \times .175 \times 20} = 0.357 \text{ cm}$$

(d) The force because of dashpot on the motor

$$F_c = c\omega A$$

$$= 125 \times \frac{2\pi \times 400}{60} \times .15 \times 10^{-2}$$

$$= 7.85 \text{ N}$$

Force because of spring $F_s = kA$

$$= 6400 \times .15 \times 10^{-2}$$

$$= 9.6 \text{ N}$$

Resultant force $$F = \sqrt{F_c^2 + F_s^2} = \sqrt{7.85^2 + 9.6^2}$$

$$= 12.4 \text{ N.}$$

EXAMPLE 4.4. *A body of mass 70 kg is suspended from a spring which deflects 2.0 cm under the load. It is subjected to a damping effect adjusted to a value 0.23 times that required for critical damping. Find the natural frequency of the undamped and damped vibrations and ratio of successive amplitudes for damped vibrations.*

If the body is subjected to a periodic disturbing force of 700 N and of frequency equal to 0.78 times the natural undamped frequency, find the amplitude of forced vibrations and the phase difference with respect to the disturbing force.

SOLUTION. Spring stiffness k = force/deflection

$$= \frac{70 \times 9.81}{2 \times 10^{-2}} = 34.335 \times 10^3 \text{ N/m}$$

$$\varepsilon = \frac{C}{C_c} = 0.23$$

Undamped natural frequency,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{34.335 \times 10^3}{70}} = 22.15 \text{ rad/sec}$$

Damped natural frequency

$$\omega_d = \omega_n \sqrt{1 - \varepsilon^2} = 22.15 \sqrt{1 - (.23)^2}$$

$$= 21.57 \text{ rad/sec}$$

Logarithmic decrement

$$\delta = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}} = \frac{2\pi \times .23}{\sqrt{1-(.23)^2}} = 1.48$$

Ratio of successive amplitudes

$$\frac{A_1}{A_2} = e^{\delta} = e^{1.48} = 4.39$$

The relation is

$$\frac{A}{F/k} = \frac{1}{\sqrt{(1-r^2)^2 + (2\varepsilon r)^2}} \qquad (r = \omega/\omega_n)$$

Given $F = 7000$ N, $k = 34.335 \times 10^3$ N/m, $r = 0.78$

$$A = \frac{700/34335}{\sqrt{(1-.78^2)^2 + (2 \times .23 \times .78)^2}}$$

$$= .038 \text{ m.}$$

$$\tan\phi = \frac{2\varepsilon r}{1-r^2} = \frac{2 \times .23 \times .78}{1-(.78)^2}$$

$$= 0.916$$

$$\phi = 42° 29'$$

EXAMPLE 4.5. *The weight of an electric motor is 125 kg and it runs at 1500 rpm. The armature weighs 35 kg and its centre of gravity lies 0.05 cm from the axis of rotation. The motor is mounted on five springs of negligible damping so that the force transmitted is one-eleventh of the impressed force. Assume that the weight of the motor is equally distributed among the five springs.*

Determine

(a) *stiffness of each spring*

(b) *dynamic force transmitted to the base at operating speed*

(c) *natural frequency of the system.* (P.U., 98)

SOLUTION. Transmissibility is given as

$$\text{T.R.} = \frac{1}{(\omega/\omega_n)^2 - 1} \text{ (when } \varepsilon \text{ is negligible)}$$

$$\frac{1}{11} = \frac{1}{(\omega/\omega_n)^2 - 1}$$

$$(\omega/\omega_n)^2 = 12$$

$$\omega/\omega_n = \sqrt{12} = 3.46$$

$$\omega_n = \omega/3.46 = \frac{1500}{3.46} = 433.5 \text{ rpm.}$$

$$f_n = \frac{433.5}{60} = 7.22 \text{ cycles/sec}$$

We know that $\omega_n = \sqrt{\dfrac{k}{m}}$

$$\omega_n = \frac{2\pi N}{60} = \frac{2\pi \times 433.5}{60} = 45.373 \text{ rad/sec}$$

$$\omega_n = \sqrt{k/m} \qquad (m = \text{Total mass} = 125 + 35 = 160 \text{ kg})$$

$$k = \omega_n^2\, m = 45.373 \times 45.373 \times \frac{160}{9.81}$$

$$= 33577.3 \text{ N/m}$$

Stiffness of each spring = 33577.3/5 = 6715.5 N/m

$$\text{T.R.} = \frac{\text{force transmitted}}{\text{impressed force}} = \frac{F_{TR}}{F}$$

$$F_{TR} = F\,\text{T.R.}$$

$$= m_1 e\omega^2 \frac{1}{11}$$

$$= \frac{35}{9.81} \times \frac{.05}{100} \times \left(\frac{2\pi \times 1500}{60}\right)^2 \times \frac{1}{11} = 3 \text{ N.}$$

EXAMPLE 4.6. *The vibrations of the platform of railway station are periodic at the frequency range of 12 – 50 Hz. A vibration measuring instrument is to be installed on some foundation independent of the platform. The small foundation is supported by four identical springs resting on the platform. The total mass of the instrument and foundation is 50 kg. What is the maximum value of spring stiffness, if the amplitude of transmitted vibration is to be less than 10% of the platform vibration over the given frequency range. Take ε = 0.20. System is treated as single degree of freedom.*

SOLUTION. Given $\varepsilon = 0.20$

$$\text{T.R.} = .10$$

$$m = 50 \text{ kg}$$

$$\text{T.R.} = \frac{[1 + 2\varepsilon r]^{1/2}}{\sqrt{[1-r^2]^2 + [2\varepsilon r]^2}} \qquad [r = \omega/\omega_n]$$

$$0.10 = \frac{[1 + .4r]^{1/2}}{\sqrt{[1-r^2]^2 + [.4r]^2}}$$

$$r = 4.15 = \omega/\omega_n$$

When $\omega = 2\pi \times 12 = 24\pi$ rad/sec

$$\omega_n = \frac{24\pi}{4.15} = 18.17 \text{ rad/sec}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\omega_n^2 = k/m$$

$$k = \omega_n^2 m = (18.17)^2 \times 50$$

$$= 16.50 \times 10^3 \text{ N/m}$$

$$\text{Stiffness of each spring} = \frac{16.50 \times 10^3}{4}$$

$$= 4.12 \times 10^3 \text{ N/m.}$$

EXAMPLE 4.7. *A vibrating body is supported by six isolators each having stiffness 32000 N/m and 6 dashpots having damping factor as 400 N-sec/m. The vibrating body is to be isolated by a rotating device having an amplitude of 0.06 mm at 600 rpm. Take m = 30 kg.*

Determine

(a) *Amplitude of vibration of the body*

(b) *Dynamic load on each isolator due to vibration.*

SOLUTION. Given $k = 6 \times 32000 = 19200$ N/m

$$c = 6 \times 400 = 2400 \text{ N-sec/m}$$

$$m = 30 \text{ kg}$$

$$N = 600 \text{ rpm}$$

$$B = 0.06 \text{ mm}$$

$$\omega = \frac{2\pi \times 600}{60} = 62.8 \text{ rad/sec}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{192000}{30}} = 80 \text{ rad/sec}$$

$$\frac{\omega}{\omega_n} = \frac{62.8}{80} = 0.785$$

$$\varepsilon = \frac{c}{2\sqrt{km}} = \frac{2400}{2\sqrt{192000 \times 30}} = 0.50$$

(a) Amplitude A can be determined as

$$\frac{A}{B} = \frac{\sqrt{1 + (2\varepsilon r)^2}}{\sqrt{[1-r^2]^2 + [2\varepsilon r]^2}} = \frac{\sqrt{1 + (2 \times .5 \times .785)^2}}{\sqrt{1 - (.6162)^2 + (.6162)}}$$

$$= 1.4549$$

$$A = .06 \times 1.4549 = .087 \text{ mm}$$

(b) The dynamic load F_D can be found as

$$F_D = Z\sqrt{k^2 + c^2\omega^2}$$

where Z is the relative amplitude of vibration, we have relation for that also

$$\frac{Z}{B} = \frac{r^2}{\sqrt{[1-r^2]^2 + [2\varepsilon r]^2}}$$

$$\frac{Z}{.06} = \frac{(.785)^2}{\sqrt{[1-(.785)^2]^2 + [2 \times .5 \times .785]^2}}$$

$$= 0.705$$

$$Z = .042 \times 10^{-3} \text{ m}$$

So $F_D = .042 \times 10^{-3}\sqrt{(192000)^2 + (2400 \times 62.8)^2}$

$$= 10.25 \text{ N}$$

Dynamic load on each isolator

$$= \frac{10.25}{6} = 1.708 \text{ N.}$$

EXAMPLE 4.8. *A vibratory body of mass 150 kg supported on springs of total stiffness 1050 k N/m has a rotating unbalance force of 525 N at a speed of 6000 rpm. If the damping factor is 0.3, determine*

(a) *the amplitude caused by the unbalance and its phase angle*

(b) *the transmissibility and*

(c) *the actual force transmitted and its phase angle.*

SOLUTION. $F = 525$ N

$$\varepsilon = .30$$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 6000}{60} = 200\pi \text{ rad/sec}$$

$$m = 150 \text{ kg}, \; k = 1050 \text{ k N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1050 \times 10^3}{150}} = 83.66 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = \frac{200\pi}{83.66} = 7.50$$

(*a*) Amplitude by unbalance force

$$\frac{A}{\frac{F}{k}} = \frac{1}{\sqrt{[1-r^2]^2 + [2\varepsilon r]^2}}$$

$$A = \frac{525/1050 \times 10^3}{\sqrt{[1-(7.5)^2]^2 + [2 \times .3 \times 7.5]^2}} = 9 \times 10^{-6} \text{ m}$$

$$\text{Phase angle, } \phi = \tan^{-1}\left[\frac{2\varepsilon r}{1-r^2}\right]$$

$$= \tan^{-1} \frac{2 \times .3 \times 7.5}{1-(7.5)^2}$$

$$\phi = 175^\circ\, 20'$$

(*b*) Transmissibility

$$\frac{F_T}{F} = \left\{\frac{1+(2\varepsilon r)^2}{[1-r^2]^2 + [2\varepsilon r]^2}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{1+(2 \times .3 \times 7.5)^2}{[1-(7.5)^2]^2 + [2 \times .3 \times 7.5]}\right\}^{\frac{1}{2}}$$

$$= .083$$

(*c*) Force transmitted

$$F_T = F \times .083$$

$$= 525 \times .083$$

$$= 43.58 \text{ N.}$$

EXAMPLE 4.9. *A vibrometer having the amplitude of vibration of the machine part as 4 mm and ε = 0.2, performs harmonic motion. If the difference between the maximum and minimum recorded value is 10 mm, determine the natural frequency of vibrometer if the frequency of the vibration part is 12 rad/sec.*

SOLUTION. $\varepsilon = 0.2$, $\omega = 12$ rad/sec

The amplitude of the recorded value is 5 mm *i.e.* $Z = 5$ mm

$$B = 4 \text{ mm}$$

Say $\quad r = \omega/\omega_n$

$$\frac{Z}{B} = \frac{r^2}{\sqrt{[(1-r^2)^2 + (2\varepsilon r)^2]}}$$

$$\left(\frac{5}{4}\right)^2 = \frac{r^4}{(1-r^2)^2 + (2\varepsilon r)^2} = \frac{r^4}{(1-r^2)^2 + (2 \times .2r)^2}$$

$$.5625\, r^4 - 2.875\, r^2 + 1.5625 = 0$$

$$r = 2.12$$

$$\frac{\omega}{\omega_n} = r$$

$$\therefore \quad \omega_n = \frac{\omega}{r} = \frac{12}{2.12} = 5.66 \text{ rad/sec}$$

EXAMPLE 4.10. *A vibrometer indicates 2 percent error in measurement and its natural frequency is 5 Hz. If the lowest frequency that can be measured is 40 Hz, find the value of damping factor (ε).*

SOLUTION. $r = \dfrac{\omega}{\omega_n} = \dfrac{40}{5} = 8$

$$\frac{Z}{B} = 1.02 \qquad \text{(since 2\% is there)}$$

$$\frac{Z}{B} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\varepsilon r)^2}}$$

$$(1.02)^2 = \frac{8^4}{(1-64)^2 + (16\varepsilon)^2}$$

$$\varepsilon = 0.35.$$

EXAMPLE 4.11. *A vibration measuring device is used to find the displacement, velocity and acceleration of a machine running at 120 rpm. If the natural frequency of the instrument is 5 Hz and it shows 0.004 cm. what are the three readings ? Assume no damping.*

SOLUTION. $\omega_n = 5 \text{ Hz} = 5 \times 2\pi = 10\pi$ rad/sec

$$\frac{Z}{B} = \frac{r^2}{1-r^2}$$

$$r = \frac{\omega}{\omega_n} = \frac{2\pi \times 120}{60 \times 10\pi} = .40$$

$$Z = .004 \text{ cm}$$

Displacement $\; B = Z\left(\dfrac{1-r^2}{r^2}\right)$

$$= .004\, \frac{(1 - .4 \times .4)}{.4 \times .4} = .021 \text{ cm}$$

Velocity $\; = \omega B = \left(\dfrac{2\pi N}{60}\right) B$

$$= \left(2\pi \times \frac{120}{60}\right) \times .021 = 0.26 \text{ cm/sec}$$

Acceleration $\; = \omega^2 B = \omega\, (\omega B)$

$$= \left(2\pi \times \frac{120}{60}\right) \times .26 = 3.265 \text{ cm/sec}^2.$$

EXAMPLE 4.12. *Prove that an undamped measuring instrument will show a true response for frequency ratio* $(\omega/\omega_n) = \dfrac{1}{\sqrt{2}}$.

SOLUTION. $\dfrac{Z}{B} = \dfrac{r^2}{1-r^2}$

Since $\quad \varepsilon = 0$

For a true response $Z = B$, so

$$1 = \frac{r^2}{1-r^2}$$

$$2r^2 = 1$$

$$r = \frac{1}{\sqrt{2}} \; i.e. \; \omega/\omega_n = \frac{1}{\sqrt{2}}$$

EXAMPLE 4.13. *For measuring the frequency of vibration of a system, a Frahm's Reed Tachometer is to be designed. A mass of 0.01 kg is to be placed at the end of one of the reeds so that the reed is in resonance at a frequency of 15 Hz. The steel reed is 40 mm long and 3 mm wide. Determine the length of the reed. Take* $E = 2 \times 10^{11}$ *N/m²*.

SOLUTION. Deflection for a cantilever is given as

$$\text{Deflection} = \frac{mgl^3}{3EI}$$

$$k = \frac{mg}{\text{deflection}} = \frac{mg.3EI}{mgl^3} = \frac{3EI}{l^3}$$

$$I = \frac{1}{12}\, bt^3 \qquad \text{where } t = \text{thickness of reed}$$

The natural frequency, f_n is given as

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{3EI}{l^3 m}}$$

or $\quad f_n^2 = \dfrac{1}{4\pi^2} \cdot \dfrac{3EI}{l^3 m}$

$$I = \frac{1}{12}\, .003\, t^3 = 2.5 \times 10^{-4} t^3$$

$$m = .01 \text{ kg}$$

$$l = .04 \text{ m}$$

$$E = 2 \times 10^{11} \text{ N/m}^2$$

$$f = 15$$

So $\quad 15 \times 15 = \dfrac{1}{4\pi^2}\, \dfrac{3 \times 2 \times 10^{11} \times 2.5 \times 10^{-4} t^3}{(.04)^3 \times .01}$

$$t^3 = 3.786 \times 10^{-11} \text{ m}^3$$

$$t = 7.2 \times 10^{-3} \text{ m.}$$

EXAMPLE 4.14. *What will be the frequency ratio when the amplitude in forced vibrations is maximum. Determine the peak amplitude and the corresponding phase angle.*

SOLUTION. We know that

$$\frac{A}{X_s} = \frac{1}{\sqrt{(1-r^2)^2 + (2\varepsilon r)^2}}$$

where $r = \omega/\omega_n$.

A will be maximum when denominator is minimum, *i.e.*

$$\frac{d}{dr}\left[(1-r^2)^2 + (2\varepsilon r)^2\right] = 0$$

$$\frac{d}{dr}\,[1 + r^4 - 2r^2 + 4\varepsilon^2 r^2] = 0$$

$$4r^3 - 4r + 8\varepsilon^2 r = 0$$

$$4r^2 - 4 + 8\varepsilon^2 = 0$$

or $\quad r^2 - 1 + 2\varepsilon^2 = 0, \qquad r^2 = 1 - 2\varepsilon^2$

$$r = (1 - 2\varepsilon^2)^{1/2}$$

And substituting the value of r^2, we get

$$\left.\frac{A}{X_s}\right|_{peak} = \frac{1}{\sqrt{(1-1+2\varepsilon^2)^2 + (2\varepsilon)^2(1-2\varepsilon^2)}}$$

$$= \frac{1}{\sqrt{4\varepsilon^4 + 4\varepsilon^2 - 8\varepsilon^4}}$$

$$= \frac{1}{2\varepsilon\sqrt{1-\varepsilon^2}}$$

Phase angle,

$$\tan\phi = \frac{2\varepsilon r}{1-r^2} = \frac{2\varepsilon\sqrt{1-2\varepsilon^2}}{1-(1-2\varepsilon^2)}$$

$$= \frac{2\varepsilon\sqrt{1-2\varepsilon^2}}{2\varepsilon^2}$$

$$= \frac{\sqrt{1-2\varepsilon^2}}{\varepsilon}$$

EXAMPLE 4.15. *Consider the spring mass system shown in figure 4.1. The mass is given a velocity of 0.1 m/sec. What will be the subsequent displacement and velocity of the mass if*

$C = 100$ *N–sec/m*

$k = 3000$ *N/m*

$m = 20$ *kg*

$F \sin \omega t = 0$

Assume the initial velocity of the mass as zero. Calculate the steady state response of the mass if $F \sin \omega t = 5 \sin 10t$.

SOLUTION. The equation of motion can be written as

$$m\ddot{x} + c\dot{x} + kx = 0$$

and
$$x = e^{-\varepsilon\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3000}{20}} = 12.24 \text{ rad/sec}$$

$$\varepsilon = \frac{c}{2m\omega_n} = \frac{100}{2\times 20\times 12.24} = 0.204$$

Damped frequency,

$$\omega_d = \sqrt{1-\varepsilon^2}\,\omega_n$$

$$= \sqrt{1-.204^2}\,.12.24$$

$$= 11.98 \text{ rad/sec}$$

Substituting the values of ω_n, ε and ω_d, we get

$$x = e^{-.204\times 12.24t}(A\cos 11.98\,t + B\sin 11.98\,t)$$

$$= e^{-2.49t}(A\cos 11.98\,t + B\sin 11.98\,t)$$

Velocity $\quad \dot{x} = \dfrac{dx}{dt}$

$$= 11.98\,e^{-2.49t}(-A\sin 11.98\,t + B\cos 11.98\,t)$$
$$-\,2.49\,e^{-2.49t}(A\cos 11.98\,t + B\sin 11.98\,t)$$

Applying boundary conditions, we have

At $\quad t = 0, x = 0$

and At $\quad t = 0, \dot{x} = .1$ m/sec

$$0 = e^0 . A, \quad A = 0$$

and
$$0.1 = 11.98\,(B) - 2.49$$

$$11.98\,B = .1 + 2.49$$

$$B = 2.59/11.98 = 0.216$$

So
$$x = 0.216e^{-2.49t}\sin 11.98\,t$$

$$\dot{x} = 11.98e^{-2.49t}\,0.216\cos 11.98t$$
$$-\,2.49\,e^{-2.49t}\,0.216\sin 11.98t$$

$$= 2.58\,e^{-2.49t}\cos 11.98t - 0.53\,e^{-2.49t}\sin 11.98t$$

$$= e^{-2.49t}(2.58\cos 11.98t - 0.53\sin 11.98t)$$

The transient is $x = x_c$ (Complementary solution)

$$x_c = 0.216\,e^{-2.49t}\sin(11.98t + \phi)$$

where $\phi = \tan^{-1} A/B$.

The steady state response of the system

$$x_p = \frac{F}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}\sin(\omega t - \alpha)$$

$$= \frac{5}{\sqrt{(3000 - 20\times 10\times 10)^2 + (100\times 10)^2}}\sin(10t - \alpha)$$

$$= 0.11\sin(10t - \alpha)$$

where
$$\alpha = \tan^{-1}\frac{2\varepsilon\omega/\omega_n}{1-(\omega/\omega_n)^2}$$

$$= \tan^{-1}\frac{2\times .204\times 10/12.24}{1-\left(\dfrac{10}{12.24}\right)^2}$$

$$= \tan^{-1}(1)$$

$$\alpha = 45°$$

$$x = x_c + x_p$$

$$= D\,e^{-2.49t}\sin(11.98t + \phi) + 0.11\sin(10t - 45)$$

Applying the boundary conditions the values of D and ϕ can be determined and then the value of x be written finally.

EXAMPLE 4.16. *Figure 4.23 shows an automobile trailer which moves over the road surface making approximately sinusoidal profile*

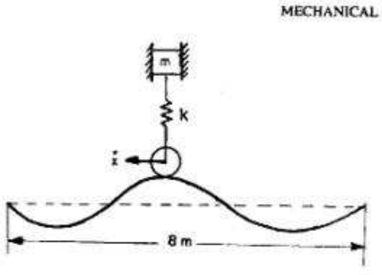


Fig.4.23

pulled on the road surface with a velocity of 60 km/hr. Calculate the critical speed of trailer if the vibration amplitude is 1.5 cm for the trailer mass of 50 kg.

SOLUTION. For negligible damping, we know

$$\frac{A}{B} = \frac{1}{1-r^2} \qquad \text{since } \varepsilon = 0 \qquad (\text{if } A > B)$$

$$r = \omega/\omega_n$$

$$\omega_n = \sqrt{\frac{k}{m}}, \quad B = 6 \text{ cm}$$

The time period of forced vibration is given as

$$\text{Time period} = \frac{\text{Wavelength}}{\text{Velocity}}$$

$$T = \frac{\lambda}{v} = \frac{8\times 60\times 60}{60\times 1000} = 0.48 \text{ sec}$$

$$T = \frac{1}{f}$$

$$\omega = 2\pi f = 2\pi\times\frac{1}{.48} = 13.08 \text{ rad/sec}$$

A is given as $A = 1.5$ cm

$$\frac{A}{B} = \frac{1}{r^2-1}\,(B > A)$$

$$\frac{1.5}{6} = \frac{1}{r^2-1}$$

$$r^2 - 1 = 4$$

$$r^2 = 5, r = 2.23$$

$$r = \omega/\omega_n$$

So $\quad \omega_n = \omega/r = \dfrac{13.08}{2.23} = 5.86$ rad/sec

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\omega_n^2 . m = k$$

$$k = (5.86)^2\times 50 = 1716.98 \text{ N/m}$$

At critical speed there will be resonance, *i.e.* $\omega = \omega_n$

$$\omega = 5.86 \text{ rad/sec}$$

$$\omega = 2\pi f$$

or
$$f = \omega/2\pi$$

or
$$T = 2\pi/\omega = 1.07 \text{ sec}$$

$$v = \frac{\lambda}{T} = \frac{8}{1.07} = 7.47 \text{ m/sec}$$

or $\quad$ 26.9 km/hr.

EXAMPLE 4.17. *A 3 kg mass is suspended in a box by a spring as shown in figure 4.24. The box is put on a platform having vibration* $y = 0.8 \sin 6t$ *cm. Determine the absolute amplitude of the mass.*

Given $k = 6000$ *N/m.* (P.U., 76 Aero)

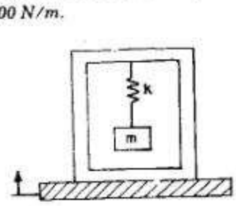


Fig. 4.24

SOLUTION. Damping because of air is neglected in engineering problems.

Then the amplitude of the relative motion of mass is given by

$$\frac{Z}{B} = \frac{r^2}{1-r^2}$$

where $r = \omega/\omega_n$.

Z and B are the relative and support amplitudes respectively.

Since $\quad y = B\sin\omega t$

So by comparison, we get

$$B = 0.8 \text{ cm}$$

and $\omega = 6$ rad/sec (By comparing with equation $A = B \sin \omega t$)

$$r = \omega/\omega_n$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{6000}{3}}$$

$$= 44.72 \text{ rad/sec}$$

$$r = \omega/\omega_n = \frac{6}{44.72} = .134$$

$$r^2 = 0.018$$

$$Z = \frac{0.8 \times .018}{1 - .018} = 0.0146 \text{ cm}$$

Thus the absolute amplitude of the mass

$$Z + B = 0.0146 + 0.8$$

$$= 0.814 \text{ cm.}$$

EXAMPLE 4.18. *Determine the relative amplitude of the end of cantilever reed with respect to base for the system shown in figure 4.25. The base is performing a harmonic motion $y = 0.8 \sin 10t$ cm in a direction perpendicular to the reed. The natural frequency of the system is twice the disturbing frequency.*

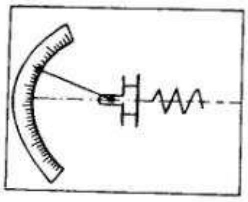


Fig. 4.25.

SOLUTION.

$$\frac{Z}{B} = \frac{r^2}{1 - r^2} \qquad (\varepsilon = 0)$$

$$r = \omega/\omega_n = 1/2$$

$$B = 0.8 \text{ cm}$$

$$Z = \frac{0.8(1/2)^2}{1 - (1/2)^2} = 0.26 \text{ cm.}$$

EXAMPLE 4.19. *Investigate the terms involved in the equations of motion of one degree of freedom system as given by*

$$5\ddot{x} + 3\dot{x} + 12x = 10 \sin \omega t.$$

SOLUTION. The general expression is given by the equation

$$m\ddot{x} + c\dot{x} + kx = F \sin \omega t$$

By comparison, we have

$$m = 5, \quad c = 3, \quad k = 12, \quad F = 10$$

With the help of above values, we find :

Natural frequency

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12}{5}} = 1.55 \text{ units}$$

Damping factor

$$\varepsilon = \frac{c}{2m\omega_n} = \frac{3}{2 \times 5 \times 1.55} = 0.19$$

Damped frequency

$$\omega_d = \sqrt{1 - \varepsilon^2}\ \omega_n = \sqrt{(1 - 0.19^2)}\ 1.55$$

$$= 1.52 \text{ units}$$

Critical damping

$$\frac{C_c}{2m} = \omega_n$$

$$C_c = \omega_n \,.\, 2m$$

$$= 1.55 \times 2 \times 5 = 15.5 \text{ units}$$

$$X_s = \text{deflection} = \frac{F}{k} = 10/12 = 0.833$$

Amplitude at resonance

$$\frac{A}{X_s} = \frac{1}{2\varepsilon}$$

$$A = \frac{X_s}{2\varepsilon} = \frac{0.833}{2 \times 0.19} = 2.19$$

Maximum frequency corresponding to maximum amplitude

$$\omega_{max} = \omega_n \sqrt{1 - 2\varepsilon^2}$$

$$= 1.55\sqrt{1 - 2 \times 0.19 \times 0.19}$$

$$= 1.49 \text{ units}$$

Magnification factor,

$$\frac{A}{X_s} = \frac{1}{2\varepsilon} = \frac{1}{2 \times 0.19} = 2.63 \text{ units}$$

Logarithmic decrement

$$\delta = \frac{2\pi\varepsilon}{\sqrt{1 - \varepsilon^2}} = \frac{1.19}{\sqrt{1 - .19^2}} = 1.21 \text{ units}$$

Phase angle

$$\phi = \tan^{-1} \frac{2\varepsilon(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}$$

$$= \tan^{-1} \frac{2 \times .19 \left(\frac{1.52}{1.55}\right)}{1 - \left(\frac{1.52}{1.55}\right)^2}$$

$$\phi = 84.1°.$$

EXAMPLE 4.20. *Determine the limiting frequency ω for an Accelerometer with 2.5% error having damping factor $\varepsilon = 0.70$ and natural frequency 80 Hz.*

SOLUTION. For accelerometer, we have

$$\frac{Z}{B} = \left(\frac{\omega}{\omega_n}\right)^2 . f, \; Z_{exact} = Br^2 f$$

where

$$f = \frac{1}{\sqrt{\{[1 - (\omega/\omega_n)^2]^2 + [2\varepsilon\, \omega/\omega_n]^2\}}}$$

In the above equation in case of Accelerometer, the ratio $\omega/\omega_n << 1$ or $\omega/\omega_n \to 0$, then the factor f can be written as unity.

The approximate expression is written by

$$\frac{Z}{B} = (\omega/\omega_n)^2 = r^2$$

$$Z_{approx} = Br^2$$

Then percentage error in the instrument

$$\% \text{ Error} = \frac{Br^2 - Br^2 f}{Br^2} \times 100 = (1 - f) \times 100$$

Putting the data for f, we get

$$f = \frac{1}{\sqrt{(1 - r^2)^2 + (2 \times .7 \times r)^2}}$$

$$f = \frac{1}{\sqrt{(1 + r^4 - 2r^2 + 1.96r^2)}}$$

$$= \frac{1}{\sqrt{(r^4 - .04r^2 + 1)}}$$

Error relation

$$2.5 = (1 - f) \times 100$$

$$= \left[1 - \frac{1}{\sqrt{(r^4 - .04r^2 + 1)}}\right] \times 100$$

$$6.25 = \left(1 + \frac{1}{r^4 - .04r^2 + 1} - \frac{2}{\sqrt{r^4 - .04r^2 + 1}}\right) \times (100)^2$$

Solving it we find $\quad r = \omega/\omega_n = 0.5$

limiting frequency, $\omega = .50 \times 80 = 40$ Hz.

EXAMPLE 4.21. *The point of suspension of a simple pendulum performs a harmonic motion as expressed by $x_0 = X_0 \sin \omega t$ along a horizontal line, as shown in figure 4.26. Derive equation of motion from the given coordinates. Find the solution for x/X and prove that when $\omega = \sqrt{2}\ \omega_n$, there is vibration at mid point of l.*

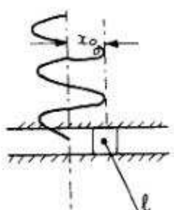


Fig. 4.26.

SOLUTION. Let any moment the pendulum makes an angle θ with the vertical, then

$$F \cos\theta = mg \qquad \text{where } F = \text{tension in string}$$

and

$$m\ddot{x} = -F \sin\theta$$

$$= -F\frac{(x - x_0)}{l} = -mg\frac{(x - x_0)}{l}$$

$$\left(\begin{array}{r}\text{For small } \theta,\ \cos\theta = 1 \\ F\cos\theta = F = mg \\ \text{see figure 4.27}\end{array}\right)$$

$$l\ddot{x} + gx - gx_0 = 0$$

This is the required equation of motion. The general forced vibration equation is

$$m\ddot{x} + kx = F \sin \omega t$$

So the solution of the above equation

$$\frac{X}{X_0} = \frac{1}{1 - \frac{l}{g}\omega^2}$$

$$\left[\because \frac{Z}{B} = \frac{1}{1-r^2} \text{ when } \varepsilon = 0\right]$$

$$\frac{X}{X_0} = \frac{1}{1 - \frac{\omega^2}{\omega_n^2}} \quad (r = \omega/\omega_n,\ \omega_n = \sqrt{g/l})$$

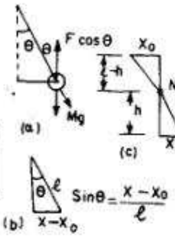


Fig. 4.27

$$\frac{h}{l-h} = \frac{-X}{X_0} = -\frac{1}{1-(\omega/\omega_n)^2}$$

or
$$\frac{l}{h} - 1 = -1 + \frac{\omega^2}{\omega_n^2}$$

$$h = l\,(\omega_n/\omega)^2$$

If $h = l/2,\ (\omega/\omega_n)^2 = 2$

$$\omega = \sqrt{2}\,\omega_n.$$

Example 4.22. *The motion of a vibratory system is to be recorded by a seismic instrument having natural frequency 3 rad/sec. What is the reading of the instrument if the motion is given by the equation*

$$Z = 2 \sin 2t + 3 \sin 3t$$

Take $\varepsilon = 0.5$

Solution. We know that

$$\frac{Z}{B} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\varepsilon r)^2}}$$

From the above equation, we have

$$\omega_1 = 2 \text{ rad/sec}$$

$$\omega_2 = 3 \text{ rad/sec}$$

Natural frequency $\omega_n = 3$ rad/sec (given)

$$r_1 = \frac{\omega_1}{\omega_n} = \frac{2}{3} = 0.67$$

$$r_2 = \frac{\omega_2}{\omega_n} = \frac{3}{3} = 1.0$$

The response of instrument to first part of motion *i.e.* 2 sin 2*t* is given as

$$\left(\frac{Z}{B}\right)_1 = \frac{r_1^2}{\sqrt{(1-r_1^2)^2 + (2\varepsilon r_1)^2}}$$

$$= \frac{(0.67)^2}{\sqrt{(1-.67^2)^2 + (2\times .5\times 0.67)^2}}$$

$$= \frac{(0.67)^2}{\sqrt{.303 + 0.448}} = 0.517 \text{ units}$$

Phase angle

$$\phi_1 = \tan^{-1}\frac{2r_1\varepsilon}{1-r_1^2} = \tan^{-1}\frac{2\times .5\times 2/3}{1-(2/3)^2}$$

$$= \tan^{-1} 1.2,\quad \phi_1 = 50.19^\circ$$

Similarly for the second part of the motion

$$\left(\frac{Z}{B}\right)_2 = \frac{r_2^2}{\sqrt{(1-r_2^2)^2 + (2\varepsilon r_2)^2}} = 1.0 \text{ units}$$

$$\phi_2 = \tan^{-1}\frac{2\varepsilon r_2}{1-r_2^2} = \tan^{-1}\frac{1}{0}$$

$$\phi_2 = 90^\circ$$

The instrument will show the reading as the combination of both the motions *i.e.*

$$Z = z_1 \sin(2t - 50.19^\circ) + z_2 \sin(3t - 90^\circ)$$

where $z_1 = 0.517$ and $z_2 = 1.0$.

Example 4.23. *A single cylinder vertical petrol engine of total mass 320 kg is mounted upon a steel chassis and causes a vertical static deflection of 2 mm. The reciprocating parts of the engine have a mass of 24 kg and move through a vertical stroke of 150 mm with simple harmonic motion. A dashpot attached to the system offers a resistance of 490 N at a velocity of 0.3 m/sec. Determine :*

(*a*) *The speed of the driving shaft at resonance ; and*

(*b*) *The amplitude of steady state vibration when the driving shaft of the engine rotates at 480 r.p.m.* (M.D.U., 94, 95)

Solution. $m = 320$ kg

Static deflection $\delta = 0.002$ m

$$\omega_n = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{0.002}} = 70 \text{ rad/sec}$$

$$\omega_n = 2\pi N/60$$

$$\text{Resonant speed, } N = \frac{\omega_n \times 60}{2\pi}$$

$$= \frac{70\times 60}{2\pi} = 668.8 \text{ r.p.m.}$$

(*b*)
$$\frac{A}{\frac{m_0 e}{m}} = \frac{(\omega/\omega_n)^2}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\varepsilon\omega/\omega_n]^2}}$$

We have to find A

$$\frac{m_0}{m} = \frac{24}{320} = 0.075$$

$$e = \frac{0.15}{2} = .075 \text{ m}$$

$$\omega = 2\pi \times \frac{480}{60} = 50.24 \text{ rad/sec}$$

$$\omega/\omega_n = \frac{50.24}{70} = 0.717$$

$$\varepsilon = \frac{C}{2m\omega_n} = \frac{490/0.3}{2\times 320\times 70} = 0.03645$$

Substituting the values in the above equation

$$\frac{A}{0.075\times 0.075} = \frac{(0.717)^2}{\sqrt{[1-(.717)^2]^2 + [2\times .03645\times .717]^2}}$$

It gives, $A = 6.04 \times 10^{-3}$ m.

Example 4.24. *The springs of an automobile trailer are compressed 0.1 m under its own weight. Find the critical speed when the trailer is passing over a road with a profile of sinewave whose amplitude is 80 mm and the wavelength is 14 m. Find the amplitude of vibration at a speed of 60 km/hr.* (M.D.U., 94 ; P.U., Aero 91)

Solution.
$$\omega_n = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{0.1}} = 9.9 \text{ rad/sec}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{9.9}{2\pi} = 1.576 \text{ cycles/sec}$$

Let us assume the critical speed of the trailer as V m/sec.

$$\frac{\text{Speed}}{\text{Wavelength}} = \text{Frequency}$$

$$\frac{V}{14} = f_n = 1.576$$

$$V = 1.576 \times 14 = 22 \text{ m/sec}$$

or
$$V = 79.2 \text{ km/hr}$$

The value of forced frequency ω at speed of 60 km/hr is

$$\omega = \frac{60\times 1000}{60\times 60} \times \frac{2\pi}{14} \qquad \left(\omega = \frac{V}{\lambda}\right)$$

$$= 7.47 \text{ rad/sec}$$

$$\omega/\omega_n = \frac{7.47}{9.9} = 0.755$$

Using equation (4.4.1.5), we get

Taking $\varepsilon = 0$

$$\frac{A}{B} = \frac{1}{1-(\omega/\omega_n)^2}$$

$$\frac{A}{0.08} = \frac{1}{1-(.755)^2}$$

$$A = 0.186 \text{ m.}$$

Example 4.25. *A seismic instrument with a natural frequency of 6 Hz is used to measure the vibration of a machine operating at 120 r.p.m. The relative displacement of the seismic mass as read from the instrument is 0.05 mm. Determine the amplitude of vibration of the machine. Neglect damping.* (M.D.U., 94)

Solution. The controlling equation is

$$\frac{Z}{B} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\varepsilon r)^2}}$$

$$= \frac{r^2}{(1-r^2)} \qquad (\text{when } \varepsilon = 0.0)$$

$$f_n = 6 \text{ Hz}$$

and
$$f = \frac{N}{60} = \frac{120}{60} = 2 \text{ Hz}$$

$$r = \frac{2}{6} = .33$$

$$Z = 0.05 \text{ m} = \text{reading of instrument}$$

Using the above equation, we get

$$\frac{.05}{B} = \frac{(.33)^2}{1-(.33)^2} = 0.122$$

$$B = 0.409 \text{ mm.}$$

Example 4.26. *An industrial machine weighing 445 kg is supported on a spring with a statical deflection of 0.5 cm. If the machine has a rotating imbalance of 25 kg-cm, determine the force transmitted at 1200 rpm and the dynamic amplitude at that speed.* (P.U., 93)

Solution. $\omega_n = \sqrt{\dfrac{g}{\delta}} = \sqrt{\dfrac{981}{0.5}} = 44.29$ rad/sec

where δ = static deflection

Forced frequency corresponding to 1200 rpm.

$$\omega = \frac{2\pi N}{60} = 2\pi \times \frac{1200}{60} = 125.6 \text{ rad/sec}$$

Frequency ratio $r = \dfrac{125.6}{44.29} = 2.84$

The dynamic amplitude A is given as

$$\frac{A}{\frac{m_0 e}{m}} = \frac{r^2}{\sqrt{(1-r^2)^2}} \qquad \text{(neglecting } \varepsilon\text{)}$$

$$A = \frac{m_0 e}{m} \cdot \frac{r^2}{\sqrt{(1-r^2)^2}} = \frac{25}{455} \frac{(2.84)^2}{\sqrt{[1-(2.84)^2]^2}}$$

$$A = .062 \text{ cm}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

So $k = \omega_n^2 . m = (44.29)^2 \times \dfrac{455}{981}$

$= 909.82$ kg/cm

Dynamic force transmitted

$$F = kA = 909.82 \times .062$$
$$= 56.41 \text{ kgf.}$$

Example 4.27. *A machine of mass one tonne is acted upon by an external force of 2450 N at a frequency of 1500 rpm. To reduce the effects of vibration, isolator of rubber having a static deflection of 2 mm under the machine load and an estimated damping $\varepsilon = 0.2$ are used.*

Determine :

(*a*) *the force transmitted to the foundation*

(*b*) *the amplitude of vibration of machine*

(*c*) *the phase lag.* (P.U., 89, 91)

Solution. Static deflection $= 2 \times 10^{-3}$ m

Given $m = 1000$ kg, $F = 2450$ N

Forcing frequency, $\omega = \dfrac{2\pi N}{60} = \dfrac{2\pi \times 1500}{60}$

$$\omega = 157 \text{ rad/sec}$$
$$\varepsilon = 0.2.$$

Force because of mass of machine

$$mg = 1000 \times 9.8 \text{ N}$$

$$K = \frac{mg}{\text{Static deflection}} = \frac{1000 \times 9.8}{2 \times 10^{-3}} = 49 \times 10^5 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{49 \times 10^5}{1000}} = 70 \text{ rad/sec}$$

$$r = \omega/\omega_n = \frac{157}{70} = 2.2428$$

$$r^2 = 5.0304$$

$$\frac{F_T}{F} = \frac{\sqrt{1+(2\varepsilon r)^2}}{\sqrt{(1-r^2)^2 + (2\varepsilon r)^2}} = \frac{\sqrt{1+(2 \times .2 \times 2.24)^2}}{\sqrt{[1-(5.03)^2]^2 + (2 \times .2 \times 2.24)^2}}$$

$$F_T = 798.8 \text{ N}$$

Using equation (4.2.10) for steady state vibrations

$$\frac{A}{X_s} = \frac{1}{\sqrt{(1-r^2)^2 + (2\varepsilon r)^2}}$$

X_s = External force/k

$$= \frac{2450}{49 \times 10^5} = 5 \times 10^{-4} \text{ m}$$

$$\frac{A}{X_s} = \frac{1}{\sqrt{(1-5.03)^2 + (2 \times .2 \times 2.24)^2}} = 0.241$$

$$A = 1.207 \times 10^{-4} \text{ m}$$

Using equation (4.2.8), we have

$$\tan\phi = \frac{2\varepsilon\omega/\omega_n}{1-(\omega/\omega_n)^2}$$

$$= \frac{2 \times .2 \times 2.24}{1-(2.24)^2} = -.223$$

$$\phi = -12.5^\circ \text{ or } 167^\circ\ 25'.$$

Example 4.28. *A weight of 2 kg is suspended by means of a spring having a stiffness of 4 kg/cm. The point of support (top of spring) is given a vertical periodic displacement (S.H.M.) at 1200 cycles per minute of maximum amplitude 0.32 cm. Determine (a) the absolute motion of the weight, and (b) the relative motion between the weight and the support.* (Roorkee, 68)

Solution. The natural frequency, $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{4 \times 981}{2}}$

$$\omega_n = 44.3 \text{ rad/sec}$$

Given $\omega = 1200 \times \dfrac{2\pi}{60} = 125.6$ rad/sec $= 40\pi$

$$B = 0.32 \text{ cm}$$

Frequency ratio, $r = \omega/\omega_n = \dfrac{125.6}{44.3}$

$$r = 2.83$$

We know that

$$\frac{A}{B} = \frac{1}{1-r^2} \qquad \text{(neglecting damping)}$$

$$A = \frac{B}{1-r^2} = \frac{0.32}{1-(2.83)^2} = -0.0457 \text{ cm}$$

The motion is harmonic

$$x = A \sin \omega t$$
$$= -.0457 \sin 40\pi t$$

The negative sign indicates that motions of support and weight are 180° out of phase.

The relative motion between support and weight

$$= 0.0457 + 0.32$$
$$= 0.3657 \text{ cm}$$

Finally, the equation of motion can be written as

$$x = 0.3657 \sin 40\pi t.$$

Example 4.29. *A mass weighing 1.93 kg is suspended in a box by vertical spring whose constant $k = 10$ kg/cm. The box is placed on the top of a shake table producing a vibration $x = 0.09 \sin 8t$. Find the absolute amplitude of mass.*

Assume no damping. (P.U., 94)

Solution. We know that if $\varepsilon = 0$

$$\frac{Z}{B} = \frac{r^2}{1-r^2}$$

Given $\omega = 8$ rad/sec

and $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{10 \times 981}{1.93}} = 71.29$ rad/sec

$$r = \frac{8}{71.29} = 0.112$$

$$r^2 = 0.01259$$

$$B = 0.09 \text{ cm} \qquad \text{(from given equation)}$$

$$\frac{Z}{0.09} = \frac{0.01259}{1-0.01259}$$

$$Z = 1.147 \times 10^{-3} \text{ cm}$$

Thus absolute amplitude of mass

$$Z + B = 1.147 \times 10^{-3} + .09 \text{ cm}$$
$$= 0.0911 \text{ cm.}$$

Example 4.30. *A vehicle weighs 490 kg. and total spring constant of its suspension system is 60 kg/cm. The profile of the road may be approximated to a line curve of amplitude 4.0 cm and wavelength of 4.0 metres. Determine :–*

(*a*) *the critical speed of the vehicle*

(*b*) *the amplitude of the steady state motion of the mass when the vehicle is driven at critical speed and the damping factor is 0.5 ; and*

(*c*) *the amplitude of the steady state motion of mass when the vehicle is driven at 57 km/hr and the damping factor same as in (b).* (P.U., 85)

Solution. (*a*) W = 490 kg

$$k = 60 \text{ kg/cm}$$

Amplitude $= 4$ cm $= B$

Wavelength $= 4.0$ m

General relation applicable

$$\frac{A}{B} = \frac{\sqrt{1+(2\varepsilon r)^2}}{\sqrt{(1-r^2)^2 + (2\varepsilon r)^2}}$$

Natural frequency of the system

$$\omega_n = \sqrt{\frac{k}{W/g}} = \sqrt{\frac{60 \times 981}{490}}$$

$$= 10.96 \text{ rad/sec}$$

The critical speed will be at resonance *i.e.* $\omega_n = \omega$

Let us represent it in km/hr.

So $V = \omega_n \times \dfrac{60 \times 60}{1000} \times \dfrac{\text{(Wavelength)}}{2\pi}$

$$= \frac{10.96 \times 3600 \times 4}{1000 \times 2\pi}$$

$$= 25.1 \text{ km/hr}$$

(*b*) $\dfrac{A}{B} = \dfrac{\sqrt{1+(2\varepsilon)^2}}{2\varepsilon}$ (at resonance $r = 1$)

$$A = B \cdot \frac{\sqrt{1+(2 \times .5)^2}}{2 \times .5} = 4\sqrt{2} = 5.66 \text{ cm}$$

(c) Time period =

$$\omega = \frac{\text{Speed (in km/hr)}}{60 \times 60} \times \frac{2\pi}{\text{Wavelength}}$$

$$= \frac{57 \times 1000}{60 \times 60} \times \frac{2\pi}{4} = 24.86 \text{ rad/sec}$$

So $\quad r = \omega/\omega_n = \frac{24.86}{10.96} = 2.268$

$$r^2 = 5.144$$

Using general relation

$$\frac{A}{B} = \frac{\sqrt{1+(2\varepsilon r)^2}}{\sqrt{(1-r^2)^2+(2\varepsilon r)^2}} = \frac{\sqrt{1+4\varepsilon^2 r^2}}{\sqrt{(1-5.144)^2+4\varepsilon^2 r^2}}$$

$$= \frac{\sqrt{1+4 \times .5 \times .5 \times 5.144}}{\sqrt{(1-5.144)^2+4 \times .25 \times 5.144}} = .5246$$

$$A = B \times .5246$$

$$= 4 \times .5246 = 2.098 \text{ cm.}$$

EXAMPLE 4.31. *The damped natural frequency of a system as obtained from a free vibration test is 9.8 c.p.s. During the forced vibration test, with constant exciting force, on the same system, the maximum amplitude of vibration is found to be at 9.6 c.p.s. Find the damping factor for the system and its natural frequency.* (P.U., 92)

SOLUTION. We know that

$$\frac{\omega_{max}}{\omega_n} = \sqrt{1-2\varepsilon^2} \qquad \text{(equation 4.3.13)}$$

We also know

$$\frac{\omega_d}{\omega_n} = \sqrt{1-\varepsilon^2}$$

From the above two equations, we find

$$\frac{\omega_{max}}{\omega_d} = \frac{\sqrt{1-2\varepsilon^2}}{\sqrt{1-\varepsilon^2}}$$

Given $\quad \omega_{max} = 9.6 \times 2\pi$ rad/sec

$$\omega_d = 9.8 \times 2\pi \text{ rad/sec}$$

So $\quad \frac{9.6 \times 2\pi}{9.8 \times 2\pi} = \frac{\sqrt{1-2\varepsilon^2}}{\sqrt{1-\varepsilon^2}}$

$$0.9596 = \frac{(1-2\varepsilon^2)}{1-\varepsilon^2}$$

which yields, $\quad \varepsilon = 0.196.$

EXAMPLE 4.32. *An instrument of 50 kg mass is located in an airplane cabin which vibrates at 2000 cpm with an amplitude of 0.1 mm. Determine the stiffness of the four steel springs required as supports for the instrument to reduce its amplitude to 0.005 mm. Also calculate the max. total load for which each spring must be designed.* (P.U., 85)

SOLUTION. $m = 50$ kg, $A = .0005$ cm, $F/k = .01$ cm

$$\frac{A}{F/k} = \frac{1}{|1-r^2|} \qquad \text{(neglecting damping)}$$

$$\omega^2 = \left(\frac{2\pi \times 2000}{60}\right)^2 = 43820.44 \text{ (rad/sec)}^2$$

$$\omega_n^2 = \frac{4k}{m} = \frac{4k}{50} = .08\,k$$

$$r^2 = \frac{\omega^2}{\omega_n^2} = \frac{43820.44}{.08k}$$

$$= \frac{547755.5}{k}$$

So using general relation

$$\frac{0.0005}{.01} = \frac{1}{1-r^2}$$

which gives, $\quad k = 26100$ N/m

Max. dynamic load on the springs

$= m\omega^2 A =$ max. inertia force on the instrument

$$= 50 \times 43820.44 \times .0005 \times 10^{-2}$$

$$= 10.96 \text{ N}$$

Hence, each spring is subjected to load $\frac{10.96}{4} = 2.74$ N.

EXAMPLE 4.33. *A machine having a mass of 100 kg and supported on springs of total stiffness 7.84×10^5 N/m has an unbalanced rotating element which results in a disturbing force of 392 N at a speed of 3000 rpm. Assuming a damping factor of $\varepsilon = 0.20$, determine*

(a) *the amplitude of motion due to unbalance,*

(b) *the transmissibility, and*

(c) *the transmitted force.* (P.U., 87, 88)

SOLUTION. $m = 100$ kg, $k = 7.84 \times 10^5$ N/m,

$$F = 392 \text{ N}, \varepsilon = 0.20$$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 3000}{60}$$

$$= 314 \text{ rad/sec}$$

$$\omega_n = \sqrt{k/m} = \sqrt{\frac{7.84 \times 10^5}{100}} = 88.59 \text{ rad/sec}$$

$$r = \omega/\omega_n = 3.54$$

$$r^2 = 12.56$$

(a) $\frac{Z}{F/k} = \frac{1 \cdot r^2}{\sqrt{(1-r^2)+(2\varepsilon r)^2}} \quad \left(\because \frac{Z}{B} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\varepsilon r)^2}} \text{ or } B = \frac{F}{\underline{K}}\right)$ k

Putting the value of various terms

$$Z = 4.29 \times 10^{-5} \text{ m}$$

(b) and transmissibility

$$\frac{F_t}{F} = 0.148$$

(c) $\quad \frac{F_t}{F} = \frac{\sqrt{1+(2\varepsilon r)^2}}{\sqrt{(1-r^2)^2+(2\varepsilon r)^2}} = \frac{\sqrt{1+(2 \times .2 \times 3.54)^2}}{\sqrt{[1-(3.54)^2]^2+(.2 \times 2 \times 3.54)}}$

$$= \frac{\sqrt{(1+2.005)}}{\sqrt{133.63+2.005}} = 0.148$$

$$F_t = 0.148 \times 392 = 58.3 \text{ N.}$$

EXAMPLE 4.34. *A machine 100 kg mass has a 20 kg rotor with 0.5 mm eccentricity. The mounting springs have $k = 85 \times 10^3$ N/m, $\varepsilon = 0.02$. The operating speed of machine is 600 rpm and the unit is constrained to move vertically. Find*

(i) *the dynamic amplitude of machine*

(ii) *the force transmitted to the supports.* (P.U., Aero 93, 92)

SOLUTION. $m_0 = 20$ kg, $m = 100$ kg

$$e = 0.5 \times 10^{-3} \text{ m}$$

$$\frac{m_0 e}{m} = \frac{20 \times 0.5 \times 10^{-3}}{100} = 10 \times 10^{-5}$$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 600}{60} = 62.8 \text{ rad/sec}$$

Natural frequency of the system, ω_n

$$\omega_n = \sqrt{k/m} = \sqrt{\frac{85 \times 10^3}{100}} = 29.1 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = \frac{62.8}{29.1} = 2.158$$

$$r^2 = 4.65$$

$$\varepsilon = 0.02$$

$$\frac{A}{\frac{m_0 e}{m}} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\varepsilon r)^2}}$$

$$\frac{A}{10 \times 10^{-5}} = \frac{4.65}{\sqrt{(1-4.65)^2+(2 \times .02 \times 2.158)^2}}$$

$$A = 1.27 \times 10^{-4} \text{ m}$$

Unbalanced force (force because of unbalance mass)

$$= m_0 \omega^2 e$$

$$= 20 \times (62.8)^2 \times .5 \times 10^{-3}$$

$$= 39.44 \text{ N}$$

The force transmitted to the foundation is given by the expression

$$\frac{F_t}{39.44} = \frac{\sqrt{1+(2\varepsilon r)^2}}{\sqrt{(1-r^2)^2+(2\varepsilon r)^2}}$$

$$= \frac{\sqrt{1+(2 \times .02 \times 2.158)^2}}{\sqrt{(1-4.65)^2+(2 \times .02 \times 2.158)^2}}$$

$$F_T = 10.84 \text{ N.}$$

EXAMPLE 4.35. *A vertical single stage air compressor having a mass of 500 kg is mounted on springs having stiffness of 1.96×10^5 N/m and $\varepsilon = 0.20$. The rotating parts are completely balanced and the equivalent reciprocating parts weigh 20 kg. The stroke is 0.2 m. Determine the dynamic amplitude of vertical motion and the phase difference between the motion and excitation force if the compressor is operated at 200 rpm.* (P.U., 90)

SOLUTION. $m = 500$ kg, $k = 1.96 \times 10^5$ N/m, $\varepsilon = 0.20$

$$m_0 = 20 \text{ kg}, \quad m = 500 \text{ kg}, \quad e = 0.1 \text{ m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1.96 \times 10^5}{500}} = 19.79 \text{ rad/sec}$$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 200}{60} = 20.93 \text{ rad/sec}$$

$$r = \omega/\omega_n = 1.057$$

$$r^2 = 1.11$$

$$\frac{A}{\frac{m_0 e}{m}} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\varepsilon r)^2}}$$

$$\frac{A}{\dfrac{20\times .1}{500}}=\frac{1.11}{\sqrt{(1-1.11)^2+(2\times .2\times 1.057)^2}}$$

$$A = 0.01 \text{ m}$$

Phase difference

$$\phi=\tan^{-1}\left(\frac{2\xi r}{1-r^2}\right)=\tan^{-1}\left(\frac{2\times .2\times 1.057}{1-1.11}\right)$$

$$\phi = 105.9°.$$

Example 4.36. *A 1000 kg machine is mounted on four identical springs of total spring constant k and negligible damping. The machine is subjected to a harmonic external force of amplitude F = 490 N and frequency 180 rpm. Determine :*

The amplitude of motion of machine and maximum force transmitted to the foundation because of unbalanced force when $k = 1.96\times 10^6$ N/m. (P.U., 78 Aero)

Solution. Natural frequency $\omega_n=\sqrt{\dfrac{k}{m}}=\sqrt{\dfrac{1.96\times 10^6}{1000}}$

$$= 44.27 \text{ rad/sec}$$

$$\omega=\frac{2\pi N}{60}=\frac{2\pi\times 180}{60}=18.84 \text{ rad/sec}$$

$$r=\omega/\omega_n=\frac{18.84}{44.27}=0.425$$

$$r^2 = .181$$

$$F = 490 \text{ N},\ \frac{F}{k}=\frac{490}{1.96\times 10^6}=2.5\times 10^{-4}\text{ m}$$

We know the relation

$$\frac{A}{F/k}=\frac{1}{\sqrt{(1-r^2)^2}}$$

$$\frac{A}{2.5\times 10^{-4}}=\frac{1}{(1-.181)}$$

$$A = 3.05\times 10^{-4}\text{ m}$$

Force transmitted

$$\frac{F_T}{F}=\frac{1}{\sqrt{(1-r^2)^2}}$$

$$F_T = 490\,\frac{1}{(1-.181)}$$

$$F_T = 598.3 \text{ N}.$$

Example 4.37. *A rotor of mass 4 kg is mounted on 1 cm diameter shaft at a point 10 cm from one end. The 25 cm long shaft is supported by bearings. Calculate the critical speed. If the centre of gravity of the disc is 0.03 mm away from the geometric centre of rotor, find the deflection of the shaft when its speed of rotation is 5000 r.p.m. Take $E = 1.96\times 10^{11}$ N/m².*

Find the critical speed when the rotor is mounted midway on the shaft.

Solution. The critical speed is given as

$$\omega_c=\sqrt{\frac{g}{\delta}}$$

and
$$\delta=\frac{Wbx}{6EIl}(l^2-x^2-b^2)$$

$$W = mg = 4\times 9.81 \text{ kg} = 39.24 \text{ N}$$

$$l = 25 \text{ cm} = .25 \text{ m}$$

$$x = 0.10 \text{ m}$$

$$b = .25 - .10 = 0.15 \text{ m}$$

$$d = 1.0 \text{ cm}$$

$$I=\frac{\pi}{64}d^4=\frac{\pi}{64}\times(0.01)^4=4.9\times 10^{-10}\text{ m}^4$$

$$\delta=\frac{39.24\times .15\times .10\,(0.25^2-0.10^2-0.15^2)}{6\times 1.96\times 10^{11}\times 4.9\times 10^{-10}\times 0.25}$$

$$= 1.2257\times 10^{-4}\text{ m}$$

Critical speed $\omega_c=\sqrt{\dfrac{g}{\delta}}=\sqrt{\dfrac{9.80}{1.2257\times 10^{-4}}}$

$$\omega_c = 282.7 \text{ rad/sec}$$

$$=\frac{282.7\times 60}{2\pi}=2701 \text{ rpm.}$$

We can find $\omega=\dfrac{2\pi N}{60}=\dfrac{2\pi\times 5000}{60}=523.33$ rad/sec

Given, $e = 0.03$ mm

$$r=\omega/\omega_c=\frac{523.33}{282.7}=1.85$$

$$r^2 = 3.42$$

$$x=\frac{er^2}{r^2-1}=\frac{.03\times 3.42}{3.42-1}=0.042 \text{ mm}$$

When the rotor is mounted midway on the shaft, the static deflection is given by

$$\delta=\frac{Wl^3}{48EI}=\frac{39.24\times(.25)^3}{48\times 1.96\times 10^{11}\times 4.9\times 10^{-10}}$$

$$\delta = 1.33\times 10^{-4}\text{ m}$$

$$\omega_c=\sqrt{\frac{g}{\delta}}=\sqrt{\frac{9.8}{1.33\times 10^{-4}}}=271.5 \text{ rad/sec.}$$

Example 4.38. *A shaft of 2.5 cm diameter, freely supported by bearings 75 cm apart, carries a single concentrated load of 20 kg midway between the bearings. Determine the first critical speed. Assume that shaft material has a density of 8 gm/cm³ and E is 2.1×10^6 kg/cm².* (Roorkee Uni., 66-67)

Solution. If a concentrated load is acting at the centre of beam whose weight is to be taken into account, the static deflection is given by

$$\delta=\frac{\left(\dfrac{W}{g}+\dfrac{17}{35}\rho l\right)l^3}{48EI}\qquad\text{(From Strength of Materials)}$$

and critical speed $\omega_c=\sqrt{\dfrac{g}{\delta}}$

Given $W = 20$ kg

$$l = 75 \text{ cm}$$

$$\rho = \text{mass per unit length}$$

$$=\frac{8}{981}\times\frac{\pi}{4}\times d^2$$

$$=\frac{8}{981}\times\frac{\pi}{4}\times(2.5)^2=0.04 \text{ gm}$$

$$= .04\times 10^{-3}\text{ kg}$$

$$I=\frac{\pi}{64}\times d^4=\frac{\pi}{64}\times(2.5)^4=1.916 \text{ cm}^4$$

$$\delta=\frac{\left(\dfrac{20}{981}+\dfrac{17}{35}\times .04\times 10^{-3}\times 75\right)75^3}{48EI}$$

$$=\frac{(.020+1.457\times 10^{-3})(75)^3}{48\times 2.1\times 10^6\times 1.916}=\frac{.02145\times 75^3}{193.13\times 10^6}$$

$$\delta = .0468\times 10^{-3}\text{ cm}$$

$$\omega_c=\sqrt{\frac{g}{\delta}}=\sqrt{\frac{981\times 10^3}{.0468}}=4578 \text{ rad/sec}$$

$$f_c=\frac{\omega_c}{2\pi}=729 \text{ Hz.}$$

Example 4.39. *The rotor of a turbo super charger weighing 9 kg is keyed to the centre of a 25 mm diameter steel shaft 40 cm between bearings. Determine :*

(a) *the critical speed of shaft,*

(b) *the amplitude of vibration of the rotor at a speed of 3200 rpm, if the eccentricity is 0.015 mm. and*

(c) *the vibratory force transmitted to the bearings at this speed.*

Assume the shaft to be simply supported and that the shaft material has a density of 8 gm/cm³.

Take $E = 2.1\times 10^6$ kg/cm². (Roorkee Uni., 69)

Solution. The total weight of rotor and shaft

$$W = 9\text{ kg}+\frac{8}{1000}\times\frac{17}{35}\times\frac{\pi}{4}d^2l\text{ kg}$$

$$=(9+8\times 10^{-3}\times .486\times\frac{\pi}{4}\times 2.5^2\times 40)\text{ kg}$$

$$= 9.763 \text{ kg}$$

Static deflection $\delta = Wl^3/48EI$

$$I=\frac{\pi}{64}d^4=\frac{\pi}{64}\times(2.5)^4=1.916 \text{ cm}^4$$

$$\delta=\frac{9.763\times(40)^3}{48\times 2.1\times 10^6\times 1.916}=3235.24\times 10^{-6}\text{ cm}$$

$$\omega_c=\sqrt{\frac{g}{\delta}}=\sqrt{\frac{981\times 10^6}{3235.24}}=550.6 \text{ rad/sec}$$

$$= 5261 \text{ r.p.m.}$$

$$\omega=\frac{2\pi N}{60}=\frac{2\pi\times 3200}{60}=334.9 \text{ rad/sec}$$

(b) $\dfrac{\omega}{\omega_c}=0.608$

The amplitude of vibration

$$x=\frac{er^2}{1-r^2}=\frac{0.015\times(.608)^2}{1-(.608)^2}=8.79\times 10^{-3}\text{ mm}$$

(c) The force transmitted to the bearings

$$F = m\,(x + e)\,\omega^2$$

$$= \frac{9}{981}\,(8.79 \times 10^{-3} + 0.015)\,(334.9)^2$$

$$= 2.44 \text{ kg.}$$

Example 4.40. *A disc of mass 4 kg is mounted midway between bearings which may be assumed to be simple supports. The bearing span is 50 cm. The steel shaft is of 10 mm diameter and is horizontal. The centre of gravity of the disc is displaced 2 mm from the geometric centre. The equivalent viscous damping at the centre of the disc-shaft may be assumed as 50 N-sec/m. If the shaft rotates at 250 rpm, determine the maximum stress in the shaft. Also find the power required to derive the shaft, at this speed.*

Take $E = 1.96 \times 10^{11}$ N/m^2. (Roorkee Uni., 90-91)

Solution. Given $m = 4$ kg

The static deflection, $\delta = \dfrac{Wl^3}{48EI}$

$$e = 2 \text{ mm}$$

$$l = 50 \text{ cm} = .50 \text{ m}$$

$$E = 1.96 \times 10^{11} \text{ N/m}^2$$

$$d = 10 \text{ mm} = 1.0 \text{ cm}$$

$$I = \frac{\pi}{64}\,d^4 = \frac{\pi}{64} \times (1 \times 10^{-2})^4 = 4.9 \times 10^{-10} \text{ m}^4$$

$$48EI = 48 \times 1.96 \times 10^{11} \times 4.9 \times 10^{-10} = 4609.9$$

$$\delta = \frac{(mg)l^3}{48EI} = \frac{4 \times 9.8 \times (.5)^3}{4609.9}$$

$$= \frac{4.9}{4609.9} = 1.0629 \times 10^{-3} \text{ m}$$

Critical speed, $\omega_c = \sqrt{\dfrac{g}{\delta}} = \sqrt{\dfrac{9.8}{1.0629 \times 10^{-3}}} = 96$ rad/sec

Angular speed of shaft, $\omega = \dfrac{2\pi N}{60} = \dfrac{2\pi \times 250}{60} = 26.16$ rad/sec

The frequency ratio $\beta = \omega/\omega_c = \dfrac{26.16}{96} = 0.2725$

The damping factor, $\varepsilon = \dfrac{C}{2\sqrt{k_t m}} = \dfrac{C}{2m\sqrt{\omega_c^2}} = \dfrac{50}{2 \times 4 \times 96}$

We can use equation (4.11.8) for finding x

$$\frac{x}{e} = \frac{\beta^2}{\sqrt{(1-\beta^2)^2 + (2\varepsilon\beta)^2}}$$

$$\frac{x}{2} = \frac{(.2725)^2}{\sqrt{[1-(.2725)^2]^2 + [2 \times .065 \times .2725]^2}} = 0.080$$

$$x = 0.16 \text{ mm} = .16 \times 10^{-3} \text{ m}$$

The dynamic load on the bearings can be determined as

$$F_d = \sqrt{(\text{Spring force})^2 + (\text{Damping force})^2}$$

$$= \sqrt{(k_t x)^2 + (c\omega x)^2}$$

$$= x\,\sqrt{k_t^2 + c^2\omega^2}$$

$$\omega_c = \sqrt{\frac{k_t}{m}}$$

$$k_t = \omega_c^2 \,.\, m = 96 \times 96 \times 4 = 36864 \text{ N/m}$$

$$k_t^2 = 1.3589 \times 10^9$$

So $F_d = .16 \times 10^{-3}\,\sqrt{(36864)^2 + (50)^2 \times (26.16)^2}$

$$= 5.90 \text{ N.}$$

The dead load on the shaft $W = mg$

$$= 4 \times 9.8 = 39.2 \text{ N}$$

Total maximum load on the shaft under the above conditions

$$= 5.90 + 39.2 = 45.1 \text{ N}$$

We know that stress relation is expressed as

$$\frac{\sigma}{y} = \frac{M}{I}$$

$$\sigma = \frac{M}{I}\,y = \frac{Fld}{4\left(\frac{\pi}{64}d^4\right)2} = \frac{8Fl}{\pi d^3} \qquad \left(\text{since } M = \frac{Fl}{4}\right)$$

$$I = \frac{\pi}{64}\,d^4$$

$$y = d/2$$

The maximum stress under dynamic conditions is given as

$$\sigma_{max} = \frac{8Fl}{\pi d^3} = \frac{8F \times .50}{\pi \times (.1 \times 10^{-2})^3}$$

$$= F_{max} \times 1.273 \times 10^9$$

$$= 5.745 \times 10^{10} \text{ N/m}^2 \qquad (F_{max} = 45.1 \text{ N})$$

The maximum stress under dead load conditions

$$\sigma_{max} = (F_{max})_{dead\ load} \times 1.273 \times 10^9$$

$$= (4 \times 9.8) \times 1.273 \times 10^9$$

$$= 4.993 \times 10^{10} \text{ N/m}^2$$

The power required to drive the shaft

$$P = \frac{2\pi NT}{60}$$

where T = damping torque

= damping force $\times x$

$$= c\omega x \times x = c\omega x^2 = 50 \times 26.16 \times (.16 \times 10^{-3})^2$$

$$T = 3.348 \times 10^{-5} \text{ N-m}$$

$$P = \frac{2\pi \times 250 \times 3.348 \times 10^{-5}}{60}$$

$$= 87.6 \times 10^{-5} \text{ watt.}$$

Example 4.41. *A rotor of mass 12 kg is mounted in the middle of 25 mm diameter shaft supported between two bearings placed at 900 mm from each other. The rotor is having 0.02 mm eccentricity. If the system rotates at 3000 rpm, determine the amplitude of steady state vibrations and the dynamic force on the bearings. Take $E = 2 \times 10^5$ N/mm^2* (P.U., 95)

Solution. Let us assume the shaft simply supported, its stiffness is given by

$$k = \frac{48\,EI}{l^3} = \frac{48 \times (2 \times 10^{11})\,\frac{\pi}{64} \times (.025)^4}{(0.9)^3}$$

$$k = 2.523 \times 10^5 \text{ N/m}$$

Natural frequency $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{2.523 \times 10^5}{12}}$

$$= 145 \text{ rad/sec}$$

Also $\omega = \dfrac{2\pi N}{60} = \dfrac{2\pi \times 3000}{60} = 100\,\pi$ rad/sec

Using equation (4.10.1), we have

$$\frac{x}{e} = \frac{1}{\left(\frac{\omega_n}{\omega}\right)^2 - 1} = \frac{1}{\left(\frac{145}{100\,\pi}\right)^2 - 1} = -1.271$$

$$x = -.02 \times 10^{-3} \times 1.271 = -2.542 \times 10^{-5} \text{ m}$$

The negative sign indicates that the displacement is out of phase with centrifugal force.

Dynamic load on bearings

$$F_D = k\,.\,x$$

$$= 2.523 \times 10^5 \times 2.542 \times 10^{-5} = 6.41 \text{ N}$$

Thus the load on each bearing will be $\dfrac{6.41}{2} = 3.206$ N

Example 4.42. *An air compressor of 450 kg operates at a constant speed of 1750 r.p.m. Rotating parts are well balanced. The reciprocating part is 10 kg and crank radius is 100 mm. The mounting introduces a viscous damping of damping factor 0.15. Specify the spring for the mounting such that only 20% of the unbalanced force is transmitted to the foundation. Find out the amplitude of transmitted force.* (A.M.I.E., 1993)

Solution. Mass of compressor = 450 kg, speed = 1750 r.p.m.

Mass of reciprocating parts = 10 kg

Crank radius = 100 mm = 10 cm = r_1

Damping factor $\xi = 0.15$, T.R. = 20%

The transmissibility ratio (T.R.) is given by

$$\text{T.R.} = \frac{\sqrt{1 + (2\,\xi\,r)^2}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \qquad ...(i)$$

where $r = \dfrac{\omega}{\omega_n}$

Putting the values in equation (*i*), we get

$$0.2 = \frac{\sqrt{1 + (2 \times 0.15 \times r)^2}}{\sqrt{(1-r^2)^2 + (2 \times .15 \times r)^2}} = \frac{\sqrt{1 + 0.09\,r^2}}{\sqrt{(1-r^2)^2 + 0.09\,r^2}}$$

Squaring both sides, we get

$$0.04 = \frac{1 + 0.09\,r^2}{1 + r^4 - 2r^2 + 0.09\,r^2}$$

$$1 + r^4 - 2r^2 + 0.09\,r^2 = 25 + 2.25\,r^2$$

$$r^4 - 4.16\,r^2 - 24 = 0$$

$$r^2 = \frac{4.16 \pm \sqrt{(4.16)^2 + 96}}{2} = 7.4022$$

$$r = 2.72$$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 1750}{60} = 183.26 \text{ rad/sec}$$

But $\dfrac{\omega}{\omega_n} = r$, $\omega_n = \dfrac{\omega}{r} = \dfrac{183.26}{2.72} = 67.375$ rad/sec

We know that $\omega_n = \sqrt{\dfrac{k}{m}}$

or $$k = \omega_n^2 \,.\, m$$

$$= (67.375)^2 \times 450 = 2042.73 \text{ kN/m}$$

The amplitude of unbalanced force because of reciprocating parts

$$= m_1 r_1 \omega^2 = 10 \times .10 \times (183.26)^2$$

$$= 33.584 \text{ kN}$$

The amplitude of force transmitted

$$= \text{T.R.} \times 33.584$$

$$= 0.20 \times 33.584 = 6.7168 \text{ kN}$$

EXAMPLE 4.43. *A trailer has 1000 kg mass when fully loaded and 250 kg when empty. The spring of the suspension is 350 kN/m. The damping factor is 0.5 when the trailer is fully loaded. The speed is 100 km/hr. The road varies sinusoidally with a wave length of 5 m. Determine the amplitude ratio of the trailer when fully loaded and empty.* (A.M.I.E., 93)

SOLUTION. Mass of empty trailer = 250 kg, $\xi = 0.50$

Mass of loaded trailer = 1000 kg, $k = 350$ kN/m

Speed of trailer = 100 km/hr = $\dfrac{100 \times 1000}{3600} = 27.77$ m/sec

Time period = $\dfrac{\text{Wave length}}{\text{Velocity}} = \dfrac{5}{27.77}$ sec

Forcing frequency, $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{5/27.77} = 34.896$ rad/sec

Natural frequency of empty trailer

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{350 \times 10^3}{250}} = 37.416 \text{ rad/sec}$$

Frequency ratio $r = \omega/\omega_n = \dfrac{34.896}{37.416} = 0.933$

The ratio of amplitude of vibration of empty trailer to that of road surface is given as

$$\frac{A}{B} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} = \frac{\sqrt{1 + (2 \times 0.5 \times 0.933)^2}}{\sqrt{[1 - (0.933)^2]^2 + (2 \times .50 \times .933)^2}}$$

$$= \frac{1.3676}{0.9419} = 1.4518$$

When the trailer is fully load, the natural frequency can be determined

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{350 \times 10^3}{1000}} = 18.708 \text{ rad/sec}$$

Frequency ratio $r = \dfrac{34.896}{18.708} = 1.8653$

Amplitude ratio in this case

$$\frac{A}{B} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} = \frac{\sqrt{1 + (2 \times 0.5 \times 1.8653)^2}}{\sqrt{[1 - (1.8653)^2]^2 + (2 \times .50 \times 1.8653)^2}}$$

$$= \frac{2.116}{3.1026} = 0.6819$$

EXAMPLE 4.44. *A machine of mass 1000 kg is supported on springs which deflect 8 mm under the static load. With negligible damping the machine vibrates with an amplitude of 5 mm when subjected to a vertical harmonic force at 80 percent of the resonant frequency when a damper is fitted it is found that the resonant amplitude is 2 mm. Find :*

(i) The amplitude of the damping force, and

(ii) The damping coefficient.

SOLUTION. Deflection = 8 mm = 8×10^{-3} m

The spring stiffness is given by

$$k = \frac{mg}{\text{deflection}} = \frac{1000 \times 9.81}{8} = 1226.25 \text{ N/mm}$$

Natural frequency of vibration

$$\omega_n = \sqrt{\frac{g}{\text{deflection}}} = \sqrt{\frac{9.81}{8 \times 10^{-3}}}$$

$$= 35.017 \text{ rad/sec}$$

Excitation frequency

$$= \frac{80}{100} \times \omega_n = 0.8 \times 35.017 = 28.0136 \text{ rad/sec}$$

Frequency ratio $r = \dfrac{\omega}{\omega_n} = 0.8$

The amplitude of vibration due to exciting force F is given by

$$Z = \frac{F/k}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \qquad ..(i)$$

Assuming damping factor $\xi = 0$, equation (*i*) can be written as

$$F = Z \,.\, k\,(1 - r^2)$$

$$= 5 \times (1 - 0.8^2) \times 1226.25 \quad (\because Z = 5 \text{ mm})$$

$$F = 2207.25 \text{ N}$$

When damper is fitted, the amplitude of resonant vibration is 2 mm.

For resonance $\omega/\omega_n = 1$

So $$Z = \frac{F/k}{2\xi r} = \frac{\dfrac{2207.25}{1226.25}}{2\xi \times 1} \qquad (\because r = 1 \,\&\, Z = 2 \text{ mm})$$

$$\xi = 0.45$$

Damping coefficient

$$C = \xi C_c = (2m\,\omega_n)\,\xi = 2 \times 1000 \times 35.017 \times .45$$

$$= 31515.3 \text{ N . sec/m}$$

EXAMPLE 4.45. *A shaft 1.5 cm dia and 1 m long is held in long bearings. The weight of the disc at the centre of the shaft if 15 kg. The eccentricity of the centre of gravity of the disc from centre of rotor is 0.03 cm. The modulus of elasticity of the material of shaft is 2×10^6 kg/cm². The permissible stress in the shaft material is 700 kg/cm².*

Find : (i) The critical speed of the shaft;

(ii) The range of speed over which it is unsafe to run the shaft. Neglect the weight of the shaft. (A.M.I.E., 1995)

SOULTION. Given : $W = 15$ kg , $l = 100$ cm , $E = 2 \times 10^6$ kg/cm²,

$$f = 700 \text{ kg/cm}^2,\; e = 0.03 \text{ cm},\; d = 1.5 \text{ cm}$$

For a shaft supported in long bearings, it may be taken as a case with fixed ends and with point load W, the deflection δ may be written as

$$\delta = \frac{Wl^3}{192\,EI} = \frac{15 \times (100)^3}{192 \times 2 \times 10^6 \times \dfrac{\pi}{64}(1.5)^4} = 0.157 \text{ cm}$$

We know that natural frequency of transverse vibration is given by

$$\omega_c = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{981}{0.157}} = 79.05 \text{ rad/sec}$$

The critical speed can be written as

$$N_c = \frac{60 \times \omega_c}{2\pi} = \frac{60 \times 79.05}{2\pi} = 754.87 \text{ r.p.m.}$$

We know the bending relation

$$\frac{M}{I} = \frac{f}{y}$$

In this case $M = \dfrac{F\,.\,l}{8}$ and $y = d/2$

where F = dynamic force causing bending in the shaft

So $$\frac{F\,.\,l}{8I} = \frac{f}{y} \text{ or } F = \frac{8I}{l}\,.\,\frac{f}{y} = \frac{8 \times \dfrac{\pi}{64}d^4}{100} \times \frac{2 \times 700}{d}$$

$$F = \frac{\pi d^3}{4} \times 7 = \pi \times \frac{1.5^3}{4} \times 7 \qquad (\because d = 1.5 \text{ cm})$$

$$F = 18.55 \text{ kgf}$$

Additional deflection due to dynamic force F, can be determined by proportion as

$$x = \delta_1 = \frac{18.55}{15} \times 0.157 = 0.194 \text{ cm}$$

When the shaft is vertical

In this case the dead weight of the disc is 15 kgf may be neglected. We know that

$$\pm\frac{x}{e} = \frac{1}{\left(\dfrac{\omega_c}{\omega}\right)^2 - 1}. \quad \text{Let } \omega/\omega_c = r = \frac{N}{N_c}$$

$$\pm\frac{x}{e} = \frac{1}{\left(\dfrac{1}{r}\right)^2 - 1} = \frac{r^2}{1 - r^2}$$

$$\pm\frac{0.194}{0.03} = \frac{r^2}{1 - r^2},\; \pm 6.47\,(1 - r^2) = r^2 \text{ or } r = 0.93,\ 1.0876$$

Thus these are two values of speed. Let N_1 and N_2 be the speeds

$$\frac{N_1}{N_c} = r \quad \text{or} \quad N_1 = N_C \,.\, r = 754.87 \times 0.93 = 702.03 \text{ r.p.m.}$$

and $$\frac{N_2}{N_c} = r \text{ or } N_2 = N_c \,.\, r = 754.87 \times 1.0876 = 821 \text{ r.p.m.}$$

When the shaft is horizontal

In this case, the total deflection will be

$$x = \delta + \delta_1 = 0.157 + 0.194 = 0.351 \text{ cm.}$$

Using the relation, $\pm\dfrac{x}{e} = \dfrac{r^2}{1 - r^2}$

$$\pm\frac{0.351}{0.03} = \frac{r^2}{1 - r^2}$$

There are two values of r i.e. $r_1 = 0.9598$ or

$$r_2 = 1.04568$$

Thus there will be two values of speed.

$$N_1 = N_c \,.\, r_1 = 754.87 \times 0.9598 = 724.52 \text{ r.p.m.}$$

and, $$N_2 = N_c \,.\, r_2 = 754.87 \times 1.04568 = 789.35 \text{ r.p.m.}$$

Problems

1. A mass of 200 kg is suspended on a spring having a scale of 30000 N/m and is acted upon by a harmonic force of 80 N at the undamped natural frequency. The damping may be considered to be viscous with a coefficient of 200 N.sec/m. Calculate.
   (*a*) the undamped natural frequency
   (*b*) the amplitude of vibration of the mass, and
   (*c*) the phase difference between the force and the displacement.
2. A reciprocating pump 300 kg is driven through a belt by an electric motor at 4000 r.p.m. The pump is mounted on isolators with total stiffness 15 M N/m and damping 6 *k* Ns/m. Determine the vibratory amplitude of the pump at the running speed due to fundamental harmonic force of excitation 2 KN. Determine the amplitude at resonance also.
3. A small motor driving a compressor weighs 35 kg and causes each of the rubber isolators to deflect by 4 mm. The motor runs at a constant speed of 2000 rpm. The compressor piston has a 60 mm stroke. The piston and reciprocating parts weigh 1 kg and perform simple harmonic motion. The amplitude of vertical motion at the operating speed is 0.50 cm. Find damping factor for rubber.
4. A spring mass system has a natural frequency of 5 Hz. When the mass is at rest, the support is made to move up with displacement $x = 24 \sin 4\pi t$ (t in seconds and x in milimeters) measured from the beginning of the motion. Determine the distance through which the mass moves in the first 0.15 second.
5. An engine is mounted on 4 rubber pads such that the static deflection is 5 mm. If the engine and coupling weigh 400 kg above, what speed must the motor run for 90% isolation.
6. An aircraft instrument of mass 15 kg is to be isolated from the engine vibrations. The engine runs at speeds ranging from 1600 rpm to 2400 rpm. Determine the rubber stiffness for 95% isolation. Neglect damping.
7. Design a vibrometer to measure amplitudes at a lowest frequency of 10 Hz with an accuracy of at least 1.5 percent. The seismic mass is to be about 1 kg. What is the stiffness of the spring? How much should be the damping in the system?
8. An accelerometer is having its natural frequency as 15×10^3 Hz. Find the amplitude and phase distortion of a signal of frequency 7×10^3 Hz. Damping ratio is 0.75.

9. An accelerometer is used to measure the motion of a structure which vibrates at 15 cpm. The static deflection of the seismic mass of the accelerometer is 1.30 cm. Determine the amplitude of the structure if the reading of instrument is 0.6 cm.
10. Determine the error in an accelerometer reading if, the natural frequency of the accelerometer is 5 times the frequency of the observed motion.
11. Resonance test on a large structure is carried out with a constant force excitation of 150 N. The test reveals that the maximum amplitude of .01 mm occurs at 10 Hz which is halved at an excitation frequency of 7.3 Hz. Find the damping ratio and identify the system.
12. Show that maximum velocity of vibration of spring-mass-dashpot system occurs at resonance and independent of damping.
13. An instrument for measuring accelerations records 35 oscillations/sec. The natural frequency of the instrument is 750 cycles/sec. Find the acceleration of the machine part attached with the instrument if the recorded amplitude is 0.03 mm. Find the amplitude of vibration of the machine part.
14. A reed techometer having 14 reeds has a frequency range of 6 to 18 Hz. Reeds are made of steel and have dimensions as 1 mm thick, 5 mm wide and 70 mm long. Calculate the end mass for the reeds corresponding to the two extreme frequencies.
15. A vertical spring-mass-dashpot system works with the following data
    $m = 2.5$ kg
    $k = 1500$ N/m
    $\varepsilon = 0.2$
    A constant force 40 N acts on the mass and the system is in equilibrium. Solve the equation of motion.
16. A machine weighting 75 kg is mounted on springs and is fitted with a deshpot to damp out vibrations. There are three springs each of stiffness 10 kg. per cm and it is found that the amplitude of vibration diminishes from 3.84 cm to 0.64 cm in two complete oscillations. Assuming that the damping force varies as velocity, find the resistance of the deshpot at unit velocity and compare the frequency of damped vibration with the frequency when deshpot is not in operation. (K.U., 99)
17. An electric motor of mass 30 kg is running at 500 r.p.m. The motor is supported on a spring of 7 kN/m and a dashpot which offers a resistance of 600 N at 0.25 m/sec. The unbalance of the rotor is equivalent to a mass of 0.8 kg located 5 cm from the axis of rotation. Knowing that the motor is constrained to move vertically, determine (*i*) the damping factor, (*ii*) amplitude of vibration and phase angle, and (*iii*) resonant speed and resonant amplitude. (A.M.I.E., 98)
18. A 15 mm diameter shaft rotates in long fixed bearing 60 cm apart and a disc of mass 20 kg is secured at mid-span. The mass centre of disc is 0.5 mm from the shaft axis. If the bending stress in the shaft is not to exceed 120 MPa, find the range of speeds over which the shaft must not run.
    $E = 2 \times 10^{11}$ N/m^2 (A.M.I.E., 94)

19. Consider the engine valve system shown in the figure 4.1 P. Assume the spring to be massless and the push rod to the infinitely rigid. Find the equation of motion of the system and the expression of the natural frequency m_p and m_v are the mass of push rod and valve and k_s is the stiffness of the spring.

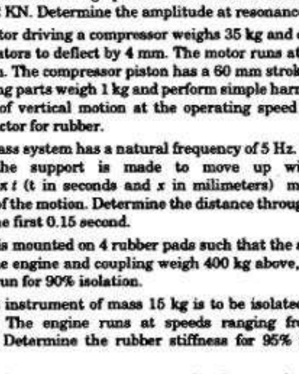


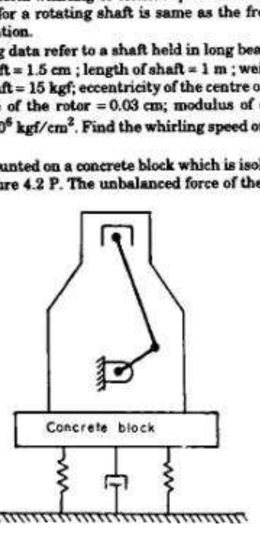
Fig.4.1 P.

(b) The idling speed of the engine is 550 r.p.m. The push rod is operated by a cam which gives rise to a periodic forcing function with the lowest harmonic having a frequency equal to half the engine speed. Therefore determine the stiffness of the spring such that the natural frequency of the mechanism is lower that half the engine idling speed. Assume mass of the valve, $m_v = 0.25$ kg, mass of the piston, $m_p = 0.3$ kg, $I = 10^{-5}$ kg-m^2, $a = 3.5$ cm and $b = 2.5$ cm.

$$\left(\text{Ans. } \omega_n = \sqrt{\frac{K_s b^2}{(I + m_v b^2 + m_p a^2)}} \quad (b)\ 708.25 \text{ N/m} \right)$$

20. In an experiment on forced vibration response of a single degree of freedom system, it is found that half power points lie at frequencies 40 and 44 Hz. Find the damping factor of the system. (U.P.S.C., 96)
21. A machine of mass 100 kg is mounted on isolators having stiffness of 1.2×10^6 N/m and a damping factor 0.1. A piston of mass 2 kg within the machine has a reciprocating motion with a stroke of 8 cm and a speed of 1800 cycles/min. Assuming the motion of piston to be simple harmonic, determine.
    (*i*) the amplitude of motion of the machine,
    (*ii*) force transmitted to the foundation,
    ... phase angles of the transmitted force and machine motion with

22. A vertical shaft of 0.5 cm diameter is 20 cm long and is supported in long bearings at its ends. A disc weighing 50 kgf is attached at the centre of the shaft. Neglecting any increase in stiffness due to the attachment of the disc to the shaft, find the critical speed of rotation and the maximum bending stress when the shaft is rotating at 75% of the critical speed. The centre of the disc is 0.25 mm from the geometric axis of the shaft. $E = 20 \times 10^5$ kgf/cm^2. (U.P.S.C., 89)
23. A shaft 12 mm in diameter and 600 mm long between long bearings carries a central load of 4 kgf. If the centre of gravity of the load is 0.2 mm from the axis of the shaft, compute the maximum flexural stress in the shaft when it is running at 90% of its critical speed. The value of E (Young's modulus) of the material of the shaft is 2×10^4 kgf/mm^2. (U.P.S.C., 86)
24. A mass weighing 10 kgf is supported by a spring of stiffness 20 kgf/cm and a viscous damper. During free vibrations, the amplitude of vibrations decreases to one-tenth of its initial value in two complete oscillations. Find the value of damping coefficient of the damper.
    If a sinusoidal force of amplitude 15 kgf and frequency 600 cycles/minute acts on the mass, find the amplitude and phase of the (*a*) motion of the mass and (*b*) force transmitted to the support. (U.P.S.C., 82)
25. (*a*) Explain the term whirling or critical speed of a shaft. Prove that the whirling speed for a rotating shaft is same as the frequency of natural transverse vibration.
    (*b*) The following data refer to a shaft held in long bearing :
    Diameter of shaft = 1.5 cm ; length of shaft = 1 m ; weight of a disc at the centre of the shaft = 15 kgf; eccentricity of the centre of gravity of the disc from the centre of the rotor = 0.03 cm; modulus of elasticity for shaft material = 2×10^6 kgf/cm^2. Find the whirling speed of the shaft in r.p.m. (U.P.S.C; 81)
26. An engine is mounted on a concrete block which is isolated from the floor as shown in figure 4.2 P. The unbalanced force of the engine in Newton



at n r.p.m. is given by :

$$F(t) = 100\left(\frac{n}{1000}\right)^2 \cos\frac{2\pi n t}{60}$$

At 1000 r.p.m., it is found that the force transmitted to the floor has an amplitude of 100 Newtons. Determine the amplitude of the transmitted force at 1500 r.p.m. when the damper is disconnected. (I.A.S., 92)

27. A shaft 75 mm dia. and 2 m long is fixed at both the ends vertically. A flywheel of mass one tonne is provided on the shaft at a distance of 1.5 m from its upper end. Find the natural longitudinal frequency of vibration of the system.

$E = 3 \times 10^5$ N/mm^2. (K.U.)

28. A periodic torque having a maximum value of 0.5 Nm at a frequency corresponding to 4 rad/s is impressed ona flywheel suspended from a wire.

The wheel has moment of inertia of 0.12 kg.m^2 and wire has stiffness of 1 Nm/rad. A viscous dashpot applies a damping couple of 0.4 Nm at an angular velocity of 1 rad/s. Find the maximum couple on dashpot.

(K.U.)

29. A spring-mass-damper system is subjected to a harmonic force. The amplitude is found to be 20 mm at resonance and 10 mm at a frequency 0.75 times the resonant frequency. Find the damping ratio of the system.

(b) The landing gear of an airplane can be idealized as the spring-mass-damper system shown in figure 4.3 P. If the runway surface is described

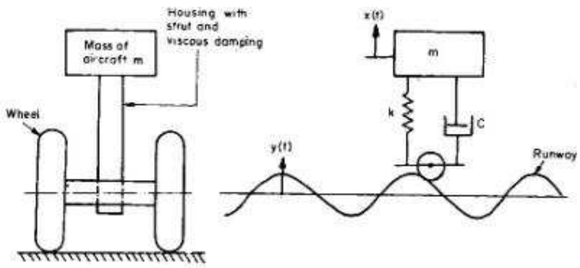


Fig.4.3 P.

$y(t) = y_0 \cos \omega t$. determine the values of stiffness and damping coefficient that limit the amplitude of vibration of the airplane (x) to 0.1m. Assume

30. A single cylinder air compressor of mass 100 kg is mounted on rubber mounts as shown in figure 4.4 P. The stiffness and damping constants of the rubber mounts are given by 10^6 N/m and 2000 Ns/m respectively. If

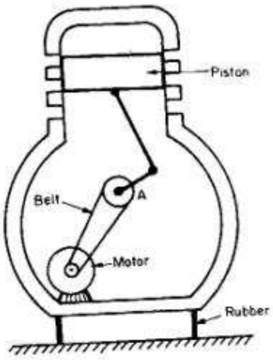


Fig.4.4 P.

the unbalance of the compressor is equivalent to a mass of 0.1 kg located at the end of the crank, determine the response of the compressor at a crank speed of 3000 r.p.m. Assume r = 10 cm. (D.U., 97)

31. An industrial machine weighing 500 kg, is on springs with a static deflection of 5 mm. If machine has rotating unbalance of 25 kg cm, determine (*a*) the force transmitted to the floor at 1200 r.p.m. (*b*) the dynamic amplitude at this speed. Assume damping factor equal to 0.3.

If the machine is mounted on a large concrete block weighing 1300 kg. What will be dynamic amplitude at the above speed. (D.U., 97)

32. Explain clearly the distortion effect observed in seismic instruments. It is desired to study the vibration of the foundation of a two cylinder diesel engine between speeds of 300 and 1200 r.p.m. by means of a vibrometer. It is known that the vibration consists of two harmonics, because of primary & secondary inertia forces in the engine. Find the maximum natural frequency that the vibrometer may have in order to keep the amplitude distortion below 5%. (Roorkee Uni.)

33. Determine the torsional stiffness of a spring for a Torsiograph with a ring having a moment of inertia of 19.6×10^{-4} kg-m^2, so that the difference in the relative motion and that of the vibrating shaft will not be greater than 3% when the shaft vibrates with a frequency of 1000 cpm or above. Neglect damping. If the shaft amplitude is 0.01 radian, determine the corresponding dynamic torque on the spring.

Determine the critical speed of a 1000 kg automobile travelling on a concrete road with expansion joints spaced 12 m apart if the static deflection of the spring system is 50 mm. (Roorkee Uni. 1999-2000)

34. A single rotor of mass 7 kg is mounted midway between bearings (each of 3 mm length) on a steel shaft 10 mm dia. The bearings span is 0.4 m. It is known that CG of the rotor is 0.025 mm from its geometric axis. If the system rotates at 1000 r.p.m, find out the amplitude of vibration, the dynamic load transmitted to the bearings and the maximum stress in the shaft, when (*a*) the shaft is vertically supported (*b*) the shaft is horizontally supported.

Neglect the weight of the shaft and the damping in the system.

(Roorkee Uni., 94-95)

35. (*a*) A piston of mass 5 kg is travelling inside a cylinder with a velocity of 15 m/s and engages a spring and damper as shown in figure 4.5 P. Determine the maximum displacement of the piston after engaging the spring-damper. How many seconds does it take ?

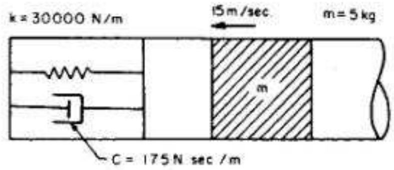


Fig.4.5 P.

(*b*) For the system shown in figure 4.6 P, determine the coefficient of friction if a tensile force P (= mg) elongates the spring by 6 mm. The initial amplitude of x_0 = 600 mm reduces to 0.8 of its value after 20 cycles.

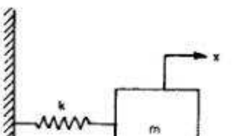


Fig.4.6 P.

(*c*) Write equivalent viscous damping coefficient for (*i*) Coulomb damping and (*ii*) material damping. (Roorkee Uni. 94-95)

36. An automobile whose weight is 150N is mounted on four identical isolators (springs & shock absorbers). Due to its weight, it sags 0.23 m. Each isolator has a damping coefficient of 0.4 N for a velocity of 3 cm per second. The car is placed on a plateform which moves vertically at resonant speed, having an amplitude of 1 cm. Find the amplitude of the car. Assume CG of the car at the centre of the wheel base. What is the dynamic load on each isolator due to vibration ? (Roorkee Uni. 94-95)

37. An aircraft instrument of mass 10 kg is to be isolated from the engine vibrations. The engine runs at speeds ranging from 1800 r.p.m. to 2500 r.p.m. Natural rubber isolators with negligible damping are used. Determine the rubber stiffness for 90% isolation.

38. The hand vibrameter is a simple seismic device used in field work for approximate measurements. The seismic mass A is suspended on a spring B as shown in figure 4.7 P. The natural frequency of the instrument is 4.5 cps. If the indicated amplitude is 0.95 mm for a known frequency of 20 cps, what is the true amplitude ? (Roorkee Uni.)

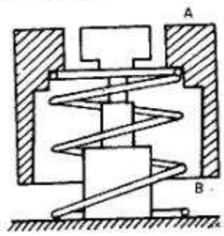


Fig.4.7 P.

5

Two Degrees of Freedom System

5.1. INTRODUCTION

The system which requires two coordinates independently to describe its motion completely, is called a two degree of freedom system. In such a system there are two masses which will have two natural frequencies. So the system will be having two equations of motion which may be treated as coupled differential equations. Sometimes, non-harmonic motion of the masses makes the system more complicated for analysis.

The system at its lowest or first natural frequency is called its first mode, at its next second higher it is called the second mode, and so on. If the two masses vibrate at the same frequency and in phase, it is called a principal mode of vibration. If at the principal mode of vibration, the amplitude of one of the masses is unity, it is known as normal mode of vibration.

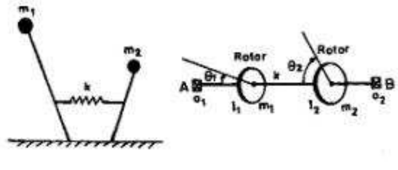


Fig. 5.1. Two degrees of freedom system.

In figure 5.1 two masses of a simple pendulum are coupled together by means of a spring. Similarly, a shaft of torsional stiffness k is having two rotors which can have angular displacements θ_1 and θ_2 independent of each other. Thus it is a two degree of freedom system.

This chapter deals with the very useful mechanical and mathematical applications of two degrees of freedom systems besides making a link between the single and multidegree of freedom systems.

5.2. TORSIONAL VIBRATIONS

Consider figure 5.1 where a shaft AB is carrying two rotors having moment of inertias as I_1 and I_2. Let θ_1 and θ_2 be the angular displace-

ments of the rotors at any instant from the mean position. The equation of motion for each rotor can be written as

$$I_1 \frac{d^2\theta_1}{dt^2} + k(\theta_1 - \theta_2) = 0$$

$$I_2 \frac{d^2\theta_2}{dt^2} + k(\theta_2 - \theta_1) = 0 \quad \ldots(5.2.1)$$

The solution may be assumed of the form

$$\theta_1 = a_1 \sin \omega t, \quad \theta_2 = a_2 \sin \omega t$$

So $$\ddot{\theta}_1 = -\omega^2\theta_1, \quad \ddot{\theta}_2 = -\omega^2\theta_2 \quad \ldots(5.2.2)$$

Substituting these values in equation (5.2.1), we get

$$-\omega^2 I_1 a_1 + k(a_1 - a_2) = 0$$

$$-\omega^2 I_2 a_2 + k(a_2 - a_1) = 0$$

$$(-\omega^2 I_1 + k)a_1 - k a_2 = 0 \quad \ldots(5.2.3)$$

$$(-\omega^2 I_2 + k)a_2 - k a_1 = 0 \quad \ldots(5.2.4)$$

Equating the determinant of the above equation equal to zero, we get

$$(-\omega^2 I_1 + k)(-\omega^2 I_2 + k) - k^2 = 0$$

$$\omega^4 I_1 I_2 - \omega^2 I_1 k - \omega^2 k I_2 + k^2 - k^2 = 0$$

$$\omega^2(\omega^2 I_1 I_2 - I_1 k - I_2 k) = 0$$

$$\omega^2\left[\omega^2 - \frac{k}{I_1 I_2}(I_1 + I_2)\right] = 0$$

So $$\omega_1 = 0$$

and $$\omega_2 = \sqrt{\frac{k(I_1 + I_2)}{I_1 I_2}} \quad \ldots(5.2.5)$$

Consider equation (5.2.3) and putting the value of ω_1 from equation (5.2.5) in it, we get

$$\frac{a_1}{a_2} = 1$$

Similarly, putting the value of ω_2 from equation (5.2.5) in equation (5.2.4), we get

$$\frac{a_1}{a_2} = \frac{-\omega^2 I_2 + k}{k} = \frac{-\omega^2 I_2}{k} + 1$$

$$= -\frac{k(I_1 + I_2)}{I_1 I_2} \cdot \frac{I_2}{k} + 1$$

$$\frac{a_1}{a_2} = -1 - \frac{I_2}{I_1} + 1 = -\frac{I_2}{I_1}$$

So $$\frac{a_1}{a_2} = -\frac{I_2}{I_1} \quad \ldots(5.2.6)$$

The section of the shaft where the angular displacement is zero, is known as node. The angular displacements of the rotors are inversely proportional to their inertias (equation 5.2.6). In figure 5.2 the first and the second mode shapes are shown.

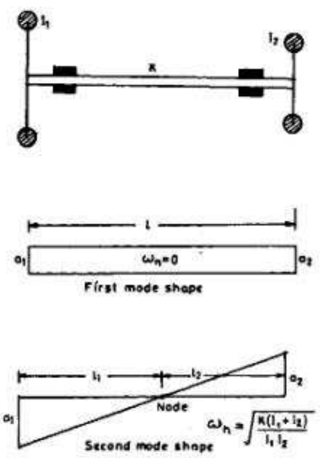


Fig. 5.2. Torsional vibration.

5.3. VIBRATIONS OF UNDAMPED TWO DEGREES OF FREEDOM SYSTEMS

Let us consider two degree spring mass system as shown in figure 5.3. The two masses m_1 and m_2 are defined by their positions x_1 and x_2 respectively at any time t.

The equations of motion for the two masses can be written as

$$m_1\ddot{x}_1 + (k_1 + k)x_1 - kx_2 = 0 \quad \ldots(5.3.1)$$

$$m_2\ddot{x}_2 + (k + k_2)x_2 - kx_1 = 0 \quad \ldots(5.3.2)$$

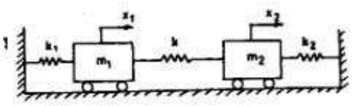


Fig. 5.3. Two degrees of freedom system.

Assume that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. These above equations can be solved for the natural frequencies and corresponding mode shapes by assuming a solution of the form

$$x_1 = A_1 \sin(\omega t + \phi) \quad \ldots(5.3.3)$$

$$x_2 = A_2 \sin(\omega t + \phi) \quad \ldots(5.3.4)$$

where A_1 and A_2 are amplitudes of vibration of the two masses and ω is one of the natural frequencies of the system. Substituting these equations into equations (5.3.1) and (5.3.2)

$$-m_1 A_1 \omega^2 \sin(\omega t + \phi) + (k_1 + k)A_1 \sin(\omega t + \phi) - kA_2 \sin(\omega t + \phi) = 0$$

or $$A_1(k + k_1 - m_1\omega^2) - A_2 k = 0 \quad \ldots(5.3.5)$$

Similarly, $$-A_1 k + A_2(k_2 + k - m_2\omega^2) = 0 \quad \ldots(5.3.6)$$

These are homogeneous linear algebraic equations in A_1 and A_2. The solution is obtained by equating to zero the determinant of the coefficients of A_1 and A_2, *i.e.*,

$$\begin{vmatrix} k_1 + k - m_1\omega^2 & -k \\ -k & k_2 + k - m_2\omega^2 \end{vmatrix} = 0$$

Expanding and solving it, we get

$$(k_1 + k - m_1\omega^2)(k_2 + k - m_2\omega^2) - k^2 = 0 \quad \ldots(5.3.7)$$

$$\omega^4 - \left[\frac{k + k_2}{m_2} + \frac{k_1 + k}{m_1}\right]\omega^2 + \frac{k_1 k_2 + k_1 k + k k_2}{m_1 m_2} = 0$$

This is the frequency equation and is a quadratic in ω^2 and gives two values of ω^2.

If we assume $k_1 = k_2 = k$ and $m_1 = m_2 = m$, frequency equation can be written as

$$\omega^4 - \left[\frac{2k}{m} + \frac{2k}{m}\right]\omega^2 + \frac{3k^2}{m^2} = 0$$

$$\omega^4 - \frac{4k}{m}\omega^2 + \frac{3k^2}{m^2} = 0$$

$$\omega^2 = \frac{\frac{4k}{m} \pm \sqrt{\frac{16k^2}{m^2} - \frac{12k^2}{m^2}}}{2}$$

$$\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{3k}{m}} \qquad ...(5.3.8)$$

vhere ω_1 and ω_2 are the frequencies of the first and second modes 'espectively. Amplitude ratios can be written with the help of equations 5.3.5) and (5.3.6) as

$$\frac{A_1}{A_2} = \frac{k}{k + k_1 - m_1\omega^2} = \frac{k}{2k - m\omega^2} \qquad ...(5.3.9)$$

ınd
$$\frac{A_1}{A_2} = \frac{k_2 + k - m_2\omega^2}{k} = \frac{2k - m\omega^2}{k} \qquad ...(5.3.10)$$

Amplitude ratio for the first mode can be written as

$$\left(\frac{A_1}{A_2}\right)_1 = \frac{k}{2k - \frac{k}{m}\cdot m} = 1 \quad \text{putting } \omega = \sqrt{\frac{k}{m}} \qquad ...(5.3.11)$$

nd
$$\left(\frac{A_1}{A_2}\right)_2 = \frac{2k - m\cdot\frac{3k}{m}}{k} = -1 \quad \text{putting } \omega = \sqrt{\frac{3k}{m}} \qquad ...(5.3.12)$$

So it can be seen from the above two equations that for the first ıode the two masses move in the same phase with equal amplitudes nd for the second mode the two masses move out of phase with equal mplitudes.

The frequency of the first and second modes can be written as

$$\omega_1 = 2\pi f_1$$

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ Hz}$$

ɔd
$$f_2 = \frac{1}{2\pi}\sqrt{\frac{3k}{m}} \text{ Hz} \qquad ...(5.3.13)$$

[atrix Form

The solutions from equations (5.3.3) and (5.3.4) can also be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \sin(\omega_1 t + \phi_1) + \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} \sin(\omega_2 t + \phi_2) \qquad ...(5.3.14)$$

where A_{11} = amplitude of x_1 at frequency $\omega = \omega_1$

A_{21} = amplitude of x_2 at frequency $\omega = \omega_1$

A_{12} = amplitude of x_1 at $\omega = \omega_2$

A_{22} = amplitude of x_2 at $\omega = \omega_2$ and so on.

So it is clear from equation (5.3.14) that the double subscripts are applied to the amplitudes, out of this the first subscript stands for coordinate and the second to the frequency.

From equations (5.3.5) and (5.3.6), relative amplitudes can be written by substituting the values of ω_1 and ω_2 in place of ω.

$$\frac{A_{11}}{A_{21}} = \frac{k}{k + k_1 - m_1\omega_1^2} = \frac{k + k_2 - m_2\omega^2}{k} = \frac{1}{\alpha_1} \qquad ...(5.3.15)$$

$$\frac{A_{12}}{A_{22}} = \frac{k}{k + k_1 - m_1\omega_2^2} = \frac{k + k_2 - m_2\omega^2}{k} = \frac{1}{\alpha_2} \qquad ...(5.3.16)$$

where α_1 and α_2 are constants which define the relative amplitudes of x_1 and x_2 at both ω_1 and ω_2.

Thus equation (5.3.14) can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} \\ \alpha_1 A_{11} \end{bmatrix} \sin(\omega_1 t + \phi_1) + \begin{bmatrix} A_{12} \\ A_{12}\alpha_2 \end{bmatrix} \sin(\omega_2 t + \phi_2)$$

$$\left[\because \frac{A_{11}}{A_{21}} = \frac{1}{\alpha_1} \quad \text{So } A_{21} = A_{11}\alpha_1 \quad \text{and } A_{22} = A_{12}\alpha_2\right.$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha_1 \end{bmatrix} A_{11} \sin(\omega_1 t + \phi_1) + \begin{bmatrix} 1 \\ \alpha_2 \end{bmatrix} A_{12} \sin(\omega_2 t + \phi_2) \qquad ...(5.3.17)$$

In the above equation A_{11}, A_{12}, ϕ_1 and ϕ_2 are the constants which can be determined from initial conditions and α_1 and α_2 are the modal ratios. The total motion of each harmonic function x_1 and x_2 can be expressed as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} A_{11} \sin(\omega_1 t + \phi_1) \\ A_{12} \sin(\omega_2 t + \phi_2) \end{bmatrix} \qquad ...(5.3.18)$$

In the above equatirn if $A_{12} = 0$, the first mode will exist.

5.4. FORCED VIBRATIONS

A viscously damped two degrees of freedom system is shown in figure 5.4. The system having two masses m_1 and m_2 is put to excitation forces F_1 and F_2. The various forces acting on the system are shown in figure 5.4(b). The equations of motion for the system can be written as

$$m_1\ddot{x}_1 + k_1x_1 + k(x_1 - x_2) + c_1\dot{x}_1 + c(\dot{x}_1 - \dot{x}_2) = F_1(t)$$

$$m_2\ddot{x}_2 + k_2x_2 + k(x_2 - x_1) + c_2\dot{x}_2 + c(\dot{x}_2 - \dot{x}_1) = F_2(t)$$

which can be written as

$$m_1\ddot{x}_1 + (c_1 + c)\dot{x}_1 + (k_1 + k)x_1 - c\dot{x}_2 - kx_2 = F_1(t) \qquad ...(5.4.1)$$

$$m_2\ddot{x}_2 + (c_2 + c)\dot{x}_2 + (k_2 + k)x_2 - c\dot{x}_1 - kx_1 = F_2(t) \qquad ...(5.4.2)$$

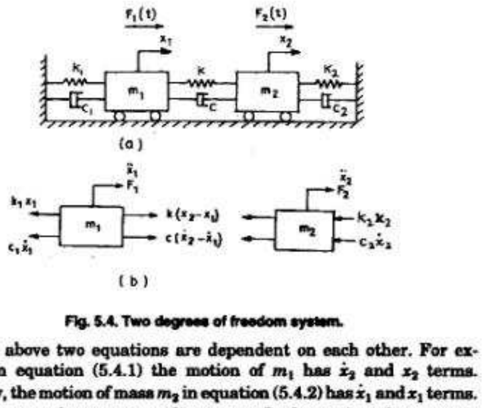


Fig. 5.4. Two degrees of freedom system.

The above two equations are dependent on each other. For example, in equation (5.4.1) the motion of m_1 has $\dot{x}_2$ and x_2 terms. Similarly, the motion of mass m_2 in equation (5.4.2) has $\dot{x}_1$ and x_1 terms. These terms are known as coupling terms. So the motion of mass m_1 is dependent on the motion of mass m_2 and vice-versa is also true. The equations so formed are called coupled equations. If in the equations (5.4.1) and (5.4.2) c and k are zero, the equations are called uncoupled equations and the masses m_1 and m_2 can move independent of each other.

The equations (5.4.1 and 5.4.2) can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c + c_1 & -c \\ -c & c + c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k + k_1 & -k \\ -k & k + k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}$$

Then it is of the form

$$[M]\,\ddot{x} + [C]\,\dot{x} + [K]\,x = F \qquad ...(5.4.3)$$

where $[M]$, $[C]$ and $[K]$ are 2×2 types matrices known as the mass matrix, damping matrix and stiffness matrix, respectively. $\ddot{x}$, $\dot{x}$, x and F are the acceleration vector, velocity vector, displacement vector and force vector, respectively. These are 2×1 type matrices

5.5. DAMPED FREE VIBRATIONS

Again consider figure 5.4(a) for convenience and forces $F_1(t)$ and $F_2(t)$ are not supposed to act on masses m_1 and m_2 respectively. Rest of the things are as usual. Writing the differential equations of motion for the two masses, we get

$$m_1\ddot{x}_1 + (c_1 + c)\dot{x}_1 + (k_1 + k)x_1 - c\dot{x}_2 - kx_2 = 0 \qquad ...(5.5.1)$$

$$m_2\ddot{x}_2 + (c_2 + c)\dot{x}_2 + (k_2 + k)x_2 - c\dot{x}_1 - kx_1 = 0 \qquad ...(5.5.2)$$

The solutions to the above equations may be of the form

$$x_1 = A_1e^{st}$$

$$x_2 = A_2e^{st}$$

Substituting the values of x_1 and x_2 in the above two equations, we get

$$[m_1s^2 + (c_1 + c)s + (k_1 + k)]A_1 - (cs + k)A_2 = 0$$

$$[m_2s^2 + (c_2 + c)s + (k_2 + k)]A_2 - (cs + k)A_1 = 0 \qquad ...(5.5.3)$$

The above equations have a non-trivial solution only if the determinants of coefficients of A_1 and A_2 are zero ; *i.e.*,

$$\begin{vmatrix} m_1s^2 + (c_1 + c)s + (k_1 + k) & -(cs + k) \\ -(cs + k) & m_2s^2 + (c_2 + c)s + (k_2 + k) \end{vmatrix} = 0$$

Expanding the determinants, we get

$$[m_1s^2 + (c_1 + c)s + (k_1 + k)]\,[m_2s^2 + (c_2 + c)s + (k_2 + k)] - (cs + k)^2 = 0$$

$$s^4 + \left[\frac{c_1 + c}{m_1} + \frac{c + c_2}{m_2}\right]s^3 + \left[\frac{k_1 + k}{m_1} + \frac{k + k_2}{m_2} + \frac{cc_1 + c_1c_2 + cc_2}{m_1m_2}\right]s^2 + \left[\frac{k_1(c + c_2) + k(c_1 + c_2) + k_2(c_1 + c)}{m_1m_2}\right]s + \left[\frac{k_1k + kk_2 + k_1k_2}{m_1m_2}\right] = 0 \qquad ...(5.5.4)$$

This is called the characteristic equation. This may be solved for the roots of s which will give its four values. The roots may be real or complex. The general solution may be written as

$$x_1 = A_{11}e^{s_1t} + A_{12}e^{s_2t} + A_{13}e^{s_3t} + A_{14}e^{s_4t} \qquad ...(5.5.5)$$

$$x_2 = A_{21}e^{s_1t} + A_{22}e^{s_2t} + A_{23}e^{s_3t} + A_{24}e^{s_4t} \qquad ...(5.5.6)$$

Matrix Form

Equations (5.5.1) and (5.5.2) can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c + c_1 & -c \\ -c & c + c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k + k_1 & -k \\ -k & k + k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This equation can be written like equation (5.4.3) as

$$[M]\ddot{x} + [c]\,x + [k]\,x = 0 \qquad ...(5.5.7)$$

The characteristic equation for an undamped system can be written as

$$[M]\ddot{x} + [k]\,x = 0$$

$$[-\omega^2 M + k]A = 0$$

where A is the displacement vector.

Thus $$|k - \omega^2 M| = 0 \qquad ...(5.5.8)$$

This is also known as frequency equation which can be solved very easily.

5.6. FORCED HARMONIC VIBRATION

A harmonic force $F \sin \omega t$ is acting on the system as shown in figure 5.5. The differential equations of motion can be written as

$$m_1\ddot{x}_1 + (k_1 + k)x_1 - kx_2 = F \sin \omega t \qquad ...(5.6.1)$$

$$m_2\ddot{x}_2 + (k + k_2)x_2 - kx_1 = 0 \qquad ...(5.6.2)$$

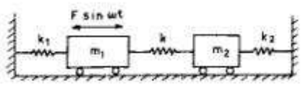


Fig. 5.5. Forced harmonic vibration.

Let us assume the solution of the form

$$x_1 = A_1 \sin(\omega t + \phi) \qquad ...(5.6.3)$$

and $$x_2 = A_2 \sin(\omega t + \phi) \qquad ...(5.6.4)$$

Equations (5.6.1) and (5.6.2) can be written with the help of equations (5.6.3) and (5.6.4) as

$$(k_1 + k - m_1\omega^2)A_1 - kA_2 = F \qquad ...(5.6.5)$$

$$-kA_1 + (k + k_2 - m_2\omega^2)A_2 = 0 \qquad ...(5.6.6)$$

Solving the above equations for A_1 and A_2, we get

$$A_1 = \frac{F(k_2 + k - m_2\omega^2)}{(k_2 + k - m_2\omega^2)(k_1 + k - m_1\omega^2) - k^2} \qquad ...(5.6.7)$$

and $$A_2 = \frac{Fk}{(k_2 + k - m_2\omega^2)(k_1 + k - m_1\omega^2) - k^2} \qquad ...(5.6.8)$$

The denominators of equations (5.6.7) and (5.6.8) are the same as the equation (5.3.7) itself. Thus the denominator of the said equations is the frequency equation itself. The denominator becomes zero when exciting frequency is equal to the natural frequency of the system, *i.e.*,

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{or} \quad \omega_2 = \sqrt{\frac{3k}{m}}$$

It is evident that the amplitudes of vibration A_1 and A_2 are infinite when denominators in equations (5.6.7) and (5.6.8) are zero. It means when exciting frequency is equal to any of the two natural frequencies, resonance in the system will occur. A_1 becomes zero when $k + k_2 = m_2\omega^2$ which is taken with the help of equation (5.6.7), thus it makes m_1 stationary. While this type of expression is not possible for mass m_2. A little consideration shows that mass m_1 has been initially excited by the force and under certain assumptions it is motionless while mass m_2 has some motion.

The equations of motion in matrix form can be written as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k & -k \\ -k & k_2 + k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F \sin \omega t \\ 0 \end{Bmatrix} \qquad ...(5.6.9)$$

5.7. SEMI-DEFINITE SYSTEMS

The systems having one of their natural frequencies equal to zero are known as semi-definite systems. The example of this type of system is shown in figure 5.6 where two masses m_1 and m_2 are connected by a spring k. The equations of motion can be written as

$$m_1\ddot{x}_1 + k(x_1 - x_2) = 0 \qquad ...(5.7.1)$$

$$m_2\ddot{x}_2 + k(x_2 - x_1) = 0 \qquad ...(5.7.2)$$

Fig. 5.6. Semi-definite system.

For free vibration, let us assume the motion to be harmonic

$$x_1 = A_1 \sin(\omega t + \phi) \qquad ...(5.7.3)$$

$$x_2 = A_2 \sin(\omega t + \phi) \qquad ...(5.7.4)$$

and so $$\ddot{x}_1 = -\omega^2 x_1$$

$$\ddot{x}_2 = -\omega^2 x_2 \qquad ...(5.7.5)$$

Substituting the above terms in equations (5.7.1) and (5.7.2), we get

$$(k - m_1\omega^2)A_1 - kA_2 = 0$$

$$-kA_1 + (k - m_2\omega^2)A_2 = 0 \qquad ...(5.7.6)$$

The determinant from the above equations can be written as

$$\begin{vmatrix} k - m_1\omega^2 & -k \\ -k & k - m_2\omega^2 \end{vmatrix} = 0 \qquad ...(5.7.7)$$

Expansion gives

$$m_1 m_2\omega^4 - k(m_1 + m_2)\omega^2 = 0 \qquad ...(5.7.8)$$

or $$\omega^2[m_1 m_2\omega^2 - k(m_1 + m_2)] = 0$$

So $$\omega_1 = 0, \quad \omega_2 = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} \qquad ...(5.7.9)$$

From equation (5.7.9) it can be seen that one of the natural frequencies of the system is equal to zero, the system is not oscillating. There is no relative motion between m_1 and m_2 and it can move as a rigid body.

5.8. CO-ORDINATE COUPLING

When we suddenly apply brakes on a moving car or automobile, two motions of car body occur simultaneously, one the translatory (x) and the other angular (θ). This type of unbalance in the system occurs because centre of gravity (G) of car and centre of rotation do not coincide. The uneven road surface may cause unwanted excessive excitation frequency but this whole is not transmitted to the car body though the front and rear wheels move up and down very frequently. This is because of springs and isolators. This type of system is shown in figure 5.7. G is the centre of gravity of car body, m the mass and I the moment of inertia. The dotted line represents the deflected form of the car body. The following coordinates are defined as : $x_1(t)$, $x_2(t)$ and $x(t)$ are the deflections of ends B, A and C.G. respectively. The compression of springs on left and right hand sides of the system are $k_2(x - l_2\theta)$ and $k_1(x + l_1\theta)$ respectively for very small value of θ. In the figure 5.7(b) and (c) the applied forces and moments are shown.

From these diagrams, the equations of motion can be written as

$$m\ddot{x} = -k_2(x - l_2\theta) - k_1(x + l_1\theta)$$

$$I\ddot{\theta} = k_2(x - l_2\theta)l_2 - k_1(x + l_1\theta)l_1$$

Thus $$m\ddot{x} + (k_1 + k_2)x - (k_2 l_2 - k_1 l_1)\theta = 0 \qquad ...(5.8.1)$$

$$I\ddot{\theta} - (k_2 l_2 - k_1 l_1)x + (k_2 l_2^2 + k_1 l_1^2)\theta = 0 \qquad ...(5.8.2)$$

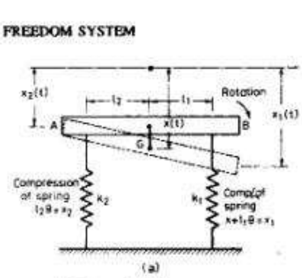


Fig. 5.7. Co-ordinate coupling.

Above both equations have x and θ terms, so they are the coupled equations. The above equations show that the system has rotary as well as translatory motion.

If $k_1 l_1 = k_2 l_2$, the above equations can be written as

$$m\ddot{x} + (k_1 + k_2)x = 0 \qquad ...(5.8.3)$$

and $$I\ddot{\theta} + (k_2 l_2^2 + k_1 l_1^2)\theta = 0 \qquad ...(5.8.4)$$

It can be seen from the above equations that translatory and angular motion can exist independently. These are called uncoupled differential equations. This is called dynamic coupling.

The natural frequencies of such a system are

$$\omega_1 = \sqrt{\frac{k_1 + k_2}{m}}$$

and $$\omega_2 = \sqrt{\frac{k_2 l_2^2 + k_1 l_1^2}{I}} \qquad ...(5.8.5)$$

If the coupling term $(k_1 l_1 - k_2 l_2)$ is non-zero, the coupling so formed is known as static or elastic coupling.

The solutions of equations (5.8.1) and (5.8.2) can be found by assuming

$$x = A_1 \sin \omega t$$

and
$$\theta = \phi_1 \sin \omega t \qquad ...(5.8.6)$$

Substituting the values of $\ddot{x}$, $\ddot{\theta}$, x and θ with the help of equation (5.8.6) in equations (5.8.1) and (5.8.2), we get

$$-A_1 m\omega^2 + (k_1 + k_2)A_1 - (k_2 l_2 - k_1 l_1)\phi_1 = 0 \qquad ...(5.8.7)$$

$$-\phi_1\omega^2 I - (k_2 l_2 - k_1 l_1)A_1 + (k_2 l_2^2 + k_1 l_1^2)\phi_1 = 0 \qquad ...(5.8.8)$$

or
$$\left(\frac{A_1}{\phi_1}\right)_1 = \frac{k_2 l_2 - k_1 l_1}{-m\omega^2 + k_1 + k_2}$$

$$\left(\frac{A_1}{\phi_1}\right)_2 = \frac{k_2 l_2^2 + k_1 l_1^2 - I\omega^2}{k_2 l_2 - k_1 l_1}$$

which gives us it in determinant form as

$$\begin{vmatrix} k_1 + k_2 - m\omega^2 & -(-k_2 l_2 + k_1 l_1) \\ -(-k_2 l_2 + k_1 l_1) & k_1 l_1^2 + k_2 l_2^2 - I\omega^2 \end{vmatrix} = 0$$

Expanding the above determinant, we get

$$(k_1 + k_2 - m\omega^2)(k_1 l_1^2 + k_2 l_2^2 - I\omega^2) - (k_1 l_1 - k_2 l_2)^2 = 0 \qquad ...(5.8.9)$$

This is the characteristic equation.

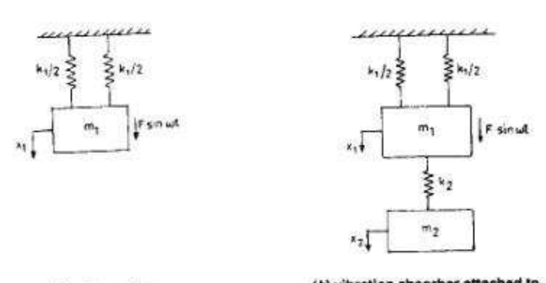
The roots of the equation can be written as

$$\omega_{1,2}^2 = \frac{1}{2}\left[\frac{k_1 + k_2}{m} + \frac{k_1 l_1^2 + k_2 l_2^2}{I} \pm \sqrt{\left(\frac{k_1 + k_2}{m} + \frac{k_1 l_1^2 + k_2 l_2^2}{I}\right)^2 - \frac{4k_1 k_2 (l_1 + l_2)^2}{mI}}\right] \qquad ...(5.8.10)$$

5.9. VIBRATION ABSORBER

When a structure externally excited has undesirable vibrations, it becomes necessary to eliminate them by coupling some vibrating system to it. The vibrating system is known as vibration absorber or dynamic vibration absorber. In such cases the excitation frequency is nearly equal to the natural frequency of the structure or machine. The mass (machine or structure) which is excited can have zero amplitude of vibration and the spring mass system (absorber) which is coupled to it vibrates freely. Vibration absorbers are used to control structural resonance.

For example, if the excitation frequency ω is nearly close to the natural frequency $\omega = \omega_n = \sqrt{k_1/m}$ of the system, the amplitude of vibration would be very large because of resonance [refer fig. 5.8(a)].



(a) main system

(b) vibration absorber attached to main system.

Fig. 5.8. Vibration absorber

in figure 5.8(b). This spring mass system acts as vibration absorber and reduces the amplitude of m_1 to zero if its natural frequency is equal to the excitation frequency *i.e.* $\omega = \sqrt{\dfrac{k_2}{m_2}}$.

Thus
$$\frac{k_1}{m_1} = \frac{k_2}{m_2}$$

When this condition is fulfilled, the absorber is called tuned absorber. Figure 5.8(a) is a single degree of freedom and after coupling the spring mass-system, it becomes two degrees of freedom system. The equations of motion can be written as

$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = F \sin \omega t \qquad ...(5.9.1)$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0 \qquad ...(5.9.2)$$

Let us assume the solution of the form

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin \omega t \qquad ...(5.9.3)$$

So
$$\ddot{x}_1 = -\omega^2 A_1 \sin \omega t$$

and
$$\ddot{x}_2 = -\omega^2 A_2 \sin \omega t \qquad ...(5.9.4)$$

Substituting the values of $\ddot{x}_1$ and $\ddot{x}_2$ from equation (5.9.4) into equations (5.9.1) and (5.9.2), we get

$$(k_1 + k_2 - m_1\omega^2)A_1 - k_2 A_2 = F \qquad ...(5.9.5)$$

$$-k_2 A_1 + (k_2 - m_2\omega^2)A_2 = 0 \qquad ...(5.9.6)$$

Solving the above equations for A_1 and A_2

$$A_1 = \frac{(k_2 - m_2\omega^2)F}{\beta} \qquad ...(5.9.7)$$

.nd
$$A_2 = \frac{k_2 F}{\beta} \qquad ...(5.9.8)$$

vhere
$$\beta = [m_1 m_2 \omega^4 - \{m_1 k_2 + m_2(k_1 + k_2)\}\omega^2 + k_1 k_2]$$

In order to have the amplitude of mass m_1 as zero, let us consider quation (5.9.7)

$$A_1 = \frac{F(k_2 - m_2\omega^2)}{\beta} = 0$$

$$\omega = \sqrt{\frac{k_2}{m_2}} = \omega_2 \quad \text{(say)} \qquad ...(5.9.9)$$

Thus if the mass and spring constant of absorber system are elected in such a way that equation (5.9.9) is satisfied, it becomes a lynamic absorber system.

Let us assume

$A_{st} = F/k_1$ = static deflection or zero frequency deflection

$\omega_1 = \sqrt{k_1/m_1}$ = natural frequency of $m_1 - k_1$ system

$\omega_2 = \sqrt{k_2/m_2}$ = natural frequency of $m_2 - k_2$ system

ınd $\mu = m_2/m_1$ = mass ratio

Equations (5.9.7) and (5.9.8) can be written in non-dimensional orm by dividing them $k_1 k_2$

$$A_1 = \frac{\left(\dfrac{k_2}{k_2} - m_2\dfrac{\omega^2}{k_2}\right)\dfrac{F}{k_1}}{\left[\dfrac{m_1 m_2 \omega^4}{k_1 k_2} - \left\{\dfrac{m_1 k_2}{k_1 k_2} + \dfrac{m_2(k_1 + k_2)}{k_1 k_2}\right\}\omega^2 + \dfrac{k_1 k_2}{k_1 k_2}\right]}$$

$$\frac{A_1}{A_{st}} = \frac{\left(1 - \dfrac{\omega^2}{\omega_2^2}\right)}{\dfrac{\omega^4}{\omega_1^2\omega_2^2} - \left\{(1+\mu)\dfrac{\omega^2}{\omega_1^2} + \dfrac{\omega^2}{\omega_2^2}\right\} + 1} \qquad ...(5.9.10)$$

Similarly, we get

$$\frac{A_2}{A_{st}} = \frac{1}{\dfrac{\omega^4}{\omega_1^2\omega_2^2} - \left\{(1+\mu)\dfrac{\omega^2}{\omega_1^2} + \dfrac{\omega^2}{\omega_2^2}\right\} + 1} \qquad ...(5.9.11)$$

Again from equation (5.9.10), we get $A_1 = 0$ when $\omega = \omega_2$. Let us see what happens to A_2 when A_1 is zero under the same conditions.

$$\frac{A_2}{A_{st}} = \frac{1}{\dfrac{\omega^4}{\omega_1^2\omega^2} - \left\{(1+\mu)\dfrac{\omega^2}{\omega_1^2} + 1\right\} + 1} \qquad \text{(Putting } \omega = \omega_2\text{)}$$

$$= \frac{1}{\dfrac{\omega^2}{\omega_1^2} - \left\{(1+\mu)\dfrac{\omega^2}{\omega_1^2}\right\}}$$

$$= -\frac{\omega_1^2}{\mu\,\omega_2^2} = -\frac{k_1}{\mu m_1}\cdot\frac{m_2}{k_2}$$

or
$$A_2 = -\frac{F}{k_1}\cdot\frac{k_1}{\mu\, m_1}\frac{m_2}{k_2} = -\frac{F\mu}{\mu k_2} = -\frac{F}{k_2}$$

$$F = -A_2 k_2 \qquad ...(5.9.12)$$

where $k_2 A_2$ is the spring force of absorber system and it is equal and opposite to exciting force when the main system is stationary *i.e.* $A_1 = 0$ when mass m_1 is moving downwards, mass m_2 moves upwards. The energy of the main system is absorbed by the absorber system which is sometimes known as auxiliary system.

When the exciting force is constant the amplitude of the auxiliary system is inversely proportional to its spring constant k_2 as can be seen from equation (5.9.12). This equation is useful in the design of absorber. Two resonant frequencies can be determined by putting $\omega_1 = \omega_2$ in the denominators of equations (5.9.10) and (5.9.11) and equating them to zero.

$$\left(\frac{\omega}{\omega_2}\right)^4 - (2+\mu)\left(\frac{\omega}{\omega_2}\right)^2 + 1 = 0$$

or
$$\left(\frac{\omega}{\omega_2}\right)^2 = \left(1 + \frac{\mu}{2}\right) \pm \sqrt{\left(\mu + \frac{\mu^2}{4}\right)} \qquad ...(5.9.13)$$

On the basis of above equation a plot has been shown in figure 5.9. The figure shows the effect of mass ratio on the natural frequencies of the system. For each mass ratio there are two natural frequencies which are above and below the natural frequency of the main system. For example, for mass ratio (μ) 0.25, resonance occurs at a frequency 0.78 and 1.28 times that of the main system. But when mass ratio is 0.5, resonance will occur at a frequency 0.707 and 1.414 times that of main system. So it is clear that for small value of mass ratio (m_2/m_1) the two values of frequency are found closer to unity *i.e.* $\omega_n = \omega_2$.

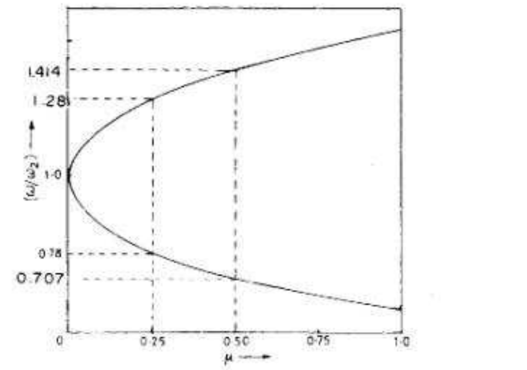


Fig. 5.9. Effect of mass ratio on natural frequency.

It means that resonance can occur if the mass m_2 of the absorber is very small. So this equation and curve help in the design of absorber.

$$\frac{A_1}{A_{st}} = \frac{\left(1-\frac{\omega^2}{\omega_2^2}\right)}{\frac{\omega^4}{\omega_2^4} - (2+\mu)\frac{\omega^2}{\omega_2^2} + 1} \quad \text{...(5.9.14)} \quad (\text{at } \omega_2 = \omega_1)$$

The amplitude ratio A_1/A_{st} will be infinite if the denominator of the above equation (5.9.10) is zero *i.e.*

$$\left(\frac{\omega}{\omega_2}\right)^4 - (2+\mu)\frac{\omega^2}{\omega_2^2} + 1 = 0 \quad \text{...(5.9.15)}$$

Let us find the solution of the above equation for $\mu = 0.2$.

$$\left(\frac{\omega}{\omega_2}\right)^4 - (2+.2)\left(\frac{\omega}{\omega_2}\right)^2 + 1 = 0$$

$$\left(\frac{\omega}{\omega_2}\right)^4 - 2.2\left(\frac{\omega}{\omega_2}\right)^2 + 1 = 0$$

$$\left(\frac{\omega}{\omega_2}\right)^2 = 1.1 \pm \sqrt{.21}$$

$$= 1.1 \pm .458$$

So $\left(\frac{\omega}{\omega_2}\right) = 1.248$ and 0.80

The two infinite amplitudes correspond to the two natural frequencies of the composite system. The two natural frequencies of the composite system are 0.80 and 1.248 times the natural frequency of the main system.

With the help of eqn. (5.9.11), we get the ratio A_2/A_{st}

$$\frac{A_2}{A_{st}} = \frac{1}{\frac{\omega^4}{\omega_2^4} - (2+\mu)\frac{\omega^2}{\omega_2^2} + 1} \quad [\text{at } \omega_2 = \omega_1] \quad \text{...(5.9.16)}$$

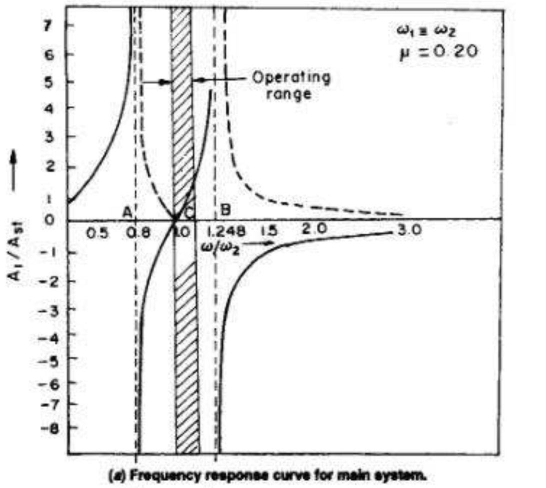
The dimensionless frequency response curves for the main and the absorber system given by eqn. (5.9.14) and eqn. (5.9.16) are shown in Fig. 5.10 (*a*) and Fig. 5.10 (*b*) respectively for a value of $\mu = 0.2$, the negative part of these curves correspond to the situation when phase difference of these amplitudes with the exciting force is 180° or they are out of phase. Comparing Fig. (5.10) (*a*) and Fig. (5.10) (*b*), we can see that for $\omega/\omega_2 < 1$, the phase difference between the two masses or their amplitudes i.e. A_1 and A_2, is zero and for $\omega/\omega_2 > 1$ the phase difference between them is 180° (as in this range A_1/A_{st} is negative and A_2/A_{st} is positive).

For the main system above without absorber we only have one resonant frequency at $\omega/\omega_1 = 1$. This can be seen from Fig. 4.4 for damping ratio $\xi = 0$. If a case arises when the exciting frequency is very close to the natural frequency of main system; then to overcome this resonant condition we attach an absorber system such that $\omega_2 = \omega_1$ to main system reducing its vibration to zero.

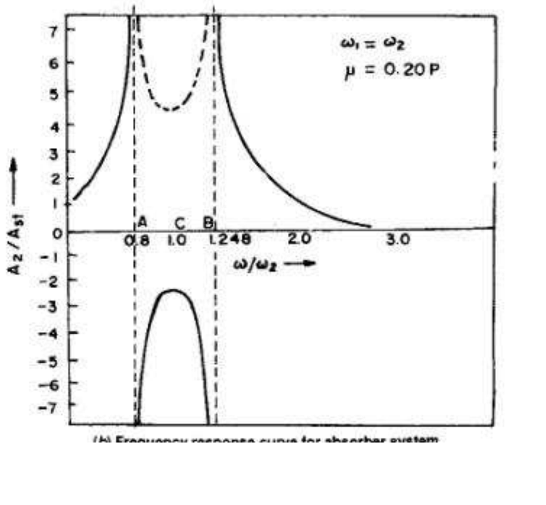
If the exciting force frequency remains constant, the system will work perfectly, if on the other hand, the exciting frequency which is dependent on the speed of machine, is not constant but varies somewhat with changes in load, then any change in the exciting frequency will shift the operating points from the optimum point and the vibrations of main system will no longer be zero.

From Fig. (5.10) (*a*) and Fig. (5.10) (*b*) we see that by adding the vibration absorber we here introduced two resonant points (i.e. $\omega/\omega_2 = 0.8$ and 1.248, *A* & *B* points) instead of one in the original system. These two resonant points are spread on either side of original resonant point ($\omega/\omega_2 = 1$, point *C*) corresponding to main system.

If there is variation of exciting frequency such that the operating point shifts near one of the new resonant points, then the system does



(a) Frequency response curve for main system.



(b) Frequency response curve for absorber system

to decide the spread between the two resonant frequencies depending on variation of exciting frequency. This gives the operating range of the frequency ratio (ω/ω_2). By locating these two values of the frequency ratio corresponding to the operating range, we can find suitable value of μ from Fig. 5.9. Thus the mass of the absorber system can be determined.

Referring to Fig. 5.10 (*a*) and 5.10 (*b*) it can be seen that to obtain different curves, two parameters can be varied. The first parameter which can be varied is the mass ratio μ. But a large mass ratio represents a practical problem of handling. An absorber system that matches the original system in size is not a good solution to any vibration problem. A smaller mass ratio on the other hand will give a narrower operating band of the absorber as can be seen from Fig. 5.9.

The second parameter is the frequency ratio ω_1/ω_2. The natural frequency of the absorber system ω_2 is the frequency at which $A_1 = 0$. It is not necessarily equal to ω_1, although the use of a vibration absorber is most wanted when the forcing frequency 'ω' is close to the natural frequency of the main system 'ω_1'. The operating restrictions make it impossible to vary either the forcing frequency ω or the natural frequency of the main system (ω_1).

From Fig. 5.9 we see that as the mass ratio increases, the separation of the two natural frequencies increases.

If damping is added to the absorber system, the amplitudes A_1 and A_2 will both get diminished at resonance. The lower natural frequency is diminished less than the higher natural frequency, and it is the lower natural frequency which must be passed through in order to reach the operating speed (Refer Fig. 5.9). To equalize the maximum amplitudes at resonance, the damped absorber is tuned to a frequency slightly lower than the natural frequency of the main system.

Optimum tuning is defined as the ratio ω_2/ω_1, when the resonant amplitudes are equal. It is sufficient here to state the result that at optimum tuning,

$$\frac{\omega_2}{\omega_1} = \frac{1}{1+\mu} \quad \text{...(5.9.17)}$$

[The derivation of eqn. (5.9.17) is beyond the scope of this book. The students can seek the derivation in books of S. Timoshenko and J.P. Den Hartog]

Damping can also be optimized. If no damping is present, the amplitude of the main system will be zero at the tuning frequency $\omega = \omega_2$. With damping, the resonant amplitudes of combined system are diminished but the minimum amplitude of the main system is no longer zero at the tuning frequency.

Optimum damping is thus defined as the amount of damping which will make the response curve nearly flat between the two natural frequencies ω_1 and ω_2. The resonant amplitudes are decreased and the amplitudes at the tuning frequency is increased.

[**Note.** If vibration absorbers are used, they are more often used without damping. Damping defeats the very purpose of an absorber, which is to eliminate unwanted vibration, and is only warranted if the frequency band in which an absorber is effective is too narrow for operation. Thus the damped dynamic vibration absorbers are not suitable for practical purposes because for them to be effective, they have to be operated in a very narrow range of natural frequencies as given in Fig. 5.9. Greater the range of natural frequency (in which the absorber can be operated), more is the practical utility of the absorber.]

5.10 TORSIONAL VIBRATION ABSORBER

As in the case of rectilinear vibrations; a torsional vibration absorber can be used to reduce or completely eliminate torsional oscillations of a system.

From Fig. 5.11 it can be seen that the main system is represented by k_{t_1} and J_1 and is subjected to a periodic torque $T \sin \omega t$. The torsional vibration absorber is represented by k_{t_2} and J_2. The analysis for dynamic vibration absorber holds good in this case also.

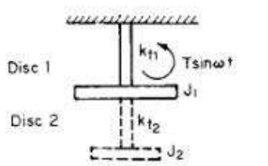


Fig. 5.11. Torsional Absorber System.

Considering the arrangement, we see that a harmonic torque $T \sin \omega t$ is impressed on the disk. The natural frequency of the main system is given by

$$\omega_n = \sqrt{\frac{k_{t_1}}{J_1}} \qquad \text{...(5.10.1)}$$

When this natural frequency of the system coincides with the impressed torque frequency, resonance occurs and the system needs some correction. One of the methods is to change the stiffness of the shaft k_{t_1} or the inertia of the disk J_1 to change the natural frequency of the system.

If the above method is somehow not possible to implement due to operating condition restrictions such as limited space etc., then an absorber disc J_2 and shaft with stiffness k_{t_2} are added to J_1 to absorb the impressed torque so that the disc 1 does not vibrate.

This absorber should be tuned to the impressed frequency 'ω' such that

$$\omega_{n_2} = \omega = \sqrt{\frac{k_{t_2}}{J_2}} \qquad \text{...(5.10.2)}$$

The absorber shaft should be strong enough to carry the impressed torque applied to the absorber disc.

Ideally any value of k_{t_2} and J_2 will meet the requirement as long as their ratio is equal to ω^2. A shaft to be added to a system may require too much axial length which may not be possible. A number of devices are used to replace it. The Ring Torsional absorber is one of them and is shown in Fig. 5.12.

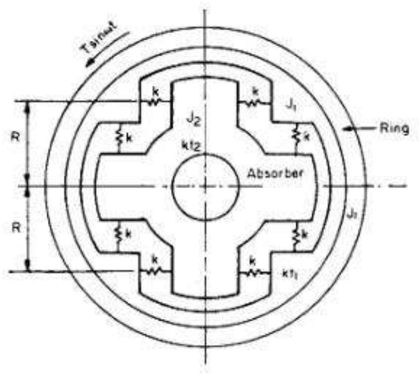


Fig. 5.12. Ring torsional absorber.

This torsional absorber consists of a ring attached to one of the discs of the original system. A mass is connected to this ring by means of springs. If no vibration is present, the entire unit rotates at a constant speed. When torsional vibrations occur in the system, the mass tends to continue to rotate at constant speed, so that the springs are deflected and it acts as an absorber. When the springs are deflected due to vibrations in the main system, there is a change in their

potential energy and thus they absorb the energy of vibration of the main system. Thus the vibrations of main system are reduced or completely eliminated. Thus coil springs replace the length of shafting.

A four spring torsional vibration absorber is also shown in Fig. 5.13.

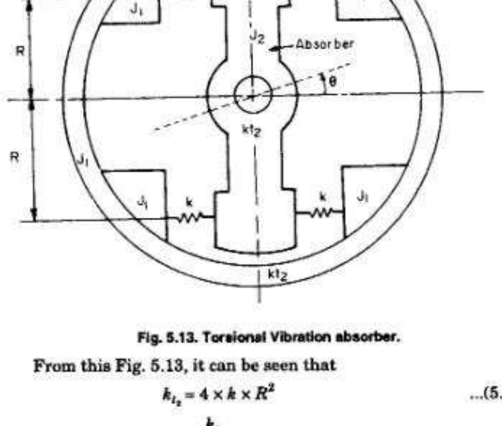


Fig. 5.13. Torsional Vibration absorber.

From this Fig. 5.13, it can be seen that

$$k_{t_2} = 4 \times k \times R^2 \qquad \text{...(5.10.3)}$$

or

$$k = \frac{k_{t_2}}{4R^2} \qquad \text{...(5.10.4)}$$

Eqn. (5.10.3) is derived as follows :

The 4 springs are in parallel with each other. Thus their equivalent stiffness is sum of their individual stiffness.

$$\therefore \quad k_e = k + k + k + k = 4\,k$$

The spring force on them is given by

$$F = k_e \times x = 4\,k \times x$$

$$= 4k \times R\theta \qquad [\because x = R\theta] \qquad \text{...(5.10.5)}$$

where x = rectilinear displacement of equivalent system

θ = angular displacement of equivalent or the absorber system.

$\therefore$ **Torque exerted on the absorber**

$$T_2 = F \times R = 4\,k \times R\theta \times R$$

$$= 4k \times R^2\,\theta \qquad \text{...(5.10.6)}$$

$\therefore$ **Torque exerted per unit twist on absorber i.e.**

$$k_{t_2} = T_2/\theta = 4\,kR^2 \qquad \text{...(5.10.7)}$$

The amplitude of vibration of the torsional absorber at the exciting frequency 'ω' is given by eqn. (5.9.12) based on a similar analysis after changing translational quantities into torsional quantities.

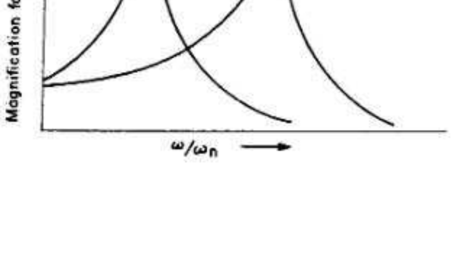
Thus $\quad T = -\beta_2\,k_{t_2} \qquad$ ...(5.10.8)

where T = **maximum value of applied torque**

β_2 = **amplitude of vibration of torsional absorber.**

5.11 DEMERIT OF DYNAMIC VIBRATION ABSORBER

The dynamic vibration absorber, whether for a torsional or rectilinear system, is only fully effective at a particular impressed frequency for which it is designed. This means that the main mass or disc is stationary (no vibration) only for this particular frequency. Thus dynamic vibration absorbers are extremely effective for constant speed machines but lose their effectiveness with any change in speed of the machines. However, it may be made reasonably effective over a fairly wide range of frequency by using a large mass ratio ($\mu = M_2/M_1$) value [$\mu = J_2/J_1$ in case of torsional absorbers]. The dynamic vibration absorber can also be employed for a small range between the two natural frequencies if conditions on the allowable amplitude of the main rotor or main system are relaxed. However, most rotors having vibration problems are likely to run through a wide range of speeds so that if dynamic vibration absorber is employed, the situation deteriorates instead of improving.



When we plot the frequency response curve of an undamped lynamic vibration absorber, we see that it has two resonance peaks orresponding to damping $\varepsilon = 0$ and damping $\varepsilon = \infty$. So its use is limited nly to the fixed-speed machines. The frequency response curve is given n Fig. 5.14.

Reproducing Fig. 5.10 (a) in Fig. 5.15, with the dotted curve epresenting the relation between A_1/A_{st} and ω/ω_2 when the absorber s not used, whereas the full line curve shows the relation with the

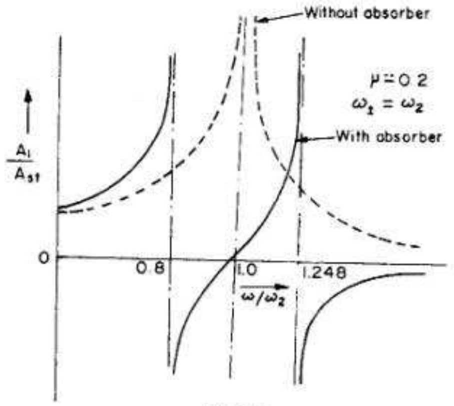


Fig. 5.15.

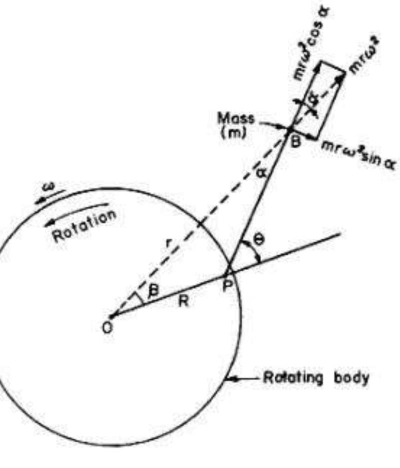
bsorber in place. It can be seen that in latter case that there are two esonant frequencies in place of one. This is the demerit of dynamic ibration absorber that it adds an additional degree of freedom.

.12 CENTRIFUGAL PENDULUM ABSORBER

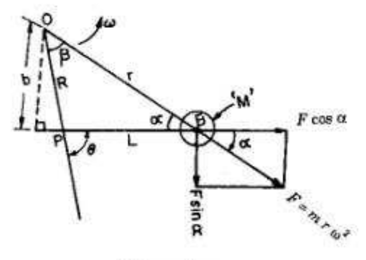
As discussed earlier, the undamped dynamic vibration absorber is ully effective only at a particular frequency for which it has been esigned. In case of torsional system, it is possible to use a dynamic ibration absorber of the pendulum type that is effective at all speeds f rotation of the system. This is the pendulum or centrifugal pendulum ype of absorber shown in Fig. 5.16 (*a*).

A pendulum *PB* of length '*L*' is attached to a rotating member at oint *P*, which is at a radius '*R*' from the centre of rotation *O*. The mass f the pendulum bob is '*m*' and the string is assumed to have negligible nass. This pendulum is subjected to a centrifugal force which is much

greater as compared to the gravitational force so that the latter is considered negligible. The rotating body rotates with a angular velocity of 'ω' and due to this rotation a centrifugal force $m\omega^2 r$ is experienced by the pendulum.



(*a*) Centrifugal Pendulum Absorber.



(*b*) Force diagram

Let us assume that at any instant, the pendulum is displaced from the radial line by a small amount θ. The bob or mass at the tip of the pendulum is then at a distance '*r*' from the centre of rotation '*O*' and is subjected to a centrifugal force of

$$F = mr\omega^2 \qquad ...(5.12.1)$$

This centrifugal force is directed outwards along *OB*. This force has one component along the pendulum line and other perpendicular to it. The latter $mr\omega^2 \sin \alpha$, is the restoring force and has to be taken into account in the differential equation of motion for the oscillation of the pendulum. Here α is the angle between the pendulum and the radial line *OB*.

The differential eqn. of motion for oscillation of pendulum is given as :

$$I\,\ddot{\theta} = -F \sin \alpha \times L \qquad ...(5.12.2)$$

$$mL^2\,\ddot{\theta} = -(mr\,\omega^2 \sin \alpha) \times L \qquad ...(5.12.3)$$

$$[\because I = mL^2]$$

[**Note.** If mass of string is also considered then $I = \left(m + \frac{M_s}{3}\right)L^2$, where M_s = mass of string]

From eqn. (5.12.3), we get

$$\ddot{\theta} + \frac{r}{L}\,\omega^2 \sin \alpha = 0 \qquad ...(5.12.4)$$

Applying the law of sines to the triangle ΔOPB, we get

$$\frac{R}{\sin \alpha} = \frac{r}{\sin (180° - \theta)}$$

or

$$r \sin \alpha = R \sin \theta \qquad ...(5.12.5)$$

Substituting the value of $r \,.\, \sin \alpha$ from equation (5.12.5) in eqn. (5.12.4), we get,

$$\ddot{\theta} + \frac{R}{L}\,\omega^2 \sin \theta = 0 \qquad ...(5.12.6)$$

For small values of θ; $\sin \theta \approx \theta$

Thus eqn. (5.12.6) becomes

$$\ddot{\theta} + \frac{R}{L}\,\omega^2\,\theta = 0 \qquad ...(5.12.7)$$

Eqn. (5.12.7) for the pendulum is that of simple harmonic motion and its natural frequency is given by

$$\omega_n = \sqrt{\frac{R\omega^2}{L}} = \omega\sqrt{\frac{R}{L}} \qquad ...(5.12.8)$$

Or the natural frequency in cycles per second is given by

$$f_n = \frac{\omega_n}{2\pi} = \frac{\omega}{2\pi}\sqrt{\frac{R}{L}} = N\sqrt{\frac{R}{L}} \qquad ...(5.12.9)$$

where N = revolutions per second of rotating body $= \frac{\omega}{2\pi}$

From eqn. (5.12.9) the principle of working of the centrifugal pendulum absorber can be seen. It is stated as the natural frequency of the pendulum absorber is always proportional to the speed of the rotating body.

Thus $\qquad f_n \propto N \qquad ...(5.12.10)$

The usual torsional system receives a certain number of disturbing torques per revolution. The no. of these torques per revolution is known as ORDER NO. of the system. A two cylinder engine working on a four stroke cycle has one disturbing torque per revolution and its order no. is one. A four and six-cylinder engines working on four stroke cycle have order number of two and three respectively.

A pendulum absorber is designed to eliminate or reduce the torsional disturbances of a particular order number. If several order numbers are present in a system, several pendulum absorbers are required.

DESIGN

For the pendulum absorber to be effective, its natural frequency f_n should be equal to the excitation frequency or frequency of disturbing torque. The most important application of this type of absorber is in I.C. engines where the frequency of torque contains the harmonics of speed 'ω'.

In such a case, let $T \sin (n\,\omega\,t)$ be the torque on the I.C. engine, where n = ORDER NO. Therefore, from the above principle, the vibration absorber must be designed for the condition that the natural frequency of the pendulum should equal the excitation frequency.

$\therefore$ From eqn. (5.12.9) we get

$$f_n = N\sqrt{\frac{R}{L}} = \frac{n\omega}{2\pi} \qquad ...(5.12.11)$$

or

$$\omega\sqrt{\frac{R}{L}} = n\omega \qquad ...(5.12.12)$$

or

$$n = \sqrt{\frac{R}{L}} = \text{ORDER No.} \qquad ...(5.12.13)$$

From eqn. (5.12.11) and eqn. (5.12.13), we get

$$n = \sqrt{\frac{R}{L}} = \frac{f_n}{N} = \frac{\text{Disturbing torque impulse/sec}}{\text{Revolutions/second}} \quad ...(5.12.14)$$

From eqn. (5.12.14)

Thus $\sqrt{\frac{R}{L}}$ = order no.

= Disturbing torque impulses/revolution.

Equation (5.12.14) provides one design criterion.

The procedure for design is to equate the order no. to $\sqrt{R/L}$ and solve for length (L) of pendulum required by choosing a value for 'R'.

The size of the pendulum mass is a function of the magnitude of disturbing torque. For a certain disturbing torque amplitude, larger the mass of the pendulum, smaller is its amplitude of vibration. Thus the pendulum mass is made as large as possible since it will then absorb the greatest amount of energy with the minimum amplitude θ. So the amplitudes of vibrations are kept small.

When used on an I.C. engine, the pendulum is usually attached to a crank web, so that R is usually about equal to the crank throw r. In such cases, the point P is never kept on the boundary of the machine and the length of the pendulum (PB) is kept small.

5.13 UNTUNED VIBRATION DAMPERS

5.13.1 Untuned Dry Friction Damper (Lanchester Damper)

This type of a damper is very advantageous to use for torsional vibrations near resonance conditions. It effectively reduces the amplitudes of torsional vibrations near resonance conditions.

Construction

It consists of two flywheels mounted freely over a hub. The hub is rigidly fixed to the shaft undergoing vibrations. There are friction plates attached to the extension of the hub. These friction plates apply pressure on the flywheels and are responsible for the driving of the flywheels. The pressure between the friction plates and the flywheels can be adjusted through the spring loaded bolts which hold both the flywheels together.

Refer figure 5.17.

Working

If there is a large frictional torque than the pressure between the friction plates and the flywheels is very large. Under such circumstances the flywheels become rigid with the shaft and possess the same oscillations as that of the shaft. Thus no energy is dissipated during vibrations since there is no relative rubbing. The energy dissipated is proportional to the frictional torque times the relative velocity.

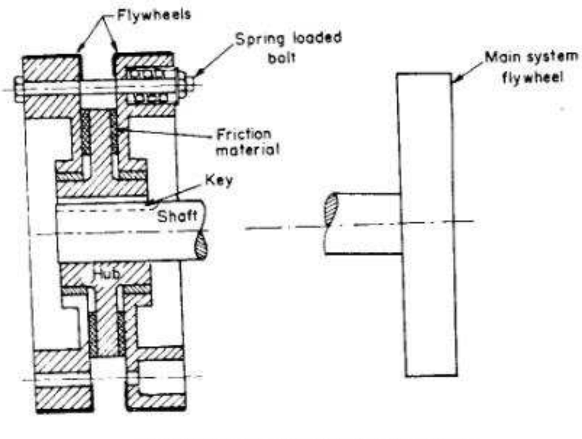
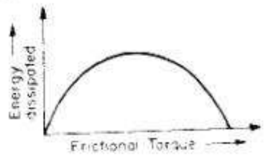


Fig. 5.17. Dry Friction Torsional vibration absorber.

When the pressure between the friction plates and flywheels becomes zero, the relative velocity is maximum but the frictional torque is zero. Again there is no energy dissipation. When the speed of the main system is such that torsional vibrations are present in the system, then the pressure between the friction material and the flywheel is such that both the frictional torque and the relative rubbing (between the friction material and the flywheels) are present. Thus there is energy dissipation in the absorber which leads to reduction in the energy of torsionally vibrating main system. Thus energy reduction causes a consequent reduction in the amplitude of vibrations of the main system.

The amplitude reduction will be greater if greater amount of energy is dissipated.

The variation of energy dissipated against the frictional torque is shown in figure 5.18.



13.2 Untuned Viscous Damper (Houdaille Damper)

This type of a damper is similar in principle to the Lanchester amper except that instead of using friction plates for dry friction imping, this system uses a viscous damping. Refer figure 5.19.

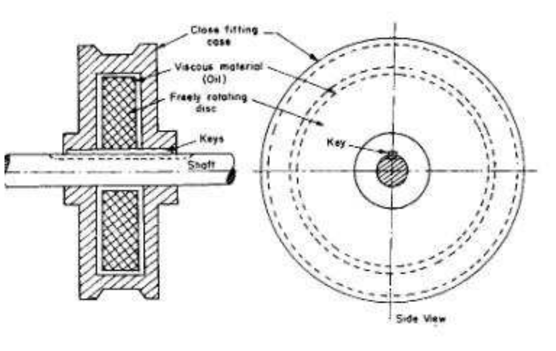


Fig. 5.19. Untuned Viscous Damper (Houdaille Damper).

It is added to a dynamic system to alter its vibrational response. consists essentially of a freely rotating disc enclosed in a close-fitting se which is keyed to the shaft. Normally the disc rotates at the shaft eed owing to viscous drag of the oil between the disc and the case. owever, if the shaft vibrates torsionally, viscous action of the oil tween the disc and casing gives a damping action.

There are two cases which arise. They are :

(1) When the damping is zero in the damper, it is ineffective and the system corresponds to a single degree of freedom.

(2) If the damping is infinite, the damper mass becomes integral with the shaft or the main mass. It still remains a single degree of freedom system when the main system rotates at such a speed that torsional vibrations are produced, then the energy is dissipated due to the viscous drag of the viscous material filled between the disc and the casing. Thus the energy of the main system is reduced thereby reducing the amplitude of torsional vibrations of the main system.

Optimum damping is introduced in the system so that the maximum response of the damper over the entire frequency range does not go beyond a certain permissible level.

Houdaille damper can be used with variable speed machines, maximum response being controlled by the ratio of damper inertia to the main system inertia i.e. $\frac{J_2}{J_1}$.

The frequency response curve for an untuned viscous damper is shown in figure 5.20.

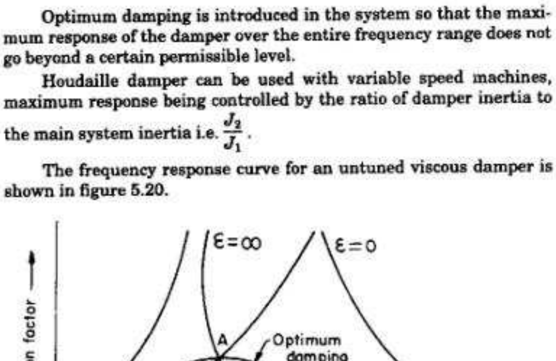


Fig. 5.20. Frequency response curve for untuned viscous damper.

It can be seen from this figure that the frequency response curve is of the same nature for the above two cases except that the peak shifts towards the left in the second case. The amount of shift depends on the ratio of damper inertia to the main system inertia.

The point of intersection of the two curves, for the above two cases corresponding to $\varepsilon = 0$ and $\varepsilon = \infty$, is the point through which response curves of different damping values pass through it.

A system having optimum damping has its response curve with A as its highest point.

5.14. TORSIONALLY EQUIVALENT SHAFT

Consider a shaft AB of varying diameter as shown in figure 5.21. It is shown that d_1, d_2, d_3 and d_4 are the diameters of shaft of lengths l_1, l_2, l_3 and l_4 respectively. Let the torque applied at the ends is T and total twist of the shaft is θ. Say θ_1, θ_2, θ_3 and θ_4 are the twists of lengths l_1, l_2, l_3 and l_4 respectively.

So $\theta = \theta_1 + \theta_2 + \theta_3 + \theta_4$ we know that $\frac{T}{I} = \frac{G\theta}{l}$

$$\frac{Tl}{GI} = \frac{Tl_1}{GI_1} + \frac{Tl_2}{GI_2} + \frac{Tl_3}{GI_3} + \frac{Tl_4}{GI_4}$$

$$\frac{l}{I} = \frac{l_1}{I_1} + \frac{l_2}{I_2} + \frac{l_3}{I_3} + \frac{l_4}{I_4}$$

$$\frac{l}{\frac{\pi}{32} d^4} = \frac{l_1}{\frac{\pi}{32} d_1^4} + \frac{l_2}{\frac{\pi}{32} d_2^4} + \frac{l_3}{\frac{\pi}{32} d_3^4} + \frac{l_4}{\frac{\pi}{32} d_4^4}$$

$$l = l_1\left(\frac{d}{d_1}\right)^4 + l_2\left(\frac{d}{d_2}\right)^4 + l_3\left(\frac{d}{d_3}\right)^4 + l_4\left(\frac{d}{d_4}\right)^4$$

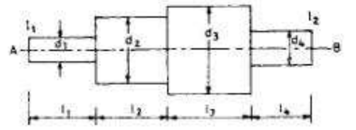


Fig. 5.21. Equivalent shaft.

If $\quad d = d_1$

$$l = l_1 + l_2\left(\frac{d}{d_2}\right)^4 + l_3\left(\frac{d}{d_3}\right)^4 + l_4\left(\frac{d}{d_4}\right)^4$$

Thus a shaft of varying diameter has been replaced by a torsionally equivalent shaft. l and $d = d_1$ are known as the length and diameter of torsionally equivalent shaft. Natural frequency of the shaft can be determined by the relation

$$\omega_n = \sqrt{\frac{k_t(I_1 + I_2)}{I_1 I_2}} \qquad ...(5.14.1)$$

where $\quad k_t$ = stiffness of the shaft

I_1, I_2 = polar moment of inertia of the shaft ends

$$= \frac{\pi}{32} d^4$$

5.15. LAGRANGE'S EQUATIONS

The equations of motion of a vibrating system are written in terms of generalised coordinates by making use of Lagrange's equations. Generalised coordinates are independent parameters which specify the system completely. If energy expressions are available, the equations of motion can be obtained with the help of Lagrange's equation. The

general form of this equation in terms of generalised coordinates is written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) - \frac{\partial T}{\partial x_j} + \frac{\partial V}{\partial x_j} = Q_j \qquad ...(5.15.1)$$

where $\quad T$ = total kinetic energy of the system

V = total potential energy of the system

$j = 1, 2, 3 .... n$

n = degree of freedom of the system

Q_j = generalised external force

For a conservative system generalised force Q_j acting on the system is zero, so equation (5.15.1) for such a system can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) - \frac{\partial T}{\partial x_j} + \frac{\partial V}{\partial x_j} = 0 \qquad ...(5.15.2)$$

For example, if the kinetic and potential energies of the system are given as

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \qquad ...(5.15.3)$$

and

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 \qquad ...(5.15.4)$$

Lagrange's equation can be used to obtain the equations of motion from the above two equations.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$$

$$\frac{\partial T}{\partial x_1} = 0$$

$$\frac{\partial V}{\partial x_1} = k_1 x_1 - k_2(x_2 - x_1)$$

First equation of motion can be written as

$$m_1 \ddot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) = 0$$

For second equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2$$

$$\frac{\partial T}{\partial x_2} = 0$$

$$\frac{\partial V}{\partial x_2} = k_2(x_2 - x_1)$$

Second equation of motion can be written as

$$m_2 \ddot{x}_2 + k_2(x_2 - x_1) = 0$$

SOLVED EXAMPLES

Example 5.1. *Figure 5.22 shows a vibrating system having two degrees of freedom. Determine the two natural frequencies of vibrations and the ratio of amplitudes of the motion of m_1 and m_2 for the two modes of vibration.*

Given : $m_1 = 1.5$ kg, $m_2 = 0.80$ kg

$k_1 = k_2 = 40$ N/m

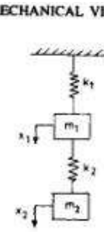


Fig. 5.22.

Solution. Let at any instant masses m_1 and m_2 are having displacements x_1 and x_2 respectively.

Equations of motion can be written as

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1)$$

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1)$$

Assuming the solution of the form

$$x_1 = A_1 \sin \omega t$$

and

$$x_2 = A_2 \sin \omega t$$

So above two equations can be written as

$$(k_1 + k_2 - \omega^2 m_1) A_1 - k_2 A_2 = 0$$

$$(k_2 - \omega^2 m_2) A_2 - k_2 A_1 = 0$$

$$\frac{A_1}{A_2} = \frac{k_2}{k_1 + k_2 - m_1\omega^2}$$

$$\frac{A_1}{A_2} = \frac{k_2 - m_2\omega^2}{k_2}$$

The frequency equation can be written as

$$(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2 = 0$$

or

$$\omega^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)\omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0$$

Putting $k_1 = k_2 = 40$ N/m and $m_1 = 1.5$ kg ; $m_2 = .80$ kg

$$\omega^4 - \left(\frac{80}{1.5} + \frac{40}{.8}\right)\omega^2 + \frac{40 \times 40}{1.5 \times .8} = 0$$

$$\omega^4 - 103.33\,\omega^2 + 1333.33 = 0$$

$$\omega_1 = 9.39 \text{ rad/sec}$$

$$\omega_2 = 3.88 \text{ rad/sec}$$

$$\text{The amplitude ratio} = \frac{k_2}{k_1 + k_2 - m_1\omega^2} = \frac{40}{40 + 4 - 1.5\,(9.39)^2}$$

$$= -0.765$$

and

$$\frac{A_1}{A_2} = \frac{40}{40 + 40 - 1.5(3.88)^2} = 0.696$$

Example 5.2. *Solve the problem shown in figure 5.23 $m_1 = 10$ kg, $m_2 = 15$ kg and $k = 320$ N/m.*

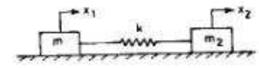


Fig. 5.23.

Solution. The equations of motion can be written as

$$m_1 \ddot{x}_1 + k(x_1 - x_2) = 0 \;;\; m_2 \ddot{x}_2 + k(x_2 - x_1) = 0$$

Assuming the solution of the form

$$x_1 = A_1 \sin \omega t \;;\; x_2 = A_2 \sin \omega t$$

$$-m_1\omega^2 A_1 + k(A_1 - A_2) = 0$$

$$-m_2\omega^2 A_2 + k(A_2 - A_1) = 0$$

Amplitude ratio $\quad \dfrac{A_1}{A_2} = \dfrac{k}{k - m_1\omega^2}$

$$\frac{A_1}{A_2} = \frac{k - m_2\omega^2}{k}$$

The frequency equation is obtained as

$$\frac{k}{k - m_1\omega^2} = \frac{k - m_2\omega^2}{k}$$

$$\omega^4 - \omega^2 k \frac{(m_1 + m_2)}{m_1 m_2} = 0$$

$$\omega^2 - \frac{k(m_1 + m_2)}{m_1 m_2} = 0$$

So $\quad \omega_1 = 0$

and

$$\omega_2 = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} = \sqrt{\frac{320(10 + 15)}{10 \times 15}} = 7.30 \text{ rad/sec.}$$

$$\left(\frac{A_1}{A_2}\right)_{\omega_1} = 1.0$$

$$\left(\frac{A_1}{A_2}\right)_{\omega_2} = \frac{320 - 15(7.30)^2}{320} = -1.49$$

Example 5.3. *Use Lagrange's equation to find equations of motion r a system shown in figure 5.23.*

Solution. There are two generalised co-ordinates x_1 and x_2 and ›th masses are connected by a spring of stiffness k.

Kinetic energy $T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$

Potential energy $V = \frac{1}{2} k(x_2 - x_1)^2$

Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) - \frac{\partial T}{\partial x_j} + \frac{\partial V}{\partial x_j} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$$

$$\frac{\partial T}{\partial x_1} = 0$$

$$\frac{\partial V}{\partial x_1} = -k(x_2 - x_1)$$

First equation of motion is

$$m_1 \ddot{x}_1 + k(x_1 - x_2) = 0$$

Similarly,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2$$

$$\frac{\partial T}{\partial x_2} = 0$$

$$\frac{\partial V}{\partial x_2} = k(x_2 - x_1)$$

Second equation of motion becomes

$$m_2 \ddot{x}_2 + k(x_2 - x_1) = 0$$

Example 5.4. *A vibratory system performs the motions as expressed the following equations :*

$$\ddot{x} + 800x + 90\theta = 0$$

$$\ddot{\theta} + 800\theta + 90x = 0$$

If the system is turned through 1.5 radians and released, find the quencies and mode shapes.

Solution. Adding both the equations, we get

$$(\ddot{x} + \ddot{\theta}) + (800 + 90)(x + \theta) = 0$$

Assuming $y_1 = x + \theta$ and substituting it in the above equation, we get

$$\ddot{y}_1 + 890 y_1 = 0$$

From this equation, the frequency can be determined as

$$\omega_1 = \sqrt{890} = 29.83 \text{ rad/sec.}$$

Subtraction of the given equations is written as

$$(\ddot{x} - \ddot{\theta}) + (800 - 90)(x - \theta) = 0$$

Let us assume $y_2 = x - \theta$ and putting in the above equation

$$\ddot{y}_2 + 710 y_2 = 0$$

Thus frequency, $\omega_2 = \sqrt{710} = 26.64$ rad/sec

Assuming the motion to be harmonic type as

$$x = x_0 \sin \omega t$$

$$\theta = \theta_0 \sin \omega t$$

$$\ddot{x} = -\omega^2 x_0 \sin \omega t$$

$$\ddot{\theta} = -\omega^2 \theta_0 \sin \omega t$$

Again rewriting the given equations and substituting the values of $\ddot{x}$ and $\ddot{\theta}$

$$-\omega^2 x_0 + 800 x_0 + 90\theta_0 = 0$$

$$(-\omega^2 + 800) x_0 = -90\theta_0$$

$$\left(\frac{x_0}{\theta_0}\right)_1 = \frac{-90}{-\omega_1^2 + 800} = \frac{-90}{-890 + 800} = 1 \quad (\text{Substituting } \omega_1^2 = 890)$$

and $\theta_0 = 1.5$ (given)

So $(x_0)_1 = 1.5 \times 1 = 1.5$ (first mode)

Similarly for the second mode shape

$$-\omega^2 \theta_0 + 800\theta_0 + 90 x_0 = 0$$

$$(-\omega^2 + 800)\theta_0 = -90 x_0$$

$$\left(\frac{x_0}{\theta_0}\right)_2 = \frac{-\omega_2^2 + 800}{-90} = \frac{-710 + 800}{-90} = -1$$

So $(x_0)_2 = -1.5$

Example 5.5. *A machine runs at 5000 rpm. Its forcing frequency is very near to its natural frequency. If the nearest frequency of the machine is to be at least 20% from the forced frequency, design a suitable vibration absorber for the system. Assume the mass of the machine as 30 kg.*

Solution. The natural frequency of the system at 5000 rpm.

$$\omega_n = \frac{2\pi N}{60} = \frac{2\pi \times 5000}{60} = 523.33 \text{ rad/sec.}$$

Assuming $\omega = \omega_2$, we can find two resonant frequencies from equation (5.9.13) as

$$\left(\frac{\omega}{\omega_2}\right)^2 = \left(1 + \frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}}$$

The resonant frequencies are at least 20% away from the forced frequency of the main system. So, we have

$$\omega/\omega_2 = 0.80$$

or

$$\omega/\omega_2 = 1.20$$

When $\omega/\omega_2 = 0.8$, the value of μ

$$(0.8)^2 = \left(1 + \frac{\mu}{2}\right) - \sqrt{\left(\mu + \frac{\mu^2}{4}\right)}$$

$$\mu = 0.2$$

and for $\omega/\omega_2 = 1.2$, the value of

$$\mu = 0.13$$

The larger value of μ is taken for design purpose.

$$\mu = 0.2 = \frac{m}{M} = \text{mass ratio}$$

$$m = 0.2 \times 30 = 6.0 \text{ kg}$$

$$\omega_1 = \sqrt{\frac{k_1}{M}} \quad \omega_1 = \omega_n$$

$$\omega_n^2 = \frac{k_1}{M}$$

$$k_1 = \omega_n^2 M = (523.33)^2 \times 30$$

$$= 8216.22 \text{ KN/m}$$

and

$$\omega_2 = \sqrt{\frac{k_2}{m}} \quad \omega_2 = \omega_n$$

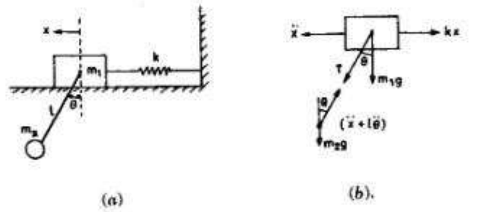
$$k_2 = \omega_2^2 m = (523.33)^2 \times 6$$

$$= 1643.24 \text{ KN/m.}$$

Example 5.6. *Find the frequencies of the system shown in figure 5.24.*

$k = 90$ *N/m*, $l = .25$ *m*

$m_1 = 2$ *kg*, $m_2 = 0.5$ *kg*



(a) (b).

Fig. 5.24.

Solution. Initially the pendulum rod is vertical and it is displaced by an angle θ as shown in figure (a) and free body diagram of forces is shown in figure (b). Let us assume that T is the tension in the pendulum rod.

Resolving the forces vertically for m_2

$$m_2 g = T \cos \theta$$

resolving the forces horizontally, $T \sin \theta$ will be known as restoring force as it moves downwards and brings m_2 to its original state. Horizontal displacement of m_2 is $x + l \sin \theta$

when θ is very small, $\sin \theta = \theta$ and $\cos \theta = 1$.

So horizontal displacement $= x + l\theta$

and acceleration $= \ddot{x} + l\ddot{\theta}$

Horizontal force $\quad m_2(\ddot{x} + l\ddot{\theta}) = -T\theta$

So $\quad m_2 g = T$ and $m_2(\ddot{x} + l\ddot{\theta}) = -T\theta$

or

$$m_2(\ddot{x} + l\ddot{\theta}) + T\theta = 0$$

$$m_2(\ddot{x} + l\ddot{\theta}) + m_2 g\theta = 0, \text{ put } T = m_2 g$$

$$(\ddot{x} + l\ddot{\theta}) + g\theta = 0$$

or

$$l\ddot{\theta} + g\theta = -\ddot{x}$$

Consider forces for mass m_1. All the forces are acting horizontally,

$$m_1 \ddot{x} = -kx + T \sin \theta$$

$$= -kx + T\theta$$

$$m_1 \ddot{x} + kx - T\theta = 0 \qquad \text{putting } T = m_2 g$$

$$m_1 \ddot{x} + kx - m_2 g\theta = 0$$

or

$$m_1 \ddot{x} + kx = m_2 g\theta$$

Let us assume the solution of the form

$$x = A \sin \omega t \qquad \text{and} \qquad \theta = \phi \sin \omega t$$

Substituting these solutions in the above two equations, we get

$$-l\omega^2\phi + g\phi - \omega^2 A = 0$$

and
$$-m_1\omega^2 A + kA - m_2 g\phi = 0$$

$$\frac{A}{\phi} = \frac{-l\omega^2 + g}{\omega^2} = \frac{m_2 g}{k - m_1\omega^2}$$

The frequency equation can be written as

$$(-l\omega^2 + g)(k - m_1\omega^2) - \omega^2 m_2 g = 0$$

$$-kl\omega^2 + m_1 l\omega^4 + gk - m_1 g\omega^2 - \omega^2 m_2 g = 0$$

$$\omega^4 - \frac{(kl + m_1 g + m_2 g)\omega^2}{m_1 l} + \frac{gk}{m_1 l} = 0$$

So
$$\omega^2 = \frac{(m_1 + m_2)g + kl \pm \sqrt{[(m_1 + m_2)g + kl]^2 - 4m_1 lkg}}{2m_1 l}$$

Substituting the numerical values in the above equation

$$\omega^2 = \frac{(2 + 0.50)9.81 + 90 \times 0.25 \pm \sqrt{[(2 + 0.5)9.81 + 90 \times 2.5]^2 - 4 \times 2 \times .25 \times 90 \times 9.81}}{2 \times .2 \times .25}$$

$$= 24.5 + 22.5 \pm \sqrt{(24.5 + 22.5)^2 - 1764}$$

$$= 47 \pm \sqrt{2209 + 1764} = 47 \pm 21.095$$

$\omega_1 = 8.25$ rad/sec, $\omega_2 = 5.08$ rad/sec.

Example 5.7. *Solve the problem of example 5.6 using Lagrange's equation.*

Solution.

$$\text{K.E.} = \frac{1}{2} m_1\dot{x}^2 + \frac{1}{2} m_2(l^2\dot{\theta}^2 + \dot{x}^2 + 2\dot{x}l\dot{\theta} \cos \theta)$$

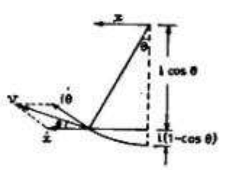
$$\text{P.E.} = \frac{1}{2} kx^2 + m_2 gl(1 - \cos \theta)$$

Applying Lagrange's equation

$$\frac{d}{dt}\frac{\partial(\text{K.E.})}{\partial \dot{x}} = m_1\ddot{x} + m_2\ddot{x} + m_2 l\ddot{\theta} \cos \theta - m_2 l\dot{\theta} \sin \theta$$

$$\frac{\partial}{\partial x}(\text{K.E.}) = 0$$

$$\frac{\partial}{\partial x}(\text{P.E.}) = kx$$



$$v^2 = (l\dot{\theta})^2 + \dot{x}^2 + 2\dot{x}l\dot{\theta} \cos \theta$$

Fig. 5.25.

So equation of motion can be written as

$$m_1\ddot{x} + m_2\ddot{x} + m_2 l\ddot{\theta} \cos \theta - m_2 l\dot{\theta} \sin \theta + kx = 0$$

when θ is very small $\sin \theta = \theta$ and $\cos \theta = 1$.

Then above equation can be written as

$$(m_1 + m_2)\ddot{x} + m_2 l\ddot{\theta} + kx = 0$$

Similarly,

$$\frac{d}{dt}\frac{\partial(\text{K.E.})}{\partial \dot{\theta}} = m_2 l^2\ddot{\theta} + m_2\ddot{x}l$$

$$\frac{\partial}{\partial \theta}(\text{K.E.}) = 0 \text{ and } \frac{\partial}{\partial \theta}(\text{P.E.}) = m_2 lg\theta$$

So the second equation of motion can be written as

$$m_2 l^2\ddot{\theta} + m_2\ddot{x}l + m_2 lg\theta = 0$$

or
$$l\ddot{\theta} + g\theta = -\ddot{x}$$

$$\left.\begin{aligned} \text{Let } x &= A \sin \omega t \\ \theta &= \phi \sin \omega t \end{aligned}\right\} \text{ for principal modes}$$

Solving the equations

$$[k - (m_1 + m_2)\omega^2]A = m_2 l\omega^2\phi$$

$$(g - l\omega^2)\phi = \omega^2 A$$

The amplitude ratio is given as

$$\frac{m_2 l\omega^2}{k - (m_1 + m_2)\omega^2} = \frac{g - l\omega^2}{\omega^2}$$

$$m_2 l\omega^4 = (g - l\omega^2)[k - (m_1 + m_2)\omega^2]$$

or
$$\omega^4 - \omega^2 \frac{[g(m_1 + m_2) + kl]}{m_1 l} + \frac{kg}{m_1 l} = 0$$

$$\omega^4 - \omega^2 \frac{[9.81(2 + .5) + 90 \times .25]}{2 \times .25} + \frac{90 \times 9.81}{2 \times .25} = 0$$

$$\omega^4 - \omega^2(94.05) + 1765.8 = 0$$

$$\omega_1 = 8.25 \text{ rad/sec} \quad \text{and} \quad \omega_2 = 5.08 \text{ rad/sec}$$

Example 5.8. *Find the natural frequencies of the system shown in figure 5.26. Assume that there is no slip between the cord and cylinder.* (M.D.U., 95)

Given

$k_1 = 40$ N/m

$k_2 = 60$ N/m

$m_1 = 2$ kg

$m_2 = 10$ kg

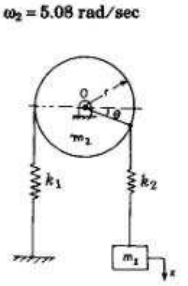


Fig. 5.26.

Solution. Let us give x vertical displacement to mass m_1 as shown. Since there is no slip between the cord and cylinder, so vertical displacement x causes the cylinder to rotate by angle θ.

Writing the equations

$$m_1\ddot{x} = -k_2(x - r\theta)$$

and
$$I\ddot{\theta} = k_2(x - r\theta)r - k_1 r^2\theta$$

where $I = \frac{1}{2} m_2 r^2$ = moment of inertia of cylinder

Above equation becomes

$$m_1\ddot{x} + k_2 x - k_2 r\theta = 0$$

$$I\ddot{\theta} + (k_1 r^2 + k_2 r^2)\theta - k_2 xr = 0$$

Let us assume the solution of the form

$$x = A \sin \omega t, \quad \ddot{x} = -\omega^2 A \sin \omega t$$

$$\theta = \phi \sin \omega t, \quad \ddot{\theta} = -\omega^2\phi \sin \omega t$$

Substituting these values in the above equations

$$-\omega^2 A m_1 + k_2 A - k_2 r\phi = 0$$

$$-\omega^2 I\phi + (k_1 r^2 + k_2 r^2)\phi - k_2 Ar = 0$$

$$(k_2 - \omega^2 m_1)A - k_2 r\phi = 0, \quad \frac{A}{\phi} = \frac{k_2 r}{k_2 - \omega^2 m_1}$$

$$\frac{A}{\ } \quad k_1 r^2 + k_2 r^2 - \omega^2 I$$

$$-k_2^2 r^2 + (k_2 - \omega^2 m_1)(k_1 r^2 + k_2 r^2 - \omega^2 I) = 0$$

Also $I = 1/2\, m_2 r^2$

$$-k_2^2 r^2 + k_1 k_2 r^2 + k_2^2 r^2 - k_2 \frac{1}{2} m_2 r^2\omega^2 - \omega^2 k_1 m_1 r^2 - \omega^2 m_1 k_2 r^2 + \omega^2 m_1 k\omega^2 \cdot \frac{1}{2} m_2 r^2 = 0$$

$$\omega^4 \frac{m_1 m_2 r^2}{2} - \omega^2\left(\frac{k_2 m_2 r^2}{2} + k_1 m_1 r^2 + m_1 k_2 r^2\right) + k_1 k_2 r^2 = 0$$

or
$$\omega^4 - \omega^2\left(\frac{k_2 m_2 r^2}{m_1 m_2 r^2} + \frac{2k_1 m_1 r^2}{m_1 m_2 r^2} + \frac{2m_1 k_2 r^2}{m_1 m_2 r^2}\right) + \frac{2k_1 k_2 r^2}{m_1 m_2 r^2} = 0$$

$$\omega^4 - \omega^2\left(\frac{k_2}{m_1} + \frac{2k_1}{m_2} + \frac{2k_2}{m_2}\right) + \frac{2k_1 k_2}{m_1 m_2} = 0$$

$$\omega^4 - \omega^2\left[\frac{2(k_1 + k_2)}{m_2} + \frac{k_2}{m_1}\right] + \frac{2k_1 k_2}{m_1 m_2} = 0$$

Substituting the values of various parameters

$$\omega^4 - \omega^2\left[\frac{2(40 + 60)}{10} + \frac{60}{2}\right] + \frac{2 \times 40 \times 60}{2 \times 10} = 0$$

$$\omega^4 - \omega^2(20 + 30) + 240 = 0$$

$$\omega^4 - 50\omega^2 + 240 = 0$$

$$\omega^2 = \frac{50 \pm \sqrt{2500 - 960}}{2} = \frac{50 \pm 39.24}{2}$$

$$\omega_1 = \sqrt{44.62} \text{ rad/sec} = 6.68 \text{ rad/sec}$$

$$\omega_2 = \sqrt{5.38} \text{ rad/sec} = 2.32 \text{ rad/sec}$$

Example 5.9. *Solve example 5.8 using Lagrange's equation.*

Solution. $\text{K.E.} = \frac{1}{2} I\omega^2 + \frac{1}{2} m_1\dot{x}^2$

Because the K.E. of a rotating body is given as $1/2\, I\omega^2$

$$= \frac{1}{2} m_2 r^2\dot{\theta}^2 + \frac{1}{2} m_1\dot{x}^2$$

and
$$\text{P.E.} = \frac{1}{2} k_2(x - r\theta)^2 + \frac{1}{2} k_1 r^2\theta^2$$

Lagrange's Equation can be written

$$\frac{d}{dt}\frac{\partial(\text{K.E.})}{\partial \dot{x}_i} - \frac{\partial(\text{K.E.})}{\partial x_i} + \frac{\partial(\text{P.E.})}{\partial x_i} = 0$$

$$\frac{d}{dt}\frac{\partial(\text{K.E.})}{\partial \dot{x}} = m_1\ddot{x}$$

$$\frac{\partial(\text{K.E.})}{\partial x} = 0$$

$$\frac{\partial(\text{P.E.})}{\partial x} = k_2(x - r\theta)$$

So first equation of motion can be written as

$$m_1\ddot{x} + k_2(x - r\theta) = 0$$

Now

$$\frac{d}{dt}\frac{\partial(\text{K.E.})}{\partial\dot{\theta}} = m_2r^2\ddot{\theta}$$

$$\frac{\partial(\text{K.E.})}{\partial\theta} = 0$$

$$\frac{\partial(\text{P.E.})}{\partial\theta} = k_1r^2\theta + k_2r^2\theta - k_2rx$$

Second equation of motion

$$m_2r^2\ddot{\theta} + (k_1r^2 + k_2r^2)\theta - k_2rx = 0$$

Now solve like previous example.

Example 5.10. *Consider two pendulums of length L as shown in figure 5.27. Determine the natural frequency of each pendulum. If $k = 100$ N/m, $m_1 = 2$ kg, $m_2 = 5$ kg, $L = .20$ m, $a = .10$ m.* (P.U., 78)

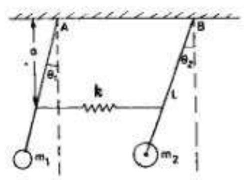


Fig. 5.27.

Solution. Let us say θ_1 and θ_2 are very small.

Taking moments about points A and B, we have

$$m_1L^2\ddot{\theta}_1 = -m_1gL\theta_1 - ka^2(\theta_1 - \theta_2)$$

$$m_2L^2\ddot{\theta}_2 = -m_2gL\theta_2 - ka^2(\theta_2 - \theta_1)$$

Let us assume the solution of the form

$$\theta_1 = \phi_1 \sin\omega t, \quad \theta_2 = \phi_2 \sin\omega t$$

Putting these values in the above equations

$$-m_1\omega^2L^2\phi_1 + m_1gL\phi_1 + ka^2(\phi_1 - \phi_2) = 0$$

$$-m_2\omega^2L^2\phi_2 + m_2gL\phi_2 + ka^2(\phi_2 - \phi_1) = 0$$

$$(-m_1\omega^2L^2 + m_1gL + ka^2)\phi_1 - ka^2\phi_2 = 0$$

$$(-m_2\omega^2L^2 + m_2gL + ka^2)\phi_2 - ka^2\phi_1 = 0$$

$$\frac{\phi_1}{\phi_2} = \frac{ka^2}{-\omega^2m_1L^2 + m_1gL + ka^2}$$

$$= \frac{-\omega^2m_2L + m_2gL + ka^2}{ka^2}$$

The frequency equation can be written as

$$(-\omega^2m_1L^2 + m_1gL + ka^2)(-\omega^2m_2L^2 + m_2gL + ka^2) - k^2a^4 = 0$$

$$\omega^4m_1m_2L^4 - \omega^2m_1m_2gL^3 - \omega^2m_1L^2ka^2 - \omega^2m_1m_2gL^3 + m_1m_2g^2L^2$$
$$+ m_1gLka^2 - \omega^2m_2L^2ka^2 + ka^2m_2gL + k^2a^4 - k^2a^4 = 0$$

$$\omega^4m_1m_2L^4 - \omega^2[m_1m_2gL^3 + m_1L^2ka^2 + m_2L^2ka^2 + m_1m_2gL^3]$$
$$+ m_1m_2g^2L^2 + m_1gLka^2 + ka^2m_2gL = 0$$

$$\omega^4 - \left[\frac{2g}{L} + \frac{ka^2}{m_2L^2} + \frac{ka^2}{m_1L^2}\right]\omega^2 + \frac{g^2}{L^2} + \frac{ka^2g}{m_2L^3} + \frac{ka^2g}{m_1L^3} = 0$$

$$\omega^4 - \omega^2\left[\frac{2 \times 9.81}{.2} + \frac{100 \times .10 \times .10}{5 \times .2 \times .2} + \frac{100 \times .1 \times .1}{2 \times .2 \times .2}\right]$$
$$+ \frac{9.81 \times 9.81}{.2 \times .2} + \frac{100 \times .1 \times .1 \times 9.81}{5 \times .2 \times .2 \times .2} + \frac{100 \times .1 \times .1 \times 9.81}{2 \times .2 \times .2 \times .2} = 0$$

$$\omega^4 - \omega^2(98.1 + 5 + 12.5) + 2405.9 + 245.25 + 613.125$$

$$\omega^4 - \omega^2(115.6) + 3264.275 = 0$$

$$\omega^2 = \frac{115.6 \pm \sqrt{13363.36 - 13057.1}}{2}$$

$$\omega^2 = \frac{115.6 \pm 17.5}{2}$$

$$\omega_1 = 8.15 \text{ rad/sec}, \quad \omega_2 = 7 \text{ rad/sec}$$

Example 5.11. *Solve example 5.10 by using Lagrange's equation.*

Solution. $\text{K.E.} = \frac{1}{2}m_1L^2\dot{\theta}_1^2 + \frac{1}{2}m_2L^2\dot{\theta}_2^2$

$$\text{P.E.} = m_1gL(1 - \cos\theta_1) + m_2gL(1 - \cos\theta_2)$$
$$+ \frac{1}{2}k(a\theta_2 - a\theta_1)^2$$

$$\frac{d}{dt}\frac{\partial(\text{K.E.})}{\partial\dot{\theta}} = m_1L^2\ddot{\theta}_1$$

$$\frac{\partial(\text{K.E.})}{\partial\theta} = 0$$

$$\frac{\partial(\text{P.E.})}{\partial\theta} = m_1gL\sin\theta_1 - ka(a\theta_2 - a\theta_1)$$

First equation of motion is

$$m_1L^2\ddot{\theta}_1 + m_1gL\sin\theta_1 - ka^2(\theta_2 - \theta_1) = 0$$

θ_1 being very small $\sin\theta_1 = \theta_1$

$$m_1L^2\ddot{\theta}_1 + m_1gL\theta_1 - ka^2(\theta_2 - \theta_1) = 0$$

Similarly, second equation of motion can be written as

$$m_2L^2\ddot{\theta} + m_2gL\theta_2 + ka^2(\theta_2 - \theta_1) = 0$$

Example 5.12. *An aerofoil wing in its first bending and torsional modes can be represented schematically as shown in figure 5.28 connected through a translational spring of stiffness k and a torsional spring of stiffness k_T. Write the equations of motion for the system and obtain the two natural frequencies. Assume the following data :*

$m = 5$ kg, $\quad I = 0.12$ kg m^2,

$k = 5 \times 10^3$ N/m, $\quad k_T = 0.4 \times 10^3$ Nm/rad,

$a = 0.1$ m

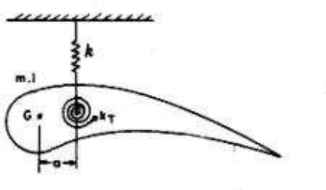


Fig. 5.28.

Solution. Let us assume that the wing is displaced by an angle θ, the equation of motions can be written as

$$m\ddot{x} = -k(x + a\theta)$$

or $$m\ddot{x} + k(x + a\theta) = 0$$

$$I\ddot{\theta} = -ak(x + a\theta) - k_T\theta$$

$$I\ddot{\theta} + ak(x + a\theta) + k_T\theta = 0$$

Let us assume the solution of the form

$$x = A\sin\omega t, \quad \ddot{x} = -\omega^2A\sin\omega t$$

$$\theta = \phi\sin\omega t, \quad \ddot{\theta} = -\omega^2\phi\sin\omega t$$

Substituting these values in the above equations

$$-\omega^2mA + k(A + a\phi) = 0$$

$$\frac{A}{\phi} = \frac{ak}{(-k + \omega^2m)}$$

and $$-\omega^2I\phi + ak(A + a\phi) + k_T\phi = 0$$

$$(-\omega^2I + a^2k + k_T)\phi + akA = 0$$

$$\frac{A}{\phi} = \frac{-\omega^2I + a^2k + k_T}{-ak}$$

Frequency equation can be written as

$$a^2k^2 + (-k + \omega^2m)(-\omega^2I + a^2k + k_T) = 0$$

$$-a^2k^2 - k\omega^2I + a^2k^2 + kk_T + \omega^4Im - \omega^2a^2mk - \omega^2mk_T = 0$$

$$\omega^4 - \omega^2\frac{(kI + a^2mk + mk_T)}{Im} + \frac{(2a^2k^2 + kk_T)}{Im} = 0$$

$$\omega^4 - \omega^2\left(\frac{k}{m} + \frac{a^2k}{I} + \frac{k_T}{I}\right) + \frac{kk_T}{Im} = 0$$

Substituting the values of various terms

$$\omega^4 - \omega^2\left(\frac{5 \times 10^3}{5} + \frac{.1 \times .1 \times 5 \times 10^3}{.12} + \frac{.4 \times 10^3}{.12}\right) + \frac{5 \times 10^6 \times .4}{.12 \times 5}$$

$$\omega^4 - 4749.99\omega^2 + 3333333.33 = 0$$

$$\omega^2 = \frac{4749.99 \pm 3037.9}{2}$$

$$\omega_1 = 62.40 \text{ rad/sec}, \quad 9.9 \text{ Hz}$$

$$\omega_2 = 29.25 \text{ rad/sec}, \quad 4.6 \text{ Hz.}$$

Example 5.13. *Solve example 5.12 by using Lagrange's equation.*

Solution. $\text{K.E.} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2$

$$\text{P.E.} = \frac{1}{2}k(x + a\theta)^2 + \frac{1}{2}k_T\theta^2$$

$$= \frac{1}{2}kx^2 + \frac{1}{2}ka^2\theta^2 + kxa\theta + \frac{1}{2}k_T\theta^2$$

Lagrange's equation

$$\frac{d}{dt}\frac{\partial(\text{K.E.})}{\partial\dot{x}} - \frac{\partial(\text{K.E.})}{\partial x} + \frac{\partial(\text{P.E.})}{\partial x} = 0$$

$$\frac{d}{dt}\frac{\partial(\text{K.E.})}{\partial \dot{x}} = m\ddot{x}$$

$$\frac{\partial(\text{K.E.})}{\partial x} = 0$$

$$\frac{\partial(\text{P.E.})}{\partial x} = kx + ka\theta$$

$$= k(x + a\theta)$$

So first equation of motion

$$m\ddot{x} + k(x + a\theta) = 0$$

And

$$\frac{d}{dt}\frac{\partial(\text{K.E.})}{\partial \dot{\theta}} = I\ddot{\theta}$$

$$\frac{\partial(\text{K.E.})}{\partial \theta} = 0$$

$$\frac{\partial(\text{P.E.})}{\partial \theta} = ka^2\theta + kxa + k_T\theta$$

Second equation of motion is

$$I\ddot{\theta} + ka^2\theta + kax + k_T\theta = 0$$

or

$$I\ddot{\theta} + k_T\theta + ka(a\theta + x) = 0$$

Example 5.14. *An electric train made of two cars each of mass 2000 kg is connected by couplings of stiffness equal to 40×10^6 N/m, as shown in figure 5.29. Determine the natural frequency of the system.*

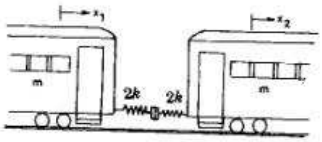


Fig. 5.29.

Solution. Equations of motion can be written as

$$m\ddot{x}_1 = k(x_2 - x_1) \quad \text{(Newton's second law of motion)}$$

$$m\ddot{x}_2 = k(x_1 - x_2)$$

Considering the solution of the form

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin \omega t$$

$$A_1(k - m\omega^2) - kA_2 = 0,$$

$$kA_1 - A_2(k - m\omega^2) = 0,$$

$$\frac{A_1}{A_2} = \frac{k - m\omega^2}{k} = \frac{k}{k - m\omega^2}$$

Frequency equation can be written as

$$k^2 - (k - m\omega^2)^2 = 0$$

$$k^2 - (k^2 + m^2\omega^4 - 2km\omega^2) = 0$$

$$k^2 - k^2 - m^2\omega^4 + 2km\omega^2 = 0$$

$$\omega^2 m(-m\omega^2 + 2k) = 0$$

$$\omega_1 = 0$$

and

$$\omega_2 = \sqrt{\frac{2k}{m}} = \sqrt{\frac{2 \times 40 \times 10^6}{2000}}$$

$$= 200 \text{ rad/sec.}$$

Example 5.15. *Two bodies having equal masses as 60 kg each and radius of gyration 0.3 m are keyed to both ends of a shaft .80 m long. The shaft is 0.08 m in diameter for 0.30 m length, .10 m diameter for 0.20 m length and .09 m diameter for rest of the length. Find the frequency of torsional vibrations.*

Take $C = 9 \times 10^{11}$ N/m² (P.U., 99)

Solution.

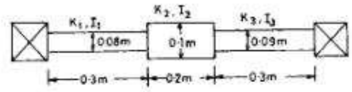


Fig. 5.30.

$$I = mk^2 = 60 \times .3 \times .3 = 5.4 \text{ kg m}^2$$

$$k_1 = \frac{GJ_1}{l_1} = \frac{9 \times 10^{11} \times (\pi/32) \times (.08)^4}{.30} = 1.2057 \times 10^7 \text{ N-m/rad}$$

$$k_2 = \frac{GJ_2}{l_2} = \frac{9 \times 10^{11} \times (\pi/32) \times (0.10)^4}{.2} = 4.415 \times 10^7 \text{ N-m/rad}$$

$$k_3 = \frac{GJ_3}{l_3} = \frac{9 \times 10^{11} \times (\pi/32) \times (.09)^4}{.3} = 1.93139 \times 10^7 \text{ N-m/rad}$$

where I = mass moment of inertia, J_1, J_2 and J_3 are polar moment of inertias.

The equivalent stiffness of the shaft is given by

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

parts of shaft are connected in series

$$= \left(\frac{1}{1.2057} + \frac{1}{4.415} + \frac{1}{1.93139}\right)10^{-7}$$

$$= (0.829 + .2265 + .51776)10^{-7}$$

$$k = 6.356 \times 10^6 \text{ N-m/rad}$$

$$\omega = \sqrt{\frac{k(I_1 + I_2)}{I_1 I_2}} = \sqrt{\frac{2k}{I}} \qquad \text{if } I_1 = I_2$$

$$= \left(\frac{2 \times 6.356 \times 10^6}{5.4}\right)^{\frac{1}{2}} = 1.534 \times 10^3 \text{ rad/sec}$$

Example 5.16. *In figure 5.31 find the natural frequencies of car with the following conditions :*

total mass of car = 300 kg

wheel base = 3.0 m

C.G. is 1.50 m from front axle

radius of gyration is 1.0 m.

Spring constants of front and rear springs are 70×10^3 N/m each.

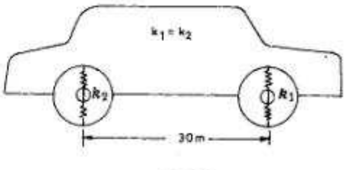


Fig. 5.31.

Solution. Given $k_1 = k_2$, $L_1 = L_2$

Writing equations of motion for the system

$$m\ddot{x} + (k_1 + k_2)x - (k_2L_2 - k_1L_1)\theta = 0$$

$$I\ddot{\theta} - (k_2L_2 - k_1L_1)x + (k_2L_2^2 + k_1L_1^2)\theta = 0$$

Since $k_1L_1 = k_2L_2$, the above equations become

$$m\ddot{x} + (k_1 + k_2)x = 0$$

and

$$I\ddot{\theta} + (k_2L_2^2 + k_1L_1^2)\theta = 0$$

or

$$I\ddot{\theta} + 2k_1L_1^2\theta = 0$$

Assuming the solution of the form

$$x = A \sin \omega t \quad ; \quad \theta = \phi \sin \omega t$$

$$\ddot{x} = -\omega^2 A \sin \omega t \; ; \; \ddot{\theta} = -\phi\omega^2 \sin \omega t$$

Substituting the values in the above equations, we get

$$-m\omega^2 A + (k_1 + k_2)A = 0$$

$$\omega^2 = \frac{k_1 + k_2}{m} = \frac{2k}{m} \quad \omega_1 = \sqrt{\frac{2k}{m}}$$

$$-\omega^2 I\phi + 2k_1L_1^2\phi = 0$$

$$\omega_2 = \sqrt{\frac{2k_1L_1^2}{I}}$$

$$\omega_1 = \sqrt{\frac{2 \times 70 \times 10^3}{300}} = 21.6 \text{ rad/sec}$$

$$I = mk^2 = 300 \times 1^2 = 300 \text{ kg-m}^2$$

$$\omega_2 = \sqrt{\frac{2 \times 70 \times 10^3 \times (1.5)^2}{300}} = 32.4 \text{ rad/sec}$$

Example 5.17. *Two pendulums of different lengths are free to rotate about y-y axis and coupled together by a rubber hose of torsional stiffness 7.35×10^3 Nm/rad as shown in figure 5.32. Determine the natural frequencies of the system if masses*

$$m_1 = 3 \text{ kg}, \; m_2 = 4 \text{ kg}, \; L_1 = .30 \text{ m}, \; L_2 = .35 \text{ m}$$

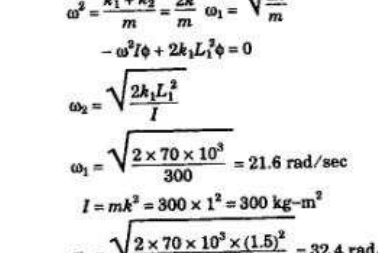


Fig. 5.32

Solution. Let pendulum m_1 and m_2 are displaced by θ_1 and θ_2

$$m_1L_1^2\,\ddot\theta_1 = -m_1gL_1\theta_1 - k_T(\theta_1-\theta_2)$$

and $$m_2L_2^2\,\ddot\theta_2 = -k_T(\theta_2-\theta_1) - m_2gL_2\theta_2$$

Let us assume the solution of the form

$$\theta_1 = \phi_1 \sin\omega t,\quad \ddot\theta_1 = -\omega^2\phi_1\sin\omega t$$

$$\theta_2 = \phi_2 \sin\omega t,\quad \ddot\theta_2 = -\omega^2\phi_2\sin\omega t$$

$$-\omega^2 m_1L_1^2\phi_1 + m_1gL_1\phi_1 + k_T(\phi_1-\phi_2) = 0$$

$$\frac{\phi_1}{\phi_2} = \frac{k_T}{-\omega^2 m_1L_1^2 + m_1gL_1 + k_T}$$

$$-\omega^2L_2^2\phi_2m_2 + k_T(\phi_2-\phi_1) + m_2gL_2\phi_2 = 0$$

$$\frac{\phi_1}{\phi_2} = \frac{-\omega^2L_2^2m_2 + m_2gL_2 + k_T}{k_T}$$

Frequency equation can be written as

$$(-\omega^2L_2^2m_2 + m_2gL_2 + k_T)(-\omega^2L_1^2m_1 + m_1gL_1 + k_T) - k_T^2 = 0$$

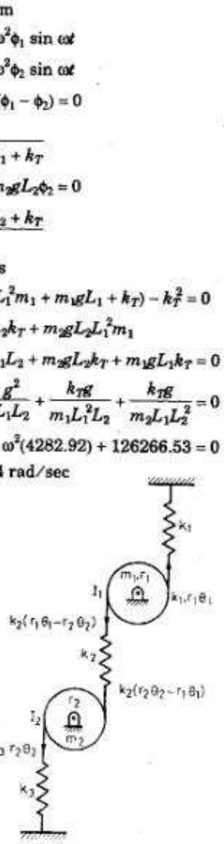
$$\omega^4 m_1m_2L_1^2L_2^2 - \omega^2(L_2^2m_2m_1gL_1 + L_2^2m_2k_T + m_2gL_2L_1^2m_1$$
$$+ k_TL_1^2m_1) + m_1m_2g^2L_1L_2 + m_2gL_2k_T + m_1gL_1k_T = 0$$

$$\omega^4 - \omega^2\left[\frac{g}{L_1} + \frac{g}{L_2} + \frac{k_T}{m_1L_1^2} + \frac{k_T}{m_2L_2^2}\right] + \frac{g^2}{L_1L_2} + \frac{k_Tg}{m_1L_1^2L_2} + \frac{k_Tg}{m_2L_1L_2^2} = 0$$

$$\omega^4 - \omega^2(4282.92) + 126266.53 = 0$$

$$\omega_1 = 65.21 \text{ rad/sec},\quad \omega_2 = 5.44 \text{ rad/sec}$$

EXAMPLE 5.18. *Determine the natural frequencies of the system as shown in the figure 5.33 if $k_1 = 40\times10^3$ N/m, $k_2 = 50\times10^3$ N/m, $k_3 = 60\times10^3$, $m_1 = 10$ kg, $m_2 = 12$ kg, $r_1 = .10$ m and $r_2 = .11$ m.*



SOLUTION. The torque equation is $\Sigma T = I\ddot\theta$.

The equations of motion can be written as

$$I_1\ddot\theta_1 = -k_1(r_1\theta_1)\,r_1 - k_2\,(r_1\theta_1 - r_2\theta_2)\,r_1$$

$$I_1\ddot\theta_1 + (k_1r_1^2 + k_2r_1^2)\theta_1 - k_2r_1r_2\theta_2 = 0$$

Similarly,

$$I_2\ddot\theta_2 = -k_3(r_2\theta_2)r_2 - k_2(r_2\theta_2 - r_1\theta_1)r_2 = 0$$

$$I_2\ddot\theta_2 + (k_3r_2^2 + k_2r_2^2)\theta_2 - k_2r_1r_2\theta_1 = 0$$

Assuming the solutions of the form

$$\theta_1 = \phi_1\sin\omega t,\quad \ddot\theta_1 = -\omega^2\phi_1\sin\omega t$$

$$\theta_2 = \phi_2\sin\omega t,\quad \ddot\theta_2 = -\omega^2\phi_2\sin\omega t$$

Substituting these values in the above equations, we get

$$-I_1\omega^2\phi_1 + (k_1r_1^2 + k_2r_1^2)\phi_1 - k_2r_1r_2\phi_2 = 0$$

$$-I_2\omega^2\phi_2 + (k_3r_2^2 + k_2r_2^2)\phi_2 - k_2r_1r_2\phi_1 = 0$$

$$\frac{\phi_1}{\phi_2} = \frac{k_2r_1r_2}{k_1r_1^2 + k_2r_1^2 - I_1\omega^2} = \frac{k_3r_2^2 + k_2r_2^2 - I_2\omega^2}{k_2r_1r_2}$$

$$(k_3r_2^2 + k_2r_2^2 - I_2\omega^2)(k_1r_1^2 + k_2r_1^2 - I_1\omega^2) - k_2^2r_1^2r_2^2 = 0$$

$$k_1k_3r_1^2r_2^2 + k_2k_3r_1^2r_2^2 - I_1\omega^2k_3r_2^2 + k_1k_2r_1^2r_2^2 + k_2^2r_1^2r_2^2$$
$$-\omega^2I_1k_2r_2^2 - I_2\omega^2k_1r_1^2 - I_2\omega^2k_2r_1^2 + I_1I_2\omega^4 = 0$$
$$-k_2^2r_1^2r_2^2 = 0$$

$$\omega^4I_1I_2 - \omega^2(I_1k_3r_2^2 + I_1k_2r_2^2 + I_2k_1r_1^2 + I_2k_2r_1^2)$$
$$+ k_1k_3r_1^2r_2^2 + k_2k_3r_1^2r_2^2 + k_1k_2r_1^2r_2^2 = 0$$

$$\omega^4 - \omega^2\left(\frac{k_3r_2^2}{I_2} + \frac{k_2r_2^2}{I_2} + \frac{k_1r_1^2}{I_1} + \frac{k_2r_1^2}{I_1}\right)$$
$$+ \frac{k_1k_3r_1^2r_2^2}{I_1I_2} + \frac{k_2k_3r_1^2r_2^2}{I_1I_2} + \frac{k_1k_2r_1^2r_2^2}{I_1I_2} = 0$$

Since $I_1 = \frac{1}{2}m_1r_1^2$ and $I_2 = \frac{1}{2}m_2r_2^2$

$$\frac{k_1k_3r_1^2r_2^2}{\frac{1}{2}m_1r_1^2\cdot\frac{1}{2}m_2r_2^2} = \frac{4k_1k_3}{m_1m_2}$$

$$\frac{k_2k_3r_1^2r_2^2}{I_1I_2} = \frac{k_2k_3r_1^2r_2^2}{\frac{1}{2}m_1r_1^2\times\frac{1}{2}m_2r_2^2} = \frac{4k_2k_3}{m_1m_2}$$

$$\frac{k_1k_3r_1^2r_2^2}{I_1I_2} = \frac{4k_1k_3}{m_1m_2}$$

$$\omega^4 - \omega^2\left(\frac{2k_3}{m_2} + \frac{2k_2}{m_2} + \frac{2k_1}{m_1} + \frac{2k_2}{m_1}\right) + \frac{4}{m_1m_2}(k_1k_3 + k_2k_3 + k_1k_2) = 0$$

This is the frequency equation.

Putting the values of various terms in the above equation, we get

$$\omega^4 - \omega^2\left(\frac{2\times60\times10^3}{12} + \frac{2\times50\times10^3}{12} + \frac{2\times40\times10^3}{10} + \frac{2\times50\times10^3}{10}\right)$$
$$+ \frac{4}{10\times12}(40\times60\times10^6 + 50\times60\times10^6 + 40\times50\times10^6) = 0$$

$$\omega^4 - 36.3333\times10^3\omega^2 + 246.66664\times10^6 = 0$$

$$\omega^2 = \frac{36.333\times10^3 \pm \sqrt{(36.333)^2\times10^6 - 4\times246.66664\times10^6}}{2}$$

$$\omega_1 = 165.2 \text{ rad/sec}$$

$$\omega_2 = 95.06 \text{ rad/sec}$$

EXAMPLE 5.19. *Derive the equations of motion for the system shown in figure 5.34 by using Lagrange's Equation.*

if $k_1 = k_2 = k_3 = 1$ and $c_1 = c_2 = c_3 = 1$ and $m_1 = m_2 = 1$.

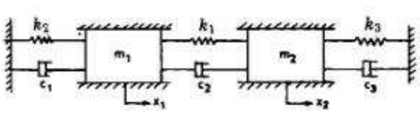


Fig. 5.34

SOLUTION. The kinetic energy of the system

$$\text{K.E.} = \frac{1}{2}m_1\dot x_1^2 + \frac{1}{2}m_2\dot x_2^2 = T$$

The potential energy

$$\text{P.E.} = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1-x_2)^2 + \frac{1}{2}k_3x_2^2 = V$$

The dissipation energy of the system

$$\text{D.E.} = \frac{1}{2}c_1\dot x_1^2 + \frac{1}{2}c_2(\dot x_1 - \dot x_2)^2 + \frac{1}{2}c_3\dot x_2^2 = D$$

Lagrange's equation for the system may be written as

$$\frac{d}{dt}\frac{\partial}{\partial\dot x_j}(\text{K.E.}) - \frac{\partial(\text{K.E.})}{\partial x_j} + \frac{\partial(\text{P.E.})}{\partial x_j} + \frac{\partial(\text{D.E.})}{\partial\dot x_j} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot x_1}\right) = m_1\ddot x_1$$

$$-\left(\frac{\partial T}{\partial x_1}\right) = 0$$

$$\frac{\partial V}{\partial x_1} = k_1x_1 + k_2(x_1 - x_2) = (k_1+k_2)x_1 - k_2x_2$$

$$\frac{\partial D}{\partial\dot x_1} = c_1\dot x_1 + c_2(\dot x_1 - \dot x_2)$$
$$= (c_1+c_2)\,\dot x_1 - c_2\dot x_2$$

Substituting the values of above terms in Lagrange's equation, we obtain the equations of motion as

$$m_1\ddot x_1 + (c_1+c_2)\dot x_1 - c_2\dot x_2 + (k_1+k_2)x_1 - k_2x_2 = 0$$

or $$\ddot x_1 + 2\dot x_2 - \dot x_2 + 2x_1 - x_2 = 0$$

It is a two degrees of freedom system, so there will be two equations of motion.

Similarly,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot x_2}\right) = m_2\ddot x_2$$

$$-\frac{\partial T}{\partial x_2} = 0$$

$$\frac{\partial V}{\partial x_2} = k_3x_2 - k_2(x_1 - x_2)$$

$$\frac{\partial D}{\partial\dot x_2} = c_3\dot x_2 - c_2(\dot x_1 - \dot x_2)$$

The second equation may be obtained by substituting the above values in Lagrange's equation as

$$m_2\ddot x_2 + (c_2+c_3)\dot x_2 + (k_2+k_3)x_2 - c_2\dot x_1 - k_2x_1 = 0$$

$$\ddot x_2 + 2\dot x_2 + 2x_2 - \dot x_1 - x_1 = 0$$

EXAMPLE 5.20. *In figure 5.35 an electrical motor-generator set is shown. Find the natural frequencies and amplitude ratios of the principal modes.*

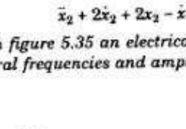


Fig. 5.35

SOLUTION. In the figure a shaft of torsional stiffness k_t carries two discs of moment of inertia I_1 and I_2. At any moment the angular displacements of discs is θ_1 and θ_2 from a certain point.

The differential equations of motion can be written as

$$I_1\ddot{\theta}_1 = k_t(\theta_2 - \theta_1)$$
$$I_2\ddot{\theta}_2 = -k_t(\theta_2 - \theta_1)$$

Rearranging the equations

$$I_1\ddot{\theta}_1 + k_t\theta_1 = k_t\theta_2$$
$$I_2\ddot{\theta}_2 + k_t\theta_2 = k_t\theta_1$$

Let us assume the solution for principal mode of vibration as

$$\theta_1 = \phi_1 \sin \omega t$$
$$\theta_2 = \phi_2 \sin \omega t$$
$$\ddot{\theta}_1 = -\omega^2\phi_1 \sin \omega t$$
$$\ddot{\theta}_2 = -\omega^2\phi_2 \sin \omega t$$

The equations of motion can be written as

$$-I_1\omega^2\phi_1 \sin \omega t + k_t\phi_1 \sin \omega t = k_t\,\phi_2 \sin \omega t$$
$$-I_2\omega^2\phi_2 \sin \omega t + k_t\phi_2 \sin \omega t = k_t\,\phi_1 \sin \omega t$$
$$(-I_1\omega^2 + k_t)\phi_1 = k_t\phi_2$$
$$(-I_2\omega^2 + k_t)\phi_2 = k_t\phi_1$$
$$\frac{\phi_1}{\phi_2} = \frac{k_t}{(k_t - \omega^2 I_1)} = \frac{k_t - \omega^2 I_2}{k_t}$$
$$k_t^2 - (k_t - \omega^2 I_1)(k_t - \omega^2 I_2)\, k_t^2 - k_t^2 - k_t\omega^2 I_2 - \omega^2 I_1 k_t + \omega^4 I_1 I_2 = 0$$
$$\omega^4 - \frac{k_t(I_1 + I_2)\,\omega^2}{I_1 I_2} = 0$$
$$\omega^2\left[\omega^2 - \frac{k_t(I_1 + I_2)}{I_1 I_2}\right] = 0$$
$$\omega_1 = 0$$

and $$\omega_2 = \sqrt{\frac{k_t(I_1 + I_2)}{I_1 I_2}}$$

The torsional stiffness of the shaft k_t can be calculated from Strength of Material formula

$$\frac{T}{I} = \frac{G\theta}{l}$$

or $$k_t = \frac{T}{\theta} = \frac{GI}{l}$$

$$\left(\frac{\phi_1}{\phi_2}\right) = +1 \qquad (\text{when } \omega_1 = 0)$$
$$\left(\frac{\phi_1}{\phi_2}\right) = -\frac{I_2}{I_1} \qquad \left(\omega_2 = \frac{k_t(I_1 + I_2)}{I_1 I_2}\right)$$

EXAMPLE 5.21. *Calculate the natural frequency of a shaft of diameter 10 cm and length 300 cm carrying two discs of diameters 125 cm and 200 cm respectively at its ends and weighing 480 kg and 900 kg respectively. Modulus of rigidity of the shaft may be taken as 2×10^6 kgf/cm².* (P.U. 93)

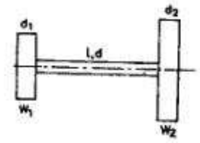


Fig. 5.36

SOLUTION. $d = 10$ cm, $l = 300$ cm

$d_1 = 125$ cm, $d_2 = 200$ cm

The system is shown in figure 5.26

$W_1 = 480$ kg, $W_2 = 900$ kg

$$I_1 = \frac{W_1}{g}\frac{r_1^2}{2} = \frac{480}{2 \times 9.81}\left(\frac{1.25}{2}\right)^2 = 9.56 \text{ kg-m}^2$$
$$I_2 = \frac{W_2}{g}\frac{r_2^2}{2} = \frac{900}{9.81}\left(\frac{2}{2}\right)^2\frac{1}{2} = 45.87 \text{ kg-m}^2$$
$$k_t = \frac{GI}{l}$$
$$I = \pi/32\, d^4 = \pi/32 \times (.10)^4 = 9.81 \times 10^{-6} \text{ m}^4$$
$$l = 300 \text{ cm}$$
$$G = 2 \times 10^6 \text{ kgf/cm}^2 = 2 \times 10^{10} \text{ kgf/m}^2$$
$$= 2 \times 9.81 \times 10^{10} \text{ N/m}^2 = 1.96 \times 10^{11} \text{ N/m}^2$$
$$k_t = \frac{GI}{l}$$
$$k_t = \frac{1.96 \times 10^{11} \times 9.81 \times 10^{-6}}{3.0}$$
$$= 6.402666 \times 10^5 \text{ N-m/rad}$$

$$\omega_n = \sqrt{\frac{k_t(I_1 + I_2)}{I_1 I_2}} = \sqrt{\frac{6.402666 \times 10^5 \times (9.56 + 45.87)}{9.56 \times 45.87}}$$
$$\omega_n = 284.48 \text{ rad/sec}$$

)r $$f_n = \frac{\omega_n}{2\pi} = \frac{284.48}{2\pi}$$
$$= 45.3 \text{ Hz or c.p.s.}$$

EXAMPLE 5.22. *A reciprocating machine weighing 25 kg running at ;000 rpm after installation has natural frequency very close to the orcing frequency of vibrating system. Design a dynamic absorber of the :earest frequency of the system which is to be at least 20% from the xcitation frequency.* (P.U., 94)

SOLUTION. Since the natural frequency is very close to the forcing requency, so 6000 rpm. may be treated as the natural frequency of the ystem

$$\omega_n = \frac{2\pi \times 6000}{60} = 628 \text{ rad/sec} = \omega_1$$

But here $\omega_1 = \omega_2$

$$\left(\frac{\omega}{\omega_2}\right)^2 = \left(1 + \frac{\mu}{2}\right) - \sqrt{\left(\mu + \frac{\mu^2}{4}\right)}$$

here μ = mass ratio

The resonant frequencies are at least 20% away from the forced equency of the main system.

So, we have

$$\omega/\omega_2 = 0.80$$
$$\omega/\omega_2 = 1.20$$

hen $\omega/\omega_2 = 0.8$, the value of $\mu = 0.2$

id for $\omega/\omega_2 = 1.2$, the value of $\mu = 0.13$

The larger value of μ is taken for design purpose.

$$\mu = 0.2 = m/M$$

So $$m = .2 \times 25 = 5 \text{ kg (weight)}$$

Using $\omega_1 = \sqrt{\dfrac{k_1 g}{W}}$ and finding $k_1 = \omega_1^2 \dfrac{W}{g}$

$$k_1 = \omega_1^2\frac{W}{g} = (628)^2 \times \frac{25}{981} = 10050.5 \text{ kg/cm}$$

d $$k_2 = \omega_1^2 \times \frac{5}{981} = 2010.1 \text{ kg/cm}$$

EXAMPLE 5.23. *The flywheel of an engine dynamo weighs 150 kg and has a radius of gyration of 25 cm. The shaft at the flywheel end has an effective length of 22 cm and is 4.5 cm in diameter. The armature weighs 90 kg and has a radius of gyration of 20 cm. The dynamo shaft has a diameter of 4 cm and an effective length of 18 cm. Neglecting the inertia of the shaft and the coupling, calculate the frequency of torsional vibrations and position of the node.*

Take $G = 0.84 \times 10^6$ kgs/cm². (Roorkee Uni. several times)

SOLUTION. Let us say suffix 1 and 2 represent flywheel and dynamo respectively.

$$I_1 = \frac{W_1}{g}k_1^2 = \frac{150}{981} \times 25^2 = 95.56 \text{ kg-cm}^2$$
$$I_2 = \frac{W_2}{g}k_2^2 = \frac{90}{981} \times 20^2 = 36.7 \text{ kg-cm}^2$$
$$k_{t_1} = \left(\frac{GI_P}{l}\right)_1$$

where I_P = Polar M.I. $= \pi/32 \times (4.5)^4$

$$G = 0.84 \times 10^6 \text{ kg/cm}^2$$
$$l = 22 \text{ cm}$$

So $$k_{t_1} = \frac{0.84 \times 10^6 \times (\pi/32) \times (4.5)^4}{22}$$
$$= 1.536 \times 10^6 \text{ kg-cm/rad}$$

Similarly,

$$k_{t_2} = \left(\frac{GI_P}{l}\right)_2 = \frac{0.84 \times 10^6 \times (\pi/32) \times (4)^4}{18}$$
$$= 1.1722 \times 10^6 \text{ kg-cm/rad}$$

The frequency ω_n is given by the expression

$$\omega_n = \sqrt{\frac{k_{t_1}k_{t_2}(I_1 + I_2)}{I_1 I_2(k_{t_1} + k_{t_2})}}$$
$$= \sqrt{\frac{1.536 \times 10^6 \times 1.1722 \times 10^6(95.96 + 36.7)}{95.96 \times 36.7 \times (1.536 + 1.1722)10^6}}$$
$$= 158.3 \text{ rad/sec}$$

The distance of node from dynamo

$$= \frac{I_2(k_{t_1} + k_{t_2})/k_{t_1}}{(I_1 + I_2)/I_1} = 46.6 \text{ cm}$$

***Example 5.24.** Two equal masses of weight 400 kg each and radius f gyration 40 cm are keyed to the opposite ends of a shaft 60 cm long. 'he shaft is 7.5 cm diameter for the first 25 cm of its length, 12.5 cm 'iameter for the next 10 cm and 8.5 cm diameter for the remaining of its ength. Find the frequency of free torsional vibrations of the system and osition of node.*

Assume $G = 0.84 \times 10^6$ kg/cm². (Roorkee Uni, AMIE)

Solution. The system is shown in figure 5.37.

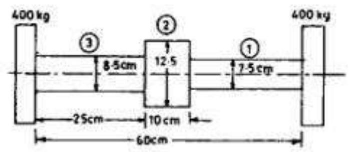


Fig. 5.37.

$$k_{t_1} = \left(\frac{GI_P}{l}\right)_1 = \frac{0.84 \times 10^6}{25} \times \pi/32 \times (7.5)^4 = 10.43 \times 10^6 \text{ kg-cm/rad}$$

$$k_{t_2} = \frac{0.84 \times 10^6}{10} \times \pi/32 \times (12.5)^4 = 10^6 \times 201.2 \text{ kg-cm/rad}$$

$$k_{t_3} = \frac{0.84 \times 10^6}{25} \times \pi/32 \times (8.5)^4 = 17.2 \times 10^6 \text{ kg-cm/rad}$$

k_{t_1}, k_{t_2}, and k_{t_3} are connected in series, so equivalent stiffness of shaft

$$\frac{1}{k_{te}} = \frac{1}{k_{t_1}} + \frac{1}{k_{t_2}} + \frac{1}{k_{t_3}}$$

$$\frac{1}{k_{te}} = 10^{-6}\left(\frac{1}{10.43} + \frac{1}{201.2} + \frac{1}{17.2}\right)$$

$$k_{te} = 6.29 \times 10^6 \text{ kg-cm/rad}$$

The expression for frequency is

$$\omega_n = \sqrt{\frac{k_e(I_1 + I_2)}{I_1 I_2}}$$

For the system $I_1 = I_2 = I$ (say)

So $$I = \frac{W}{g}k^2 = \frac{400}{981} \times (40)^2$$

$$= 652.4 \text{ kg-cm sec}^2$$

$$\omega_n = \sqrt{\frac{6.29 \times 10^6 \times 2 \times 652.4}{652.2 \times 652.4}}$$

$$= 138.86 \text{ rad/sec}$$

Since $I_1 = I_2$, so the node will lie in the middle of the equivalent shaft.

Let us find the length of equivalent shaft

$$l_e = l_1 + l_2\left(\frac{d_1}{d_2}\right)^4 + l_3\left(\frac{d_1}{d_3}\right)^4$$

$$= 25 + 10\left(\frac{7.5}{12.5}\right)^4 + 25\left(\frac{7.5}{8.5}\right)^4$$

$$= 41.45 \text{ cm}$$

The middle of equivalent shaft is 20.72 cm from the left hand side.

***Example 5.25.** Determine the frequency of the system shown in figure 5.38.*

Given

$k_1 = k_2 = 40$ N/m

$k = 60$ N/m, $m_1 = m_2 = 10$ kg

(P.U. Aero, 94)

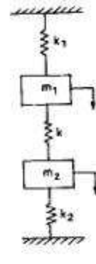


Fig. 5.38.

Solution. The equations of motion can be written as

$$m_1\ddot{x}_1 = -k_1x_1 - k(x_1 - x_2)$$

$$m_2\ddot{x}_2 = -k_2x_2 + k(x_1 - x_2)$$

Rearranging the above equations, we get

$$m_1\ddot{x}_1 + (k_1 + k)x_1 - kx_2 = 0$$

$$m_2\ddot{x}_2 - kx_1 + (k + k_2)x_2 = 0$$

Let us assume the motion of the form

$$x_1 = A_1 \sin(\omega t + \phi)$$

$$x_2 = A_2 \sin(\omega t + \phi)$$

where A_1, A_2 and ϕ are arbitrary constants. Putting these terms in the above equations, we get

$$(k_1 + k - m_1\omega^2)A_1 - kA_2 = 0$$

$$-kA_1 + (k + k_2 - m_2\omega^2)A_2 = 0$$

From these two equations frequency equation can be written as

$$\begin{bmatrix} k_1 + k - m_1\omega^2 & -k \\ -k & k + k_2 - m_2\omega^2 \end{bmatrix} = 0$$

After the expansion of determinant, we get frequency equation

$$\omega^4 - \left[\frac{k_1 + k}{m_1} + \frac{k + k_2}{m_2}\right]\omega^2 + \frac{k_1k + kk_2 + k_1k_2}{m_1m_2} = 0$$

$$\frac{k_1 + k}{m_1} = \frac{40 + 60}{10} = 10$$

$$\frac{k + k_2}{m_2} = \frac{60 + 40}{10} = 10$$

$$k_1k_2 = 40 \times 40 = 1600$$

$$k_1k = 40 \times 60 = 2400$$

$$kk_2 = 60 \times 40 = 2400$$

$$m_1m_2 = 10 \times 10 = 100$$

Substituting the values of various terms in the frequency equation, we get

$$\omega^4 - [10 + 10]\omega^2 + \frac{1600 + 2400 + 2400}{100} = 0$$

$$\omega^4 - 20\omega^2 + 64 = 0$$

$$\omega_1 = 4 \text{ rad/sec}$$

$$\omega_2 = 2 \text{ rad/sec.}$$

***Example 5.26.** Derive the equation of motion of the vibratory ystem shown in figure 5.39.*

Determine the natural frequencies for given data

$k_1 = 98000$ N/m, $m_1 = 196$ kg

$k_2 = 19600$ N/m, $m_2 = 49$ kg (P.U. 85, 89, 94)

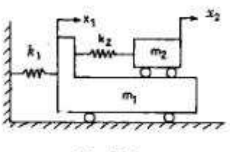


Fig. 5.39.

Solution. The equations of motion for the system shown in figure 5.39 can be written as

$$m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0$$

$$m_2\ddot{x}_2 - k_2(x_1 - x_2) = 0$$

Rearranging the above equations, we can write them as

$$m_1\ddot{x}_1 + k_1x_1 + k_2x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0$$

Let us assume the solution of the form

$$x_1 = A_1 \sin \omega t$$

and $$x_2 = A_2 \sin \omega t$$

So $$\ddot{x}_1 = -\omega^2 A_1 \sin \omega t$$

$$\ddot{x}_2 = -\omega^2 A_2 \sin \omega t$$

The above equations can be written as

$$-m_1\omega^2A_1 + k_1A_1 + k_2A_1 - k_2A_2 = 0$$

$$-m_2\omega^2A_2 + k_2A_2 - k_2A_1 = 0$$

or $$(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2 = 0$$

$$\omega^4 m_1m_2 - m_1k_2\omega^2 - k_1m_2\omega^2 + k_1k_2 - k_2m_2\omega^2 = 0$$

So $$\omega^4 - \omega^2\left(\frac{k_2}{m_2} + \frac{k_1}{m_1} + \frac{k_2}{m_1}\right) + \frac{k_1k_2}{m_1m_2} = 0$$

Substituting the values of various terms, we get

$$\omega^4 - 1000\omega^2 + 2 \times 10^5 = 0$$

Thus $\omega_1 = 26.9$ rad/sec

$\omega_2 = 16.6$ rad/sec.

***Example 5.27.** Determine the two natural frequencies and mode shapes for the system shown in figure 5.40. The string is stretched with a large tension T.* (P.U., 91)

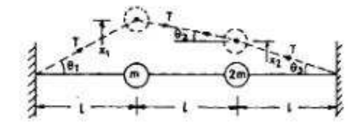


Fig. 5.40.

Solution. Assume the tension in the string as T and it does not change for small values of oscillation.

Equation of motion for left ball (mass)

$$m\ddot{x}_1 = -T\sin\theta_1 - T\sin\theta_2$$

$$= -T\frac{x_1}{l} + T\frac{(x_2 - x_1)}{l}$$

$$m\ddot{x}_1 + \frac{2Tx_1}{l} - \frac{Tx_2}{l} = 0 \qquad ...(1)$$

Similarly, equation of motion for right mass can be written as

$$2m\ddot{x}_2 = +T\sin\theta_2 - T\sin\theta_3$$

$$= -\frac{T(x_2 - x_1)}{l} - \frac{Tx_2}{l}$$

$$2m\ddot{x}_2 + \frac{2Tx_2}{l} - \frac{Tx_1}{l} = 0 \qquad ...(2)$$

Assuming the solution of equation (1) and (2) of the form

$$x_1 = A_1\sin\omega t$$

$$x_2 = A_2\sin\omega t$$

We get,

$$\left(-m\omega^2 + \frac{2T}{l}\right)A_1 - \frac{T}{l}A_2 = 0$$

$$\left(-2m\omega^2 + \frac{2T}{l}\right)A_2 - \frac{T}{l}A_1 = 0$$

$$\frac{A_1}{A_2} = \frac{\frac{T}{l}}{-m\omega^2 + \frac{2T}{l}} = \frac{-2m\omega^2 + \frac{2T}{l}}{\frac{T}{l}}$$

$$\left(-m\omega^2 + \frac{2T}{l}\right)\left(-2m\omega^2 + \frac{2T}{l}\right) - \frac{T^2}{l^2} = 0$$

$$\omega^4 - \omega^2\frac{3T}{ml} + \frac{3}{2}\frac{T^2}{m^2l^2} = 0$$

So

$$\omega^2 = \frac{T}{ml}\,\frac{3 \pm \sqrt{3}}{2}$$

$$\left(\frac{A_1}{A_2}\right)_{\omega_1} = \frac{\frac{T}{l}}{-m\omega^2 + \frac{2T}{l}}$$

$$= \frac{\frac{T}{l}}{-\frac{mT}{ml}\left(\frac{3 \pm \sqrt{3}}{2}\right) + \frac{2T}{l}}$$

$$= \frac{2}{1 + \sqrt{3}} = \frac{2(1 - \sqrt{3})}{1 - 3} = -1 + \sqrt{3}$$

Similarly, $\left(\frac{A_1}{A_2}\right)_{\omega_2} = -1 - \sqrt{3}$

Example 5.28. *Find out the two natural frequencies of vibration for the system shown in figure 5.41.*

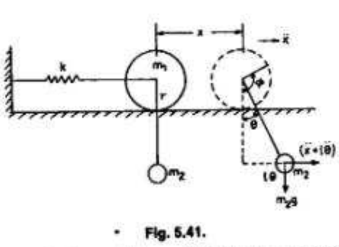


Fig. 5.41.

The light pendulum rod is provided at the centre of the roller. The spring acts through the centre of the roller. What happens to the two natural frequencies when (i) $k = 0$ (ii) $l = 0$.

Solution. Let us assume the friction force as F_f between the roller and the rough surface.

The equations of motion can be written as

$$m_1\ddot{x} = -kx + m_2g\theta + F_f$$

$$= -kx + m_2g\sin\theta + F_f$$

[$\sin\theta = \theta$ for small oscillation]

$$= -kx + m_2g\theta + F_f \qquad ...(1)$$

where F_f = friction force

For the roller motion

$$m_2(\ddot{x} + l\ddot{\theta}) = -m_2g\sin\theta \qquad ...(2)$$

$$I\ddot{\theta} = -F_f r \qquad (\sin\theta = \theta)$$

$$F_f = -\frac{I\ddot{\theta}}{r} = -\frac{\frac{1}{2}m_1r^2\ddot{\theta}}{r} = -\frac{m_1r\ddot{\theta}}{2}$$

Putting the value of friction force F_f into equation (1), we get

$$m_1\ddot{x} = -kx + m_2g\theta - \frac{m_1\ddot{x}}{2}$$

$$\frac{3}{2}m_1\ddot{x} + kx - m_2g\theta = 0 \qquad ...(3)$$

$$m_2(\ddot{x} + l\ddot{\theta}) + m_2g\theta = 0 \qquad ...(4)$$

Let us say

$$x = A\sin\omega t, \qquad \theta = B\sin\omega t$$

$$\ddot{x} = -\omega^2A\sin\omega t \qquad \ddot{\theta} = -\omega^2B\sin\omega t$$

Then equations (3) and (4) can be written as

$$-\frac{3}{2}m_1\omega^2A + kA - m_2gB = 0$$

$$\left(-\frac{3}{2}m_1\omega^2 + k\right)A - m_2gB = 0$$

$$-m_2\omega^2A - m_2l\omega^2B + m_2gB = 0$$

$$-m_2\omega^2A + (m_2g - m_2l\omega^2)B = 0$$

$$\frac{A}{B} = \frac{m_2g}{-3/2\,m_1\omega^2 + k} = \frac{g - l\omega^2}{\omega^2}$$

$$\omega^4 - \frac{\omega^2\left(\frac{3}{2}m_1g + kl + m_2g\right)}{\frac{3}{2}m_1l} + \frac{k\cdot g}{\frac{3}{2}m_1l} = 0$$

$$\omega^2 = \frac{2m_2g + 3m_1g + 2kl \pm \sqrt{(2m_2g + 3m_1g + 2kl)^2 - 24\,m_1lkg}}{6m_1l}$$

Example 5.29. *An automobile weighs 2000 kg and has a wheel base of 3.0 metres. Its centre of gravity is located 1.4 metre behind the front wheel axis and has a radius of gyration about its C.G. as 1.1 metre. The front springs have a combined stiffness of 6000 kg/cm and rear springs 6500 kg/cm. Find the principle mode of vibration of the automobile, and locate the nodal points for each mode.* (P.U., 87)

Solution. Radius of gyration k = 110 cm

$$W_1 = 2000 \text{ kg}$$

$$m = \frac{W_1}{g}$$

$$k_1 = 6000 \text{ kg/cm}$$

$$k_2 = 6500 \text{ kg/cm}$$

$$l_1 + l_2 = 300 \text{ cm}$$

or

$$l_1 = 140 \text{ cm}, \; l_2 = 160 \text{ cm}$$

$$I = mk^2 = \frac{W}{g}k^2$$

$$= \frac{2000}{981} \times (110)^2 = 24668.7$$

Using equation (5.8.10)

$$\omega_{1,2} = \frac{1}{2}\left[\frac{k_1 + k_2}{m} + \frac{k_1l_1^2 + k_2l_2^2}{I} \pm \sqrt{\left(\frac{k_1 + k_2}{m} + \frac{(k_1l_1^2 + k_2l_2^2)}{I}\right)^2 - \frac{4k_1k_2(l_1 + l_2)^2}{mI}}\right]$$

$$\frac{k_1 + k_2}{m} = \frac{(6000 + 6500)\,981}{2000} = 6131.25$$

$$\frac{k_1l_1^2 + k_2l_2^2}{I} = \frac{6000 \times 140^2 + 6500 \times 160^2}{\frac{2000}{981} \times (110 \times 110)}$$

$$= 11512.56$$

$$\frac{4k_1k_2(l_1 + l_2)^2}{MI} = \frac{4 \times 6000 \times 6500 \times (140 + 160)^2}{\frac{2000}{981} \times \frac{2000}{981} \times 110 \times 110}$$

$$= 2.79164 \times 10^8$$

Putting the values of various terms in the above frequency equation, we get

$$\omega_{1,2} = \frac{1}{2}\left[6131.25 + 11512.56 \pm \sqrt{(6131.25 + 11512.56)^2 - 2.79164 \times 10^8}\right]$$

$$= \frac{1}{2}[17643.8 \pm 5669.2]$$

$$\omega_1 = 107.96 \text{ rad/sec}$$

$$\omega_2 = 77.37 \text{ rad/sec}$$

The amplitude ratios for the two modes of vibration

$$\left(\frac{A_1}{\phi_1}\right)_{1} = \frac{k_2l_2 - k_1l_1}{k_1 + k_2 - m\omega_1^2} = \frac{6500 \times 160 - 6000 \times 140}{6000 + 6500 - \frac{2000}{981}(107.96)^2}$$

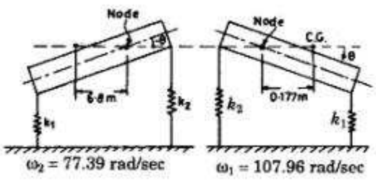
$$= -0.177 \text{ m/rad}$$

and

$$\left(\frac{A_1}{\phi_1}\right)_2 = \frac{k_2 l_2^2 + k_1 l_1^2 - I\omega_2^2}{k_2 l_2 - k_1 l_1}$$

$$= \frac{6500 \times 160^2 + 6000 \times 140^2 - \frac{2000}{981} \times 110^2 \times 77.37^2}{6500 \times 160 - 6000 \times 140}$$

$= 6.8$ m/rad



$\omega_2 = 77.39$ rad/sec $\omega_1 = 107.96$ rad/sec

Fig. 5.42

The two principal modes of vibration as shown in figure 5.42.

For the first mode, the amplitude ratio is $\left(\frac{A_1}{\phi_1}\right)_1 = -\,0.177$ m/rad

This indicates that when A_1 is positive ϕ_1 is negative from any assumed direction of rotation. This happens at frequency $\omega_1 = 107.96$ rad/sec. This means the node is 0.177 m from the C.G. of the car body. The second nodal point in 6.8 m to the right of the C.G. of the car body.

EXAMPLE 5.30. *Find the natural frequency of the system shown in figure 5.43.* (P.U., 89)

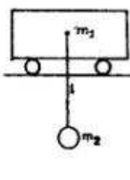


Fig. 5.43.

SOLUTION. Let us assume the whole system is moved to the right by x. The ball m_2 is displaced by θ as shown in figure 5.44. The total movement of ball m_2 is $x + l\theta$.

The equations of motion are

For pendulum,

$$m_2(\ddot{x} + l\ddot{\theta}) = -T\theta \quad (\sin\theta = \theta) \quad (T = m_2 g)$$

$$m_2(\ddot{x} + l\ddot{\theta}) + m_2 g\theta = 0$$

$$\ddot{x} + l\ddot{\theta} + g\theta = 0 \qquad ...(1)$$

For mass m_1, $m_1\ddot{x} = T\theta$

$$m_1\ddot{x} - m_2 g\theta = 0$$

or

$$\ddot{x} = \frac{m_2}{m_1} g\theta \qquad ...(2)$$

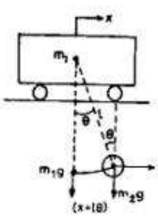


Fig. 5.44.

Putting the value of $\ddot{x}$ from equation (2) in equation (1), we get

$$\frac{m_2}{m_1} g\theta + l\ddot{\theta} + g\theta = 0$$

$$l\ddot{\theta} + \left(g + \frac{m_2 g}{m_1}\right)\theta = 0$$

$$\ddot{\theta} + \left(\frac{g}{l} + \frac{m_2 g}{m_1 l}\right)\theta = 0$$

$$\ddot{\theta} + \frac{g}{m_1 l}(m_1 + m_2)\,\theta = 0$$

So

$$\omega_n = \sqrt{\frac{g}{m_1 l}(m_1 + m_2)}$$

EXAMPLE 5.31. *Determine the natural frequency of oscillation of the double pendulum as shown in figure 5.45. Find its value when $m_1 = m_2 = 5$ kg, $l_1 = l_2 = 25$ cm.*

Given

$$m_1 = m_2 = m$$

$$l_1 = l_2 = l$$

(P.U., Aero 76)

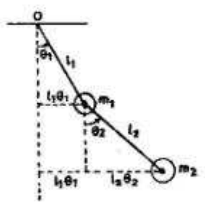


Fig. 5.45

SOLUTION. The equations of motion for two masses are given as

$$I_1\ddot{\theta}_1 = -m_1 g l_1\theta_1 - m_2 g l_1\theta_1 - m_2(l_1\ddot{\theta}_1 + l_2\ddot{\theta}_2)l_1$$

$$m_1 l_1^2\ddot{\theta}_1 = -m_1 g l_1\theta_1 - m_2 g l_1\theta_1 - m_2(l_1\ddot{\theta}_1 + l_2\ddot{\theta}_2)l_1 \qquad ...(1)$$

(when angle θ_1 is small $\sin\theta_1 = \theta_1$)

and

$$m_2 l_2^2\ddot{\theta}_2 = -m_2 g l_2\theta_2 - m_2 l_1 l_2\ddot{\theta}_1$$

$$\ddot{\theta}_2 l_2^2 + g l_2\theta_2 + l_1 l_2\ddot{\theta}_1 = 0$$

$$\ddot{\theta}_2 + \frac{l_1 l_2}{l_2^2}\ddot{\theta}_1 + \frac{g l_2\theta_2}{l_2^2} = 0$$

$$\ddot{\theta}_2 + \frac{l_1}{l_2}\ddot{\theta}_1 + \frac{g}{l_2}\theta_2 = 0 \qquad ...(2)$$

Equation (1) can be written in simplified form as

$$\ddot{\theta}_1 + \frac{m_2 l_2}{(m_1 + m_2)l_1}\ddot{\theta}_2 + \frac{g}{l_1}\theta_1 = 0 \qquad ...(3)$$

Let us assume the solution of the form

$$\theta_1 = A_1 \sin\omega t$$

$$\theta_2 = A_2 \sin\omega t$$

So

$$\ddot{\theta}_1 = -\omega^2 A_1 \sin\omega t$$

$$\ddot{\theta}_2 = -\omega^2 A_2 \sin\omega t$$

Putting these terms in equations (2) and (3), we get

$$-\omega^2 A_2 + \frac{l_1}{l_2}(-\omega^2 A_1) + \frac{g}{l_2}A_2 = 0$$

$$\left(-\omega^2 + \frac{g}{l_2}\right)A_2 - \omega^2\frac{l_1}{l_2}A_1 = 0$$

or

$$\frac{A_1}{A_2} = \frac{-\omega^2 + \frac{g}{l_2}}{\omega^2\frac{l_1}{l_2}} \qquad ...(4)$$

and

$$-\omega^2 A_1 + \frac{m_2 l_2}{(m_1 + m_2)l_1}(-\omega^2 A_2) + \frac{g}{l_1}A_1 = 0$$

$$\left(-\omega^2 + \frac{g}{l_1}\right)A_1 - \frac{m_2 l_2\omega^2}{(m_1 + m_2)l_1}A_2 = 0$$

$$\frac{A_1}{A_2} = \frac{m_2 l_2\omega^2}{(m_1 + m_2)\,l_1\left(-\omega^2 + \frac{g}{l_1}\right)} \qquad ...(5)$$

The frequency equation can be written with the help of equations (4) and (5) as

$$\frac{-\omega^2 + \frac{g}{l_2}}{\omega^2\frac{l_1}{l_2}} = \frac{m_2 l_2\omega^2}{(m_1 + m_2)\,l_1\left(-\omega^2 + \frac{g}{l_1}\right)}$$

$$m_2\omega^4 = \left(-\omega^2 + \frac{g}{l_2}\right)(m_1 + m_2)\left(-\omega^2 + \frac{g}{l_1}\right)$$

$$\omega^4 = \frac{m_1 + m_2}{m_2}\left(\omega^4 - \omega^2\frac{g}{l_1} - \omega^2\frac{g}{l_2} + \frac{g^2}{l_1 l_2}\right) = 0$$

$$\omega^4 - \frac{m_1 + m_2}{m_2}\left(\omega^4 - \omega^2\frac{g}{l_1} - \omega^2\frac{g}{l_2} + \frac{g^2}{l_1 l_2}\right) = 0$$

$$\omega^4\left(1 - \frac{m_1 + m_2}{m_2}\right) + \frac{m_1 + m_2}{m_2}\omega^2 g\left(\frac{1}{l_1} + \frac{1}{l_2}\right) - \left(\frac{m_1 + m_2}{m_2}\right)\frac{g^2}{l_1 l_2} = 0$$

$$\omega^4\frac{m_1}{m_2} - \left(\frac{m_1 + m_2}{m_2}\right)\omega^2\frac{(l_1 + l_2)}{l_1 l_2}g + \frac{(m_1 + m_2)}{m_2}\frac{g^2}{l_1 l_2} = 0$$

$$\omega^4 - \frac{(m_1 + m_2)\,\omega^2}{m_1}\frac{(l_1 + l_2)}{l_1 l_2}g + \frac{(m_1 + m_2)g^2}{m_1 l_1 l_2} = 0$$

This is the frequency equation.

$$m_1 = m_2 = 5\text{ kg}, \quad l_1 = l_2 = 25\text{ cm}$$

$$\omega^4 - \frac{2m}{m}\omega^2\frac{2l}{l^2}g + \frac{2mg^2}{ml^2} = 0$$

$$\omega^4 - 4\omega^2\frac{g}{l} + \frac{2g^2}{l^2} = 0$$

Given $l = 0.25$ m

$$\omega^4 - \frac{4 \times \omega^2 \times 9.8}{.25} + \frac{2 \times 9.8^2}{(.25)^2} = 0$$

$$\omega^4 - 156.8\,\omega^2 + 3073.28 = 0$$

$$\omega^2 = \frac{156.8 \pm \sqrt{(156.8)^2 - 4 \times 3073.28}}{2}$$

$$\omega_{1,2}^2 = \frac{156.8 \pm 110.87}{2}$$

$$\omega_1 = 11.56 \text{ rad/sec}$$

$$\omega_2 = 4.8 \text{ rad/sec}$$

Example 5.32. *Find the natural frequency and amplitude ratio of the system shown in figure 5.46.* (P.U., Aero 91)

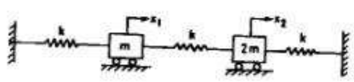


Fig. 5.46.

Solution. The equation of motion can be obtained by using Lagrange's equation.

Lagrange's equation is :

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}(\text{K.E.}) - \frac{\partial}{\partial x_i}(\text{K.E.}) + \frac{\partial}{\partial x_i}(\text{P.E.}) = 0$$

$$\text{K.E.} = T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}2m\,\dot{x}_2^2$$

$$= \frac{1}{2}m\dot{x}_1^2 + m\dot{x}_2^2$$

$$\text{P.E.} = V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_1 - x_2)^2 + \frac{1}{2}kx_2^2$$

Now $\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m\ddot{x}_1$

$$-\frac{\partial T}{\partial x_1} = 0$$

$$\frac{\partial V}{\partial x_1} = kx_1 + k(x_1 - x_2)$$

The first equation of motion is

$$m\ddot{x}_1 + kx_1 + k(x_1 - x_2) = 0$$

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \qquad \text{...(1)}$$

In the same way,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = 2m\ddot{x}_2$$

$$\frac{\partial T}{\partial x_2} = 0$$

$$\frac{\partial V}{\partial x_2} = kx_2 + k(x_1 - x_2)(-1)$$

$$= 2kx_2 - kx_1$$

The equation of motion can be written as

$$2m\ddot{x}_2 + 2kx_2 - kx_1 = 0 \qquad \text{...(2)}$$

Let us assume the motion is periodic and is composed of harmonic motion of various amplitudes and frequencies.

Say $\quad x_1 = A_1 \sin \omega t$

$$x_2 = A_2 \sin \omega t$$

So equation (1) can be written as

$$-m_1\omega^2 A_1 + 2kA_1 - kA_2 = 0$$

$$(-m\omega^2 + 2k)A_1 = kA_2$$

So $\quad \frac{A_1}{A_2} = \frac{k}{(-m\omega^2 + 2k)} = \frac{-2m\omega^2 + 2k}{k}$

$$2(-m\omega^2 + 2k)(-m\omega^2 + k) - k^2 = 0$$

$$2(m^2\omega^4 - mk\omega^2 - 2mk\omega^2 + 2k^2) - k^2 = 0$$

$$2m^2\omega^4 - 6mk\omega^2 + 4k^2 - k^2 = 0$$

$$\omega^4 - \frac{3k}{m}\omega^2 + \frac{3}{2}\left(\frac{k}{m}\right)^2 = 0$$

$$\omega^2 = \frac{\frac{3k}{m} \pm \sqrt{\left(\frac{3k}{m}\right)^2 - 4 \times \frac{3}{2}\left(\frac{k}{m}\right)^2}}{2}$$

$$= \frac{\frac{3k}{m} \pm \sqrt{\frac{9k^2}{m^2} - \frac{6k^2}{m^2}}}{2}$$

$$\omega^2 = \frac{3k}{2m} \pm \frac{k}{2m}\sqrt{3}$$

$$= 1.5\,k/m \pm .86\,k/m$$

$$\omega_1 = 1.5\sqrt{\frac{k}{m}}$$

$$\omega_2 = 0.8\sqrt{\frac{k}{m}}$$

Amplitude ratio

$$\left(\frac{A_1}{A_2}\right)_{\omega_1} = \frac{k}{-m\omega^2 + 2k} = \frac{k}{-m\left(1.5\sqrt{\frac{k}{m}}\right)^2 + 2k}$$

$$= -4$$

$$\left(\frac{A_1}{A_2}\right)_{\omega_2} = \frac{-2m\omega^2 + 2k}{k}$$

$$= \frac{-2m\left(.8\sqrt{\frac{k}{m}}\right)^2 + 2k}{k}$$

$$= \frac{-2 \times .64 + 2}{1} = 0.72$$

Example 5.33. *Find the natural frequencies and mode shapes for the torsional system shown in figure 5.47.*

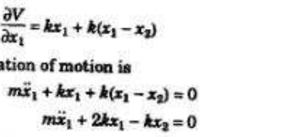


Fig. 5.47

Given $J_1 = J_0,\ J_2 = 2J_0$

and $\quad k_{t_1} = k_{t_2} = k_{t_3} = k_t$ (P.U., 77)

Solution. The equations of motion for the two discs may be written as

$$J_1\ddot{\theta}_1 = -k_{t_1}\theta_1 + k_{t_2}(\theta_2 - \theta_1)$$

$$J_1\ddot{\theta}_1 + k_t\theta_1 + k_t(\theta_1 - \theta_2) = 0$$

and $\quad J_0\ddot{\theta}_1 + 2k_t\theta_1 - k_t\theta_2 = 0 \qquad \text{...(1)}$

$$J_2\ddot{\theta}_2 = -k_{t_2}(\theta_2 - \theta_1) - k_{t_3}\theta_2$$

$$J_2\ddot{\theta}_2 + k_t(\theta_2 - \theta_1) + k_t\theta_2 = 0$$

$$J_2\ddot{\theta}_2 + k_t\theta_2 + k_t\theta_2 - k_t\theta_1 = 0$$

$$J_2\ddot{\theta}_2 + 2k_t\theta_2 - k_t\theta_1 = 0$$

$$2J_0\ddot{\theta}_2 + 2k_t\theta_2 - k_t\theta_1 = 0 \qquad \text{...(2)}$$

Let us assume the solution of equations (1) and (2) in the form

$$\theta_1 = A_1 \sin \omega t, \quad \theta_2 = A_2 \sin \omega t$$

Then equations (1) and (2) can be written as

$$-\omega^2 J_0 A_1 + 2k_t A_1 - k_t A_2 = 0$$

$$(-\omega^2 J_0 + 2k_t)A_1 - k_t A_2 = 0$$

$$(-2\omega^2 J_0 + 2k_t)A_2 - k_t A_1 = 0$$

$$\frac{A_1}{A_2} = \frac{k_t}{-\omega^2 J_0 + 2k_t} = \frac{(-2\omega^2 J_0 + 2k_t)}{k_t}$$

The frequency equation can be written as

$$2(-\omega^2 J_0 + 2k_t)(-\omega^2 J_0 + k_t) - k_t^2 = 0$$

$$2[\omega^4 J_0^2 - \omega^2 J_0 k_t - 2k_t\omega^2 J_0 + 2k_t^2] - k_t^2 = 0$$

$$2\omega^4 J_0^2 - 6k_t J_0\omega^2 + 3k_t^2 = 0$$

$$\omega^4 - \frac{3k_t}{J_0}\omega^2 + \frac{3k_t^2}{2J_0^2} = 0$$

$$\omega^2 = \frac{\frac{3k_t}{J_0} \pm \sqrt{\left(\frac{3k_t}{J_0}\right)^2 - 4 \cdot \frac{3}{2}\frac{k_t^2}{J_0^2}}}{2}$$

$$= \frac{1.5k_t}{J_0} + \frac{\sqrt{3}}{2}\frac{k_t}{J_0}$$

So $\quad \omega_1 = 1.5\sqrt{k_t/J_0}$

$$\omega_2 = .80\sqrt{k_t/J_0}$$

The amplitude ratios are given by

$$\left(\frac{A_1}{A_2}\right)_{\omega_1} = \frac{k_t}{-\omega_1^2 J_0 + 2k_t} = \frac{k_t}{-\left(1.5\sqrt{\frac{k_t}{J_0}}\right)^2 J_0 + 2k_t}$$

$$= -4$$

$$\left(\frac{A_1}{A_2}\right)_{\omega_2} = \frac{-2\omega_2^2 J_0 + 2k_t}{k_t} = \frac{-2(.8)^2\frac{k_t}{J_0}J_0 + 2k_t}{k_t}$$

$$= \frac{-2 \times .64 + 2}{1} = +0.72$$

Example 5.34. *Determine the natural frequency of torsional vibrations of a shaft with two circular discs of uniform thickness at the ends.*

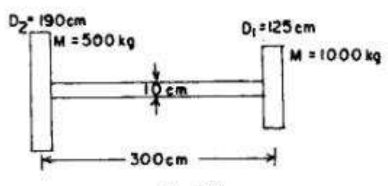


Fig. 5.48

The masses of the discs are $M_1 = 500$ kg and $M_2 = 1000$ kg and their outer diameters are $D_1 = 125$ cm and $D_2 = 190$ cm. The length of the shaft is $l = 300$ cm and its diameter $d = 10$ cm. $G = 0.83 \times 10^{11}$ N/m².

(M.D.U., 95)

SOLUTION. The system is shown in figure 5.48.

For smaller disc, $r_1 = \frac{1.25 \text{ m}}{2} = 0.625$ m

$$I_1 = \frac{1}{2} M_1 r_1^2 = \frac{1}{2} \times 500 \times (0.625)^2$$

$$= 97.65 \text{ kg-m}^2$$

For bigger disc, $r_2 = \frac{1.90}{2} = 0.95$ m

$$I_2 = \frac{1}{2} M_2 r_2^2 = \frac{1}{2} \times 1000 \times (0.95)^2 = 451.25 \text{ kg-m}^2$$

$$K_t = \frac{GI_P}{l} = \frac{0.83 \times 10^{11}}{3.00} \times \pi/32 \times (.10)^4$$

$$= 2.71 \times 10^5 \text{ N-m/rad}$$

We know that

$$\omega_n = \sqrt{\frac{K_t (I_1 + I_2)}{I_1 I_2}} = \sqrt{\frac{2.71 \times 10^5 \,(97.65 + 451.25)}{97.65 \times 451.25}}$$

$$= 58.1 \text{ rad/sec}$$

or $$f_n = \frac{\omega_n}{2\pi} = \frac{58.1}{2\pi}$$

$$= 9.25 \text{ Hz.}$$

EXAMPLE 5.35. *A reciprocating engine has a mass of 40 kg and runs at a constant speed of 3000 rpm. After it was installed it vibrated with a large amplitude at operating speed. What dynamic vibration absorber should be coupled to the system if the nearest resonant frequency of the combined system has to be at least 25% away from the operating speed.*

(P.U., 88)

SOLUTION. $$\left(\frac{\omega}{\omega_2}\right)^2 = 1 + \frac{\mu}{2} \pm \sqrt{\mu + \frac{\mu^2}{4}}$$

$$\omega = \frac{2\pi f}{60} = \frac{2\pi \times 3000}{60} = 314.15 \text{ rad/sec}$$

$$\omega_2 = 0.75\omega = 0.75 \times 314.15 = 235.6 \text{ rad/sec}$$

Using the above relation

$$\left(\frac{314.15}{235.6}\right)^2 = 1 + \frac{\mu}{2} \pm \sqrt{\mu + \frac{\mu^2}{4}}$$

or $$1.77 = 1 + \frac{\mu}{2} \pm \sqrt{\mu + \frac{\mu^2}{4}}$$

$$\left(.77 - \frac{\mu}{2}\right)^2 = \left(\mu + \frac{\mu^2}{4}\right)$$

$$1.77\,\mu = .6049$$

$$\mu = .3417$$

and we know that mass ratio

$$\mu = \frac{m_2}{m_1}$$

$$m_1 = 40 \text{ kg}$$

$$m_2 = .3417 \times 40 = 13.67 \text{ kg}$$

For vibration absorber, we have

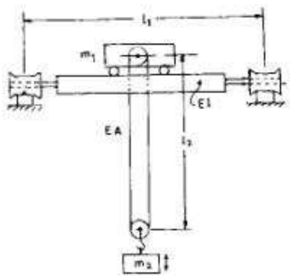
$$\omega_1 = \omega_2$$

$$\frac{k_1}{m_2} = \frac{k_2}{m_2}$$

$$(314.15)^2 = \frac{k_2}{13.67}$$

$$k_2 = 1.34 \times 10^6 \text{ N/m.}$$

EXAMPLE 5.36. *Figure 5.49 shows an overhead crane schematically. The cabin is at the centre of the beam of length l_1.* Reduce the system to



an equivalent two degrees of freedom system and determine, the natural frequencies.

Assume $EI = 21 \times 10^6$ Nm²

$m_1 = 3000$ kg

$l_1 = 5$ m

$EA = 82.47 \times 10^6$ N

$m_2 = 700$ kg

$l_2 = 6$m (U.O.R, Roorkee 84, 85)

SOLUTION. Let us say k_1 is the stiffness of beam l_1 and k_2 the stiffness of beam l_2. Then the equivalent system becomes as shown in figure 5.50.

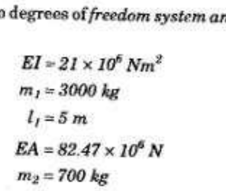


Fig. 5.50.

The stiffness k_1 is given by the expression as (for a simply supported beam)

$$k_1 = \frac{48\,EI}{l_1^3} = \frac{48 \times 21 \times 10^6}{(5)^3} = 8.064 \times 10^6 \text{ N/m}$$

Cables are subjected to axial loads, the stiffness of each cable is given as

$$k_2 = \frac{AE}{l_2} = \frac{82.47 \times 10^6}{6} = 13.745 \times 10^6 \text{ N/m}$$

The total stiffness of the two parallel cables is $2k_2$ *i.e.*,

$$2k_2 = 2 \times 13.745 \times 10^6$$

$$= 27.490 \times 10^6 \text{ N/m}$$

The equations of motion can be written as

$$m_1 \ddot{x}_1 = -k_1 x_1 - 2k_2(x_1 - x_2)$$

$$m_2 \ddot{x}_2 = 2k_2(x_1 - x_2)$$

$$m_1 \ddot{x}_1 + k_1 x_1 + 2k_2 x_1 - 2k_2 x_2 = 0$$

$$m_2 \ddot{x}_2 + 2k_2 x_2 - 2k_2 x_1 = 0$$

Finally, we get the frequency equation as

$$\omega^4 - \omega^2 \frac{(2k_2 m_1 + m_2 k_1 + 2k_2 m_2)}{m_1 m_2} + \frac{2k_2 k_1}{m_1 m_2} = 0$$

$$2k_2 m_1 = 2 \times 13.745 \times 10^6 \times 3000 = 8.247 \times 10^{10}$$

$$m_2 k_1 = 700 \times 8.064 \times 10^6 = .5644 \times 10^{10}$$

$$2k_2 m_2 = 2 \times 13.745 \times 700 \times 10^6 = 1.9243 \times 10^{10}$$

$$m_1 m_2 = 3000 \times 700 = 21 \times 10^5$$

$$\frac{2k_1 k_2}{m_1 m_2} = \frac{2 \times 8.064 \times 10^6 \times 13.745 \times 10^6}{3000 \times 700}$$

$$= 1.0556 \times 10^8$$

Putting these values in frequency equation, we get

$$\omega^4 - \omega^2 \frac{(8.247 + .5644 + 1.9243)}{21 \times 10^5} 10^{10} + 1.0556 \times 10^8 = 0$$

$$\omega^4 - .5112 \times 10^5\, \omega^2 + 1.0556 \times 10^8 = 0$$

$$\omega^2 = \frac{.5112 \times 10^5 \pm \sqrt{(.5112 \times 10^5)^2 - 4 \times 1.0556 \times 10^8}}{2}$$

$$= \frac{.5112 \times 10^5 \pm .4681 \times 10^5}{2}$$

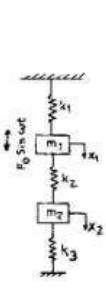
$$\omega_1 = 221.28 \text{ rad/sec}$$

$$= 35.23 \text{ Hz}$$

and $$\omega_2 = 46.42 \text{ rad/sec}$$

$$= 7.39 \text{ Hz}$$

EXAMPLE 5.37. *For the harmonically excited two degrees of freedom system shown in figure 5.51, set up differential equations of motions.* (P.U. 92)



SOLUTION. The equations of motions can be written as

$$m_1 \ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2) + F_0 \sin \omega t$$

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1) - k_3 x_2$$

or $$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = F_0 \sin \omega t$$

$$m_2 \ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 = 0$$

Let us assume the motion of the form

$$x_1 = A_1 \sin \omega t$$
$$x_2 = A_2 \sin \omega t$$
$$\ddot{x}_1 = -\omega^2 A_1 \sin \omega t$$
$$\ddot{x}_2 = -\omega^2 A_2 \sin \omega t$$

Substituting these values in the above equations, we get

$$(k_1 + k_2 - m_1\omega^2)A_1 - k_2 A_2 = F_0$$
$$-k_2 A_1 + (k_2 + k_3 - m_2\omega^2)A_2 = 0$$

Solving for A_1 and A_2 from the above two equations, we get

$$A_1 = \frac{(k_2 + k_3 - m_2\omega^2)F_0}{[m_1 m_2 \omega^4 - \{m_1(k_2 + k_3) + m_2(k_1 + k_2)\}\omega^2 + \{k_1k_2 + k_2k_3 + k_1k_3\}]}$$

$$A_2 = \frac{k_2 F_0}{[m_1 m_2 \omega^4 - \{m_1(k_2 + k_3) + m_2(k_1 + k_2)\}\omega^2 + \{k_1k_2 + k_2k_3 + k_1k_3\}]}$$

The steady state vibration can be written as

$$x_1 = A_1 \sin \omega t$$
$$x_2 = A_2 \sin \omega t$$

where A_1 and A_2 can be substituted from the above equations.

EXAMPLE 5.38. *A torsional system has an inertia of 1.5 kg-m² and a torsional stiffness of 4.36 × 10³ N-m/radian. It is acted upon by a torsional excitation of 54 rad/s. Determine the parameter of the absorber to be fixed to the main system if it is desired to keep the natural frequency atleast 20% away from the impressed frequency.*

SOLUTION. The natural frequency of the main system is

$$\omega_1 = \sqrt{\frac{K_{t_1}}{J_1}} = \sqrt{\frac{4.36 \times 10^3}{1.5}} = 53.95 \text{ rad/s} = \omega$$

For the undamped vibration absorber here, the excitation frequency is equal to the main system natural frequency of 54 rad/s.

Assuming $\omega_1 = \omega_2$, we can find the two resonant frequencies from eqn. (5.9.13) as

$$\left(\frac{\omega}{\omega_2}\right)^2 = \left(1 + \frac{\mu}{2}\right) \pm \sqrt{\left(\mu + \frac{\mu^2}{4}\right)}$$

Now the resonant frequencies are atleast 20% away from the excitation frequency. So we have,

$$\omega/\omega_2 = 0.80$$

When $(\omega/\omega_2) = 0.8$, the value of μ

$$(0.8)^2 = \left(1 + \frac{\mu}{2}\right) - \sqrt{\left(\mu + \frac{\mu^2}{4}\right)}$$
$$\mu = 0.2$$

and for $\omega/\omega_2 = 1.20$, the value of

$$\mu = 0.13$$

The larger value of μ (i.e. $\mu = 0.20$) is taken for the design purpose.

Now $\mu = \dfrac{J_2}{J_1}$

$\therefore$ $J_2 = \mu J_1 = 0.2 \times 1.5 = 0.3$ kg-m²

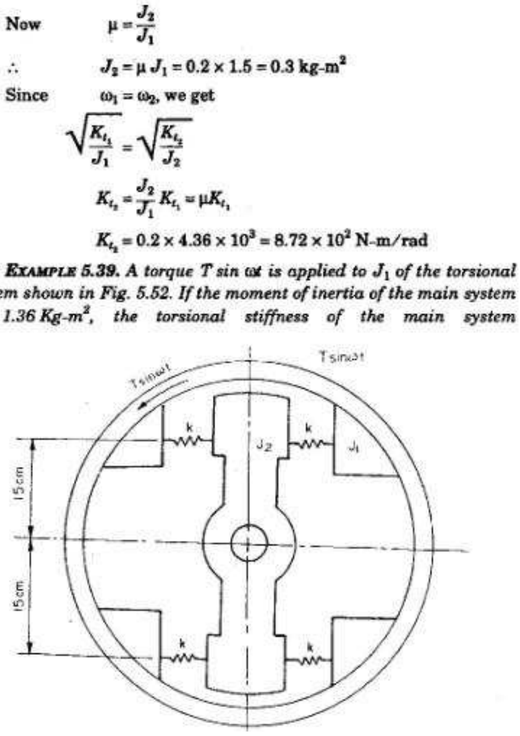
Since $\omega_1 = \omega_2$, we get

$$\sqrt{\frac{K_{t_1}}{J_1}} = \sqrt{\frac{K_{t_2}}{J_2}}$$

or $$K_{t_2} = \frac{J_2}{J_1} K_{t_1} = \mu K_{t_1}$$

or $$K_{t_2} = 0.2 \times 4.36 \times 10^3 = 8.72 \times 10^2 \text{ N-m/rad}$$

EXAMPLE 5.39. *A torque T sin ωt is applied to J_1 of the torsional system shown in Fig. 5.52. If the moment of inertia of the main system J_1 = 1.36 Kg-m², the torsional stiffness of the main system K_{t_1} = 9.826 × 10⁵ N-m/rad, T = 300 N-m and ω = 850 rad/s, specify the minimum size J_2 and K_{t_2} of the absorber. Also calculate the stiffness K of each of the 4 absorber springs such that the resonant frequencies are at least 20% from excitation frequency. What will be the amplitude of vibration of this absorber. The distance of the springs from the centre of the ring is 15 cm on both sides of the centre. Refer Fig. 5.52.*



SOLUTION. The natural frequency of the main system is

$$\omega_1 = \sqrt{\frac{K_{t_1}}{J_1}} = \sqrt{\frac{9.826 \times 10^5}{1.36}} = 850 \text{ rad/s} = \omega$$

For the undamped vibration absorber here, the excitation frequency is equal to the main system natural frequency of 850 rad/s.

Assuming $\omega_1 = \omega_2$, we can find the two resonant frequencies from eqn. (5.9.13) as

$$\left(\frac{\omega}{\omega_2}\right)^2 = \left(1 + \frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}}$$

Now the resonant frequencies are atleast 20% away from the excitation frequency. So, we here

$$\omega/\omega_2 = 0.80$$

or $$\omega/\omega_2 = 1.20$$

When $\omega/\omega_2 = 0.8$, the value of μ

$$(0.8)^2 = \left(1 + \frac{\mu}{2}\right) - \sqrt{\left(\mu + \frac{\mu^2}{4}\right)}$$
$$\mu = 0.2$$

and for $\omega/\omega_2 = 0.2$, the value of

$$\mu = 0.13$$

The larger value of μ (i.e. $\mu = 0.20$) is taken for the design purpose.

Now $\mu = \dfrac{J_2}{J_1}$ [for a torsional system]

$\therefore$ $J_2 = \mu J_1 = 0.2 \times 1.36 = 0.272$ kg-m²

Since $\omega_1 = \omega_2$, we get

$$\sqrt{\frac{K_{t_2}}{J_2}} = \sqrt{\frac{K_{t_1}}{J_1}}$$

or $$K_{t_2} = \frac{J_2}{J_1} \cdot K_{t_1} = \mu K_{t_1}$$

or $$K_{t_2} = 0.2 \times 9.826 \times 10^5$$
$$= 1.9652 \times 10^5 \text{ N-m/rad}$$

The 4 absorber springs are in parallel and so their equivalent stiffness is sum of stiffness of individual springs.

Applying eqn. (5.10.7), we get

$$K_{t_2} = 4K \times R^2$$

or $$K = \frac{K_{t_2}}{4 \times R \times R} = \frac{1.9652 \times 10^5}{4 \times 0.15 \times 0.15} = 2.1835 \times 10^6 \text{ N/m}$$

The amplitude of vibration of the absorber at an exciting frequency of 850 rad/s is given by eqn. (5.9.12) after changing the translational quantities into torsional quantities.

$\therefore$ $$T = -\beta_2 K_{t_2}$$

or $$\beta_2 = -\frac{T}{K_{t_2}} = -\frac{300}{1.9652 \times 10^5} = -1.53 \times 10^{-3} \text{ radians}$$

EXAMPLE 5.40. *For a four-cylinder engine working on four-stroke cycle, the crank throw is 100 mm. What should be the length of the equivalent pendulum suspended at a radius of 80 mm to serve as a centrifugal absorber. If the disturbing torque on the main system is 150 N-m, what size of the pendulum should be taken so that its amplitude is limited to 15° ? Consider the angular velocity of the machine as 60 rad/s. Refer Fig. 5.16 (a).*

SOLUTION. Referring Fig. 5.16 (a), we get

$$r = 100 \text{ mm}$$
$$R = 80 \text{ mm}$$

For a four cylinder engine working on four-stroke cycle, the order no. is 2

From eqn. (5.12.13), we get

$$n = \sqrt{\frac{R}{L}}$$
$$2 = \sqrt{\frac{80}{L}}$$

or $$L = 20 \text{ mm}$$

Disturbing torque = 150 N-m

Restoring torque = $Mr\omega^2 \sin \alpha \times L$

The amplitude 'θ' of pendulum is limited to 15°

From eqn. (5.12.5), we get

$$\sin \alpha = \frac{R}{r} \sin \theta = \frac{80}{100} \sin 15°$$

or $$\alpha = 12°$$

But the disturbing torque equals the restoring torque in magnitude.

$$\therefore \quad 150 = Mr\,\omega^2 \times \sin\alpha \times L$$

$$= M \times 0.1 \times (60)^2 \times \sin 12^\circ \times 0.02$$

or $\quad M = 100.2$ kg

EXAMPLE 5.41. *Two rotors A and B are attached to the end of a shaft 50 cm long. Weight of the rotor A is 300 kg and its radius of gyration is 30 cm and the corresponding values of B are 500 kg and 45 respectively. The shaft is 7 cm in diameter for the first 25 cm, 12 cm diameter for the next 10 cm and 10 cm diameter for the remainder of its length. Modulus of rigidity for the shaft material is 8×10^6 kg/cm^2.*

Find (i) the position of the node and (ii) the frequency of torsional vibration. (A.M.I.E., 1994)

SOLUTION. The configuration diagram is shown in Fig. 5.53.

$$W_A = 300 \text{ kg}, \quad W_B = 500 \text{ kg}$$

or $\quad m_A = \dfrac{W_A}{g} = \dfrac{300}{9.81}$ kg,

and $\quad m_B = \dfrac{W_B}{g} = \dfrac{500}{9.81}$ kg,

$$K_A = 30 \text{ cm}, \quad K_B = 45 \text{ cm}$$

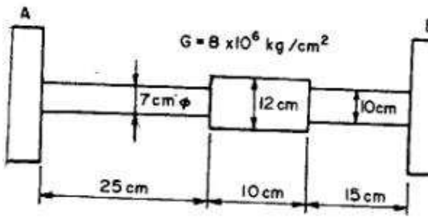


Fig. 5.53.

The shaft may be converted into a torsionally equivalent shaft the length of which is given by (Assuming $d = 7$ cm)

$$l = l_1 + l_2\left(\frac{d}{d_2}\right)^4 + l_3\left(\frac{d}{d_3}\right)^4 = 29.76 \text{ cm}$$

Let N be the position of node for the two rotor system and the length of two parts of equivalent shaft be l_A and l_B as shown in Fig. 5.54.

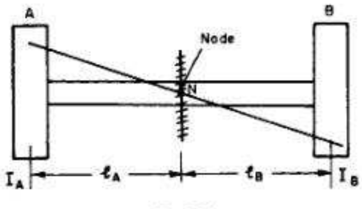


Fig. 5.54.

So, $\quad l = l_A + l_B \qquad \ldots(i)$

We know that, $\quad \omega_A = \omega_B = \sqrt{\dfrac{K_{t_A}}{I_A}} = \sqrt{\dfrac{K_{t_B}}{I_B}}$

and $\quad K_t = \dfrac{T}{\theta} = \dfrac{GJ}{l}$

(where J = Polar Moment of inertia of shaft)

$$J = \pi/32\, d^4$$

or $\quad \dfrac{G}{I_A l_A}\left(\dfrac{\pi}{32}d^4\right) = \dfrac{G}{I_B l_B}\left(\dfrac{\pi}{32}d^4\right)$

$$\frac{l_A}{l_B} = \frac{I_B}{I_A} = \frac{m_B K_B^2}{m_A K_A^2}$$

$$= \frac{500\,(0.45)^2 \times 9.81}{9.81 \times 300 \times (.30)^2}$$

or $\quad \dfrac{l_A}{l_B} = 3.75 \qquad \ldots(ii)$

From equations (*i*) and (*ii*), we get

$$l_A = 23.5 \text{ cm}, \quad l_B = 6.26 \text{ cm}$$

Now $\quad \omega_B = \sqrt{\dfrac{G}{I_B l_B}\cdot\dfrac{\pi}{32}d^4}$

$$= \sqrt{\frac{8 \times 10^6}{\frac{500}{981}(45)^2 \times 6.26} \times \frac{\pi}{32}(7)^4}$$

$$= 540.24 \text{ rad/sec}$$

Since $\quad \omega_B = \omega_A = 540.24$ rad/sec

or $\quad f_B = f_A = \dfrac{540.24}{2\pi} = 85.98$ cycles/sec

Problems

1. A governor is shown in figure 5.1 P schematically. The two links which carry the balls of mass m each are connected by a spring of stiffness k and has a natural length of $2e$. Find out the expression for the inclination of the links with vertical when the governor rotates at a speed ω.

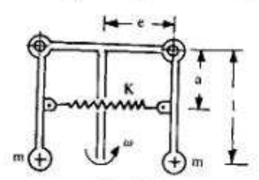


Fig. 5.1 P

2. Figure 5.2 P shows two light and rigid rods pivoted at points O_1 and O_2 respectively, and both are horizontal under the action of three springs as shown. Obtain the frequency equation.

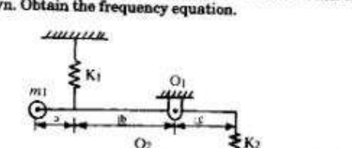


Fig. 5.2 P

Find two natural frequencies when

$$a = b = c = d = e = l; \quad f = 2\,l$$

$$m_1 = m_2 = m\;; \quad k_1 = k_2 = k_3 = k$$

3. Derive an expression for the frequency equation of the system as shown in figure 5.3 P.

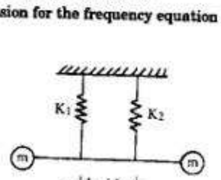


Fig. 5.3 P

4. Restraining motion of the rod in the vertical plane, find the frequencies of the system as shown in figure 5.4 P.

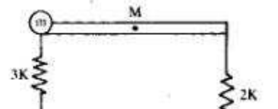


Fig. 5.4 P

5. A small reciprocating machine of 30 kg mass runs at a constant speed of 5000 rpm. After installation the forcing frequency was found to be too close to the natural frequency of the system. Design a dynamic absorber if the closest frequency of the system is to be at least 20% from the disturbing frequency.

6. Find the two natural frequencies of the system shown in figure 5.5 P.

$m_1 = 50$ tonne

$m_2 = 90$ tonne

$m_3 = 250$ tonne

$k_1 = 12$ MN/m

$k_2 = 15$ MN/m

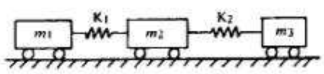
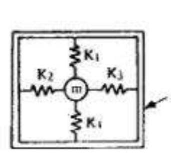


Fig. 5.5 P

7 Determine the natural frequencies of vibration of the system shown in figure 5.6 P



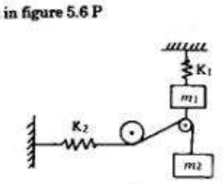
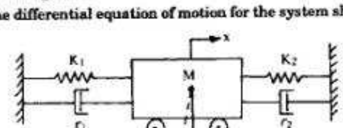


Fig. 5.6 P

Fig. 5.7 P

8. Derive the frequency equation of the system shown in figure 5.7 P. The pulleys are weightless.

9. Derive the differential equation of motion for the system shown in figure 5.8. P.



10. An airfoil of mass m and mass moment of inertia I_c about the mass centre C is put for testing in a wind tunnel. Derive the differential equations of motion. Refer figure 5.9 P.

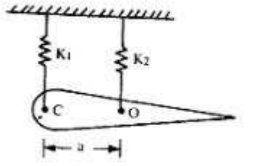


Fig. 5.9 P

11. Two rotors A and B are attached to the ends of a shaft 600 mm long. The mass of the rotor A is 400 kg and its radius of gyration is 400 mm. The corresponding values of rotor B are 500 kg and 500 mm respectively. The shaft is 80 mm diameter for the first 250 mm, 120 mm for next 150 mm length and 100 mm for the remaining length. Modulus of rigidity of the shaft material is 0.8×10^5 MN/m². Find :

(a) The position of the node.

(b) The frequency of torsional vibrations. (K.U., 98)

12. Two rotors of mass moment of inertia J_1, J_2 are connected to the ends of a shaft of length l, diameter d and modulus of rigidity G. The shaft is appropriately supported to permit the rotation of the shaft about its axis. Find the natural frequency of the free torsional vibration of the system. (U.P.S.C., 85)

13. A car model figure 5.10 P, simplified by considering its rigid body supported on rear and front springs, is considered to study vertical linear vibration and angular oscillations. Write equation of motion for the car and determine natural frequencies. Car parameters are $W = 150$ N, $L_1 = 1.35$ m, $L_2 = 1.65$ m, $K_1 = 360$N/m, $K_2 = 370$ N/m and $J_c = 27$ m⁴.

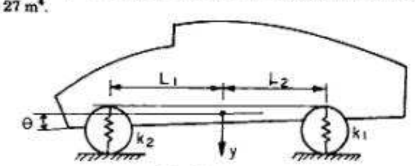


Fig. 5.10 P

(b) A jig contains a screen that reciprocates with a frequency of 600 cpm. The jig weighs 230 N and has a fundamental frequency of 400 cpm. If an absorber weighing 60 N is to be installed to eliminate the vibration of the jig frame determine the absorber spring stiffemess. What will be resulting two natural frequencies of the system ? (Roorkee Uni. 94, 95)

6

Several Degrees of Freedom System

6.1. INTRODUCTION

The systems having more than one degree of freedom are known as several or multi degrees of freedom systems. We have already discussed two degrees of freedom systems. A system must have as many equations of motion and as many natural frequencies as the number of degrees of freedom. In principle, the vibration analysis of two degrees of freedom system is not much different to that of multi degree of freedom systems except that the latter requires much more mathematical analysis. As the number of degrees of freedom increases, it becomes very tedious solving the equations of motion and to determine the natural frequencies and mode shapes. The natural frequencies and mode shapes can be determined easily and quickly with the help of computers. It will be quite relevant here to discuss some approximate methods like Rayleigh's method, Holzer's method, Dunkerley's method, Stodola method, Matrix method, Rayleigh-Ritz method, etc. In this chapter, the use of different methods to determine the natural frequencies has been made.

6.2. INFLUENCE COEFFICIENT

The equations of motion of several degrees of freedom system can be expressed in terms of influence coefficients. The influence coefficient a_{ij} is defined as the static deflection at point i because of unit load acting at point j. Similarly, a_{ji} is the deflection at point j due to unit load at point i.

According to Maxwell's Reciprocal Theorem

$$a_{ij} = a_{ji} \quad \text{...(6.2.1)}$$

For example,

$$a_{12} = a_{21}$$

$$a_{13} = a_{31} \text{ and so on.}$$

If a system made of several points is acted by several forces $F_1, F_2, F_3 ... F_n$ causing respective deflections $x_1, x_2, x_3 ... x_n$, it can be expressed mathematically as

$$x_1 = a_{11}F_1 + a_{12}F_2 + a_{13}F_3 + ...a_{1n}F_n$$

$$x_2 = a_{21}F_1 + a_{22}F_2 + a_{23}F_3 + ...a_{2n}F_n$$

The general expression can be written as

$$x_n = a_{n1}F_1 + a_{n2}F_2 + ... a_{nn}F_n \quad \text{...(6.2.2)}$$

For any principal mode of vibration, the amplitude of various masses $m_1, m_2 ..... m_n$ can be expressed as

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin \omega t$$

$$\vdots$$

$$x_n = A_n \sin \omega t \quad \text{...(6.2.3)}$$

But the inertia force F_1 at any point can be written as

$$F_1 = -m_1\ddot{x}_1 = m_1 \omega^2 A_1 \sin \omega t$$

$$F_2 = m_2 \omega^2 A_2 \sin \omega t \text{ and so on} \quad \text{...(6.2.4)}$$

Substituting the values of forces from equation (6.2.4) into equation (6.2.2), we get

$$A_1 \sin \omega t = a_{11}m_1 \omega^2 A_1 \sin \omega t + a_{12}m_2 \omega^2 A_2 \sin \omega t - ..... + a_{1n}m_n \omega^2 A_n \sin \omega t$$

$$A_2 \sin \omega t = a_{21}m_1 \omega^2 A_1 \sin \omega t + a_{22}m_2 \omega^2 A_2 \sin \omega t - ..... + a_{2n}m_n \omega^2 A_n \sin \omega t$$

$$A_3 \sin \omega t = a_{31}m_1 \omega^2 A_1 \sin \omega t + a_{32}m_2 \omega^2 A_2 \sin \omega t - ..... + a_{3n}m_n \omega^2 A_n \sin \omega t \quad \text{...(6.2.5)}$$

Rearranging and cancelling out the common term $\sin \omega t$, we get

$$(a_{11}m_1 \omega^2 - 1)A_1 + a_{12}m_2 \omega^2 A_2 + ...a_{1n}m_n \omega^2 A_n = 0$$

$$a_{21}m_1 \omega^2 A_1 + (a_{22}m_2 \omega^2 - 1)A_2 + ...a_{2n}m_n \omega^2 A_n = 0$$

General expression can be written as

$$a_{n1}m_1 \omega^2 A_1 + a_{n2}m_2 \omega^2 A_2 + ...(a_{nn}m_n \omega^2 - 1)A_n = 0 \quad \text{...(6.2.6)}$$

If the determinant of the above equations is equated to zero, it gives the frequency equation.

Equation (6.2.2) may be written in matrix form as

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{Bmatrix}$$

In short form, the above equation may be written as

$$\{x\} = [a] \{F\} \quad \text{...(6.2.7)}$$

here [a] = Flexibility Matrix

The forces can also be written as

$$F_1 = k_{11}x_1 + k_{12}x_2 + k_{13}x_3$$

$$F_2 = k_{21}x_1 + k_{22}x_2 + k_{23}x_3$$

$$F_3 = k_{31}x_1 + k_{32}x_2 + k_{33}x_3 \text{ and so on.} \quad \text{...(6.2.8)}$$

The above equation may be written as

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad \text{...(6.2.9)}$$

or $\{F\} = [k]\{x\}$

where $[k]$ = stiffness matrix

Then equation (6.2.7) can be written as

$$\{x\} = [a][k]\{x\} \quad (6.2.10)$$

or $[a] = [k]^{-1}$

Thus $[a][k] = [I]$

where $[I]$ = unit matrix.

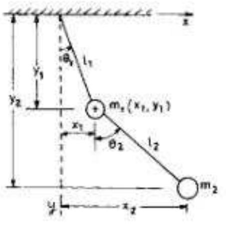


Fig. 6.1. Double pendulum.

EXAMPLE. *Prove Maxwell's reciprocal theorem $a_{12} = a_{21}$ for the simply supported beam with concentrated loads acting as shown in figure 6.1(A).* (M.D.U., 90)

Fig. 6.1(A)

SOLUTION. The work done will be obtained when the system is deformed by the application of force. First of all, force W_1 is applied at point 1 gradually from zero to its full value. The deflection at point 1 is $a_{11}W_1$.

The work done at point $1 = \frac{1}{2}W_1(a_{11}W_1)$

$$= \frac{1}{2}W_1^2\, a_{11}$$

Similarly, when force W_2 is gradually applied at point 2, the work done will be

$$= 1/2\ W_2^2\, a_{22}$$

But when force W_2 is applied the additional deflection at point 1 due to force W_2 is $a_{12}W_2$. But already a force W_1 is acting at point 1. The work done by force W_1 corresponding to deflection $a_{12}W_2$ at point 1 will be $W_1W_2\, a_{12}$. So total work done in the first mode

$$(W.D.)_1 = \frac{1}{2}\, W_1^2 a_{11} + \frac{1}{2}\, W_2^2 a_{22} + W_1W_2 a_{12}$$

Similarly, the work done in the second mode can be written as

$$(W.D.)_2 = \frac{1}{2}\, W_2^2 a_{22} + \frac{1}{2}\, W_1^2 a_{11} + W_1W_2 a_{21}$$

From the above two equations, it is clear that $a_{12} = a_{21}$.

5.3. GENERALIZED COORDINATES

The configuration of a system is completely specified by certain ndependent parameters or coordinates which are known as generalzed coordinates. These parameters specify the motion of a system :ompletely. If a system has n degrees of freedom it will have n generalzed coordinates. The generalized coordinates are generally denoted by $q_1, q_2, q_3 \ldots q_n$.

We consider double pendulum in figure 6.1 where

$$x_1 = l_1 \sin\theta_1 \quad x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

$$y_1 = l_1 \cos\theta_1 \quad y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$$

Here x_1, y_1, x_2 and y_2 are not the generalized coordinates. Only θ_1 nd θ_2 are the independent coordinates which specify the system comdetely. These are the generalized coordinates.

So $\quad \theta_1 = q_1$ and $\theta_2 = q_2$.

.4. MATRIX METHOD

This method is very convenient way to solve the equations of otion. The lowest natural frequency of the system can be determined

very quickly by this method. Matrix method is very important to analyse as it is the basis of many computer solutions.

The procedure is explained with the help of following solved example. Consider figure 6.2 for analysis.

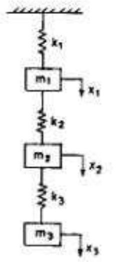


Fig. 6.2 Spring-mass system.

The equations of motion can be written as

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2)$$

$$m_2\ddot{x}_2 = k_2(x_1 - x_2) - k_3(x_2 - x_3)$$

$$m_3\ddot{x}_3 = k_3(x_2 - x_3)$$

Rearranging the above equations, we get

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 - k_2x_1 + (k_3 + k_2)x_2 - k_3x_3 = 0$$

$$m_3\ddot{x}_3 - k_3x_2 + k_3x_3 = 0 \quad \ldots(6.4.1)$$

Equation (6.4.1) may be written in matrix form as

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_1 & k_3+k_2 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0 \quad \ldots(6.4.2)$$

The above equations can be written in matrix form as

$$[m]\{\ddot{x}\} + [k]\{x\} = 0 \quad \ldots(6.4.3)$$

where $[m]$ = mass matrix

and $\quad [k]$ = stiffness matrix

or

$$\{\ddot{x}\} + [m]^{-1}[k]\{x\} = 0$$

$$\{\ddot{x}\} + [c]\{x\} = 0 \quad \ldots(6.4.4)$$

where $\quad [c] = [m]^{-1}[k]$ = Dynamic matrix

and $\quad [m]^{-1} = \dfrac{adj\ m}{|\,m\,|}$

For harmonic oscillations at frequency ω, $\{\ddot{x}\} = -\,\omega^2\{x\}$, so equation (6.4.4) reduces to $\quad [c]\{x\} - \omega^2\{x\} = 0$

If $\omega^2 = \lambda$ say, then the above equation may be written as

$$[c]\{x\} - \lambda\,\{x\} = 0$$

or

$$[\lambda I - c]\ \{x\} = 0 \quad \ldots(6.4.5)$$

where $\quad [I]$ = identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The solution of equation (6.4.5) may be obtained by

$$|\,\lambda I - c\,| = 0$$

For simplicity of expression, let us say that

$$m_1 = m_2 = m_3 = m$$

$$k_1 = k_2 = k_3 = k$$

Again equation (6.4.2) may be written as

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0 \quad \ldots(6.4.6)$$

$$|\,m\,| = m^3$$

$$m^{-1} = \frac{adj\ m}{|\,m\,|} = \frac{1}{m^3}\begin{bmatrix} m^2 & 0 & 0 \\ 0 & m^2 & 0 \\ 0 & 0 & m^2 \end{bmatrix}$$

$[c] = [m]^{-1}[k]$ (Using equation 6.4.6 for $[k]$)

$$= \frac{1}{m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

$$[c] = \frac{k}{m}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$|\,\lambda I - c\,| = 0$$

$$\begin{vmatrix} (\lambda - 2k/m) & +k/m & 0 \\ +k/m & \lambda - 2k/m & +k/m \\ 0 & +k/m & \lambda - k/m \end{vmatrix} = 0$$

Solving it, we get

$$\lambda^3 - 5\,\lambda^2\,\frac{k}{m} + 6\,\lambda\,\frac{k^2}{m^2} - \frac{k^3}{m^3} = 0$$

From the above equation, we obtain

$$\lambda_1 = 0.198\ k/m = \omega_1^2$$

r

$$\omega_1 = \sqrt{0.198\ k/m} = .44\sqrt{\frac{k}{m}}\ \text{rad/sec}$$

$$\omega_2 = \sqrt{1.555\ k/m} = 1.24\sqrt{\frac{k}{m}}\ \text{rad/sec}$$

$$\omega_3 = \sqrt{3.247\ k/m} = 1.80\sqrt{\frac{k}{m}}\ \text{rad/sec}$$

6.5. ORTHOGONALITY PRINCIPLE

For a system with three-degree of freedom the orthogonality principle may be written as

$$\left.\begin{aligned} m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0 \\ m_1A_2A_3 + m_2B_2B_3 + m_3C_2C_3 = 0 \\ m_1A_1A_3 + m_2B_1B_3 + m_3C_1C_3 = 0 \end{aligned}\right\} \quad \ldots(6.5.1)$$

where m_1, m_2, m_3 are masses.

$A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$ and C_3 are the amplitudes of vibration of the system. We will make use of equation (6.5.1) in matrix iteration method to find the natural frequencies and mode shapes of the system.

6.6. MATRIX ITERATION METHOD

With the help of this method the natural frequencies and corresponding mode shapes are determined. Use of influence coefficients is made in the analysis. The method can best be understood by solving the problem of figure 6.3 by matrix iteration method. (P.U. 85)

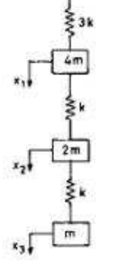


Fig. 6.3.

SOLUTION. The equations for the above system in terms of influence coefficients can be written as

$$x_1 = a_{11}4mx_1\,\omega^2 + a_{12}2mx_2\,\omega^2 + a_{13}mx_3\,\omega^2$$

$$x_2 = a_{21}4mx_1\,\omega^2 + a_{22}2mx_2\,\omega^2 + a_{23}mx_3\,\omega^2$$

$$x_3 = a_{31}4mx_1\omega^2 + a_{32}2mx_2\,\omega^2 + a_{33}mx_3\,\omega^2$$

Influence coefficients are

$$a_{11} = a_{12} = a_{13} = a_{31} = a_{21} = \frac{1}{3k}$$

$$a_{22} = \frac{1}{3k} + \frac{1}{k} = \frac{4}{3k} = a_{23} = a_{32}$$

$$a_{33} = \frac{1}{3k} + \frac{1}{k} + \frac{1}{k} = \frac{7}{3k}$$

Substituting these terms into above equations, we get

$$x_1 = \frac{4m}{3k}\,x_1\,\omega^2 + \frac{2m}{3k}\,x_2\,\omega^2 + \frac{m}{3k}\,x_3\,\omega^2$$

$$x_2 = \frac{4m}{3k}\,x_1\,\omega^2 + \frac{8m}{3k}\,x_2\,\omega^2 + \frac{4m}{3k}\,x_3\,\omega^2$$

$$x_3 = \frac{4m}{3k}\,x_1\,\omega^2 + \frac{8m}{3k}\,x_2\,\omega^2 + \frac{7m}{3k}\,x_3\,\omega^2$$

This can be written in matrix form as

$$\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \frac{\omega^2 m}{3k}\begin{bmatrix}4&2&1\\4&8&4\\4&8&7\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix}$$

To start with the iteration process, let us assume

$$\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \begin{Bmatrix}1\\2\\3\end{Bmatrix}$$

First iteration

$$\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \frac{\omega^2 m}{3k}\begin{bmatrix}4&2&1\\4&8&4\\4&8&7\end{bmatrix}\begin{Bmatrix}1\\2\\3\end{Bmatrix}$$

$$= \frac{\omega^2 m}{3k}\begin{bmatrix}11\\32\\41\end{bmatrix} = \frac{\omega^2 m}{3k}(11)\begin{bmatrix}1\\2.91\\3.73\end{bmatrix}$$

Now assume

$$\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \begin{Bmatrix}1\\2.91\\3.73\end{Bmatrix}$$

Second iteration

$$\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \frac{\omega^2 m}{3k}\begin{bmatrix}4&2&1\\4&8&4\\4&8&7\end{bmatrix}\begin{Bmatrix}1\\2.91\\3.73\end{Bmatrix}$$

$$= \frac{\omega^2 m}{3k}\begin{bmatrix}13.55\\42.2\\53.39\end{bmatrix} = \frac{\omega^2 m}{3k}(13.55)\begin{bmatrix}1\\3.11\\3.94\end{bmatrix}$$

Third iteration

$$\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \frac{\omega^2 m}{3k}\begin{bmatrix}4&2&1\\4&8&4\\4&8&7\end{bmatrix}\begin{Bmatrix}1\\3.11\\3.94\end{Bmatrix} = \frac{\omega^2 m}{3k}\begin{bmatrix}14.16\\44.64\\56.46\end{bmatrix}$$

$$= \frac{\omega^2 m}{3k}(14.16)\begin{bmatrix}1\\3.15\\3.98\end{bmatrix}$$

Fourth iteration

$$\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \frac{\omega^2 m}{3k}\begin{bmatrix}4&2&1\\4&8&4\\4&8&7\end{bmatrix}\begin{Bmatrix}1\\3.15\\3.98\end{Bmatrix}$$

$$= \frac{\omega^2 m}{3k}\begin{bmatrix}14.28\\45.12\\57.06\end{bmatrix} = \frac{\omega^2 m}{3k}(14.28)\begin{bmatrix}1\\3.15\\3.99\end{bmatrix}$$

The ratio obtained in the fourth iteration is very close to the initial value, so

$$\begin{Bmatrix}1\\3.15\\3.99\end{Bmatrix} = \frac{m\omega^2}{3k}(14.28)\begin{Bmatrix}1\\3.15\\3.99\end{Bmatrix}$$

or

$$\frac{14.28}{3k}m\omega^2 = 1$$

$$\omega^2 = \frac{3k}{14.28\,m}$$

$$\omega_1 = \sqrt{\frac{3}{14.28}\frac{k}{m}} = .458\sqrt{\frac{k}{m}}\ \text{rad/sec}$$

To find the second principal mode, the orthogonality principle is used as

$$\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \frac{\omega^2 m}{3k}\begin{bmatrix}4&2&1\\4&8&4\\4&8&7\end{bmatrix}\begin{bmatrix}0 & -\frac{m_2}{m_1}\left(\frac{x_2}{x_1}\right) & -\frac{m_3}{m_1}\left(\frac{x_3}{x_1}\right)\\ 0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix}$$

$$= \frac{\omega^2 m}{3k}\begin{bmatrix}4&2&1\\4&8&4\\4&8&7\end{bmatrix}\begin{bmatrix}0 & -\frac{2}{4}\left(\frac{3.15}{1}\right) & -\frac{1}{4}\left(\frac{3.99}{1}\right)\\ 0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix}$$

$$\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4&2&1\\4&8&4\\4&8&7\end{bmatrix}\begin{bmatrix}0&-1.58&-1\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix}$$

$$\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}0&-4.32&-3.0\\0&1.67&0\\0&1.67&3.0\end{bmatrix}\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix}$$

Now assume

$$\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \begin{Bmatrix}1\\1\\1\end{Bmatrix}$$

First iteration

$$\begin{Bmatrix}1\\1\\1\end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}0&-4.32&-3\\0&1.67&0\\0&1.67&3\end{bmatrix}\begin{Bmatrix}1\\1\\1\end{Bmatrix}$$

$$= \frac{m\omega^2}{3k}\begin{bmatrix}-7.32\\1.67\\4.67\end{bmatrix} = \frac{m\omega^2}{3k}(7.32)\begin{bmatrix}-1\\.23\\.64\end{bmatrix}$$

Second iteration

$$\begin{Bmatrix}-1.0\\0.23\\0.64\end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}0&-4.32&-3\\0&1.67&0\\0&1.67&3\end{bmatrix}\begin{Bmatrix}-1.0\\.23\\.64\end{Bmatrix}$$

$$= \frac{m\omega^2}{3k}\begin{bmatrix}-0.3036\\0.3841\\2.3041\end{bmatrix}$$

$$= \frac{m\omega^2}{3k}(2.3)\begin{bmatrix}-0.132\\0.167\\1.0\end{bmatrix}$$

Third iteration

$$\begin{Bmatrix}-0.132\\0.167\\1.0\end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}0&-4.32&-3\\0&1.67&0\\0&1.67&3\end{bmatrix}\begin{Bmatrix}-0.132\\0.167\\1.0\end{Bmatrix}$$

$$= \frac{m\omega^2}{3k}\begin{bmatrix}-3.72\\0.28\\3.28\end{bmatrix} = \frac{m\omega^2}{3k}(3.72)\begin{bmatrix}-1.0\\0.07\\0.88\end{bmatrix}$$

Fourth iteration

$$\begin{Bmatrix}-1.0\\0.07\\0.88\end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}0&-4.32&-3\\0&1.67&0\\0&1.67&3\end{bmatrix}\begin{Bmatrix}-1.0\\0.07\\0.88\end{Bmatrix}$$

$$= \frac{m\omega^2}{3k}\begin{bmatrix}-2.94\\0.12\\2.76\end{bmatrix} = \frac{m\omega^2}{3k}(2.94)\begin{Bmatrix}-1.0\\0.04\\0.94\end{Bmatrix}$$

The value is converging to $\begin{Bmatrix}-1\\0\\1\end{Bmatrix}$

Hence $\frac{m\omega^2}{3k}\times 2.94 = 1$ or $\omega_2 = \sqrt{\frac{k}{m}}$ rad/sec

To obtain the third mode, let us use orthogonality principle as

$$m_1A_2A_3 + m_2B_2B_3 + m_3C_2C_3 = 0$$

$$m_1A_1A_3 + m_2B_1B_3 + m_3C_1C_3 = 0$$

Substituting $\begin{Bmatrix}A_1\\B_1\\C_1\end{Bmatrix} = \begin{Bmatrix}x_1\\x_2\\x_2\end{Bmatrix} = \begin{Bmatrix}1\\3.15\\3.99\end{Bmatrix}$

$$A_1 = 1,\ B_1 = 3.15,\ C_1 = 3.99$$

and

$$A_2 = -1.0,\ B_2 = 0,\ C_2 = 1.0$$

So $\quad 4m(-1)A_3 + 2m(0)B_3 + m(1)C_3 = 0$

$$4m(1)A_3 + 2m(3.15)B_3 + m(3.99)C_3 = 0$$

Solving for B_3 and C_3

$$2\times 3.15B_3 + 4.99\,C_3 = 0$$

$$6.3\,B_3 + 4.99\,C_3 = 0$$

$$B_3 = \frac{-4.99}{6.3}C_3 = -.79\,C_3$$

Solving for A_3 and C_3

$$4A_3 + 2\times 3.15\times -.79C_3 = -3.99\,C_3$$

$$A_3 = .246\,C_3$$

$$\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \begin{bmatrix}0&0&.246\\0&0&-.79\\0&0&1.0\end{bmatrix}\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix}$$

So, using sweeping matrix, we have

$$\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}0&-4.32&-3.0\\0&1.67&0\\0&1.67&3.0\end{bmatrix}\begin{bmatrix}0&0&.246\\0&0&-.79\\0&0&1.0\end{bmatrix}\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix}$$

$$= \frac{m\omega^2}{3k}\begin{bmatrix}0&0&.413\\0&0&-1.32\\0&0&1.68\end{bmatrix}\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix}$$

First iteration,

Assuming $\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \begin{Bmatrix}1\\2\\3\end{Bmatrix}$

$$= \frac{m\omega^2}{3k}\begin{bmatrix}0&0&.413\\0&0&-1.32\\0&0&1.68\end{bmatrix}\begin{Bmatrix}1\\2\\3\end{Bmatrix}$$

$$= \frac{m\omega^2}{3k}\begin{bmatrix}.413\times 3\\-1.32\times 3\\1.68\times 3\end{bmatrix} = \frac{m\omega^2}{3k}(1.239)\begin{bmatrix}1\\-3.19\\4.06\end{bmatrix}$$

Second iteration

$$= \frac{m\omega^2}{3k}\begin{bmatrix}0&0&.413\\0&0&-1.32\\0&0&1.68\end{bmatrix}\begin{Bmatrix}1\\-3.19\\4.06\end{Bmatrix}$$

$$= \frac{m\omega^2}{3k}(1.676)\begin{bmatrix}1\\-3.19\\4.06\end{bmatrix}$$

Hence the third mode is $\begin{Bmatrix} 1 \\ -3.19 \\ 4.06 \end{Bmatrix}$ and the natural frequency of it

$$1 = \frac{m}{3k}\omega^2\, 1.676$$

$$\omega^2 = \frac{3k}{1.676\,m}, \quad \omega_3 = \sqrt{\frac{3k}{1.676\,m}} = 1.34\sqrt{\frac{k}{m}} \text{ rad/sec.}$$

Thus

First mode is $\begin{Bmatrix} 1 \\ 3.15 \\ 3.99 \end{Bmatrix}$, $\omega_1 = 0.458\sqrt{\frac{k}{m}}$ rad/sec

Second mode is $\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$, $\omega_2 = \sqrt{\frac{k}{m}}$ rad/sec

Third mode is $\begin{Bmatrix} 1 \\ -3.19 \\ 4.06 \end{Bmatrix}$, $\omega_3 = 1.34\sqrt{\frac{k}{m}}$ rad/sec

6.7. DUNKERLEY'S METHOD

Natural frequencies of structures are evaluated by this method. This method is used to find the natural frequency of transverse vibrations. The load of the system is uniformly distributed. Dunkerley's equation can be written as

$$\frac{1}{\omega^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \dots \frac{1}{\omega_s^2} \qquad \text{...(6.7.1)}$$

where ω = natural frequency of transverse vibration of shaft for many point loads. $\omega_1, \omega_2, \omega_3$, etc = natural frequency of individual point loads.

ω_s = natural frequency of transverse vibration because of the weight of shaft.

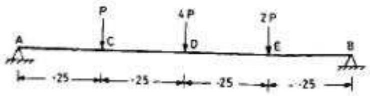


Fig. 6.4.

This approach can best be understood with the help of figure shown in figure 6.4. Let us neglect the weight of the beam AB. Say the natural frequency of the system is ω and corresponding to three loads being ω_1, ω_2 and ω_3.

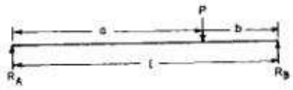


Fig. 6.5.

So $\quad \frac{1}{\omega^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2}$

If the beam is loaded as shown in figure 6.5, then

$$R_A = \frac{P.b}{l},\; R_B = \frac{P.a}{l}$$

To determine the deflection of the beam at a distance x from left end, A.

$$EI\frac{d^2y}{dx^2} = -\frac{Pb}{l}x + P(x-a) \qquad \text{(from strength of materials)}$$

or $$EIy = -\frac{Pbx^3}{6l} + C_1x + C_2 + \frac{P(x-a)^3}{6}$$

Applying boundary conditions as

$y = 0$ at $x = 0$

and $y = 0$ at $x = l$

So $C_2 = 0$

$$C_1 = \frac{Pbl}{6} - \frac{P(l-a)^3}{6l}$$

So $$y = \frac{1}{EI}\left[-\frac{Pbx^3}{6} + x\left(\frac{Pbl}{6} - \frac{(l-a)^3}{6l}\right)\right] \qquad (\text{at } x = a)$$

$$= \frac{Pbx}{6EIl}(l^2 - x^2 - b^2) \qquad (0 < x < a)$$

Deflection at point C, y_1

$$y_1 = \frac{P \times .75 \times .25}{6EI \times 1}\left[1 - .25^2 - .75^2\right] \qquad b = .75 \text{ m},\; x = .25 \text{ m}$$

$$= \frac{P(.03125)}{EI}(1 - .0625 - .5625)$$

$$= \frac{P(.0117187)}{EI} \text{ m}$$

Deflection at point D, y_2

$$y_2 = \frac{Pbx}{6EIl}(l^2 - x^2 - b^2)$$

$$= \frac{P \times 0.5 \times 0.5}{6EI \times 1}(1 - .5^2 - .5^2)$$

$$= \frac{P(.020833)}{EI} \text{ m}$$

But here P = 4 times of it

So $\quad y_2 = \frac{P(.08333)}{EI}$

Similarly, deflection at E

$$y_3 = \frac{P}{EI}(.0234375) \text{ m}$$

$$\omega_1 = \sqrt{\frac{g}{y_1}},\; \omega_2 = \sqrt{\frac{g}{y_2}},\; \omega_3 = \sqrt{\frac{g}{y_3}}$$

or $$\frac{1}{\omega_1^2} = \frac{y_1}{g},\; \frac{1}{\omega_2^2} = \frac{y_2}{g},\; \frac{1}{\omega_3^2} = \frac{y_3}{g}$$

$$\frac{1}{\omega^2} = \frac{(y_1 + y_2 + y_3)}{g}$$

$$= \frac{P}{gEI}(.0117187 + .0234375 + .08333)$$

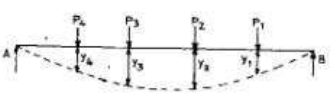
$$= \frac{P}{gEI}(0.1184)$$

or $$\omega = 2.91\sqrt{\frac{gEI}{P}}$$

6.8. RAYLEIGH METHOD

This is the energy method to find the frequency. This method is used to find the natural frequency of the system when transverse point loads are acting on the beam or shaft. Good estimate of fundamental frequency can be made by assuming the suitable deflection curve for the fundamental mode. The maximum kinetic energy is equated to maximum potential energy of the system to determine the natural frequency.

Let us consider a shaft AB of negligible weight as shown in figure 6.6 (a). Several point loads P_1, P_2, P_3, P_4, etc. are acting transversely. Suppose y_1, y_2, y_3, y_4, etc., be the maximum deflections under the influence of point loads.



The maximum potential energy of the system can be written as

$$\text{P.E.} = \frac{1}{2}P_1y_1 + \frac{1}{2}P_2y_2 + \frac{1}{2}P_3y_3 + \frac{1}{2}P_4y_4$$

$$\text{P.E.} = \frac{1}{2}\sum Py \qquad \text{...(6.8.1)}$$

The maximum kinetic energy of the system can be written as

$$\text{K.E.} = \frac{1}{2g}P_1(\omega y_1)^2 + \frac{1}{2g}P_2(\omega y_2)^2 + \frac{1}{2g}P_3(\omega y_3)^2 + \frac{1}{2g}P_4(\omega y_4)^2$$

$$= \frac{\omega^2}{2g}\Sigma Py^2 \qquad \text{...(6.8.2)}$$

where ω = natural frequency of vibration.

Equating the maximum kinetic energy to maximum potential energy, we have

$$\frac{\omega^2}{2g}\Sigma Py^2 = \frac{1}{2}\Sigma Py$$

$$\omega = \sqrt{\frac{g\,\Sigma Py}{\Sigma Py^2}} \qquad \text{...(6.8.3)}$$

The above equation can be written in a more generalised way by including the distributed mass of the beams. If m is the mass of the beam per unit length and y is the assumed deflection curve, the maximum potential energy of beam of length l is expressed as

$$\text{P.E.} = \frac{1}{2}\int_0^l M\, d\theta \qquad \text{...(6.8.4)}$$

where M = bending moment

$d\theta$ = change in slope over a distance dx

From beam theory, we know that

$\frac{M}{I} = \frac{E}{R}$ where R is the radius of curvature and EI is the flexural rigidity.

Also $\quad \frac{1}{R} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$

Thus $$M = \frac{EI}{R} = EI\frac{d^2y}{dx^2} \qquad \text{...(6.8.5)}$$

and $$\frac{d\theta}{dx} = \frac{d^2y}{dx^2}, \text{ or } d\theta = \left(\frac{d^2y}{dx^2}\right)dx \qquad \text{...(6.8.6)}$$

So from equations (6.8.5 and 6.8.6) equation (6.8.4) can be written as

$$P.E. = \frac{1}{2}\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx \qquad ...(6.8.7)$$

The maximum kinetic energy of the system can be written as

$$K.E. = \frac{1}{2}\int_0^l m(\omega y)^2\, dx \qquad ...(6.8.8)$$

Equating maximum kinetic energy to maximum potential energy, e get

$$\frac{1}{2}\int_0^l m(\omega y)^2\, dx = \frac{1}{2}\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx$$

So

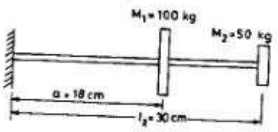
$$\omega^2 = \frac{\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^l my^2 dx}$$

$$= \frac{EI}{m}\,\frac{\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^l y^2 dx} \qquad ...(6.8.9)$$

EXAMPLE. Find the lower natural frequency of vibration for the em shown in figure 6.6 (b) by Rayleigh's method.

$E = 1.96 \times 10^{11}\ N/m^2$

$I = 4 \times 10^{-7}\ m^4$ (P.U. 91)



SOLUTION. With the help of equation (6.8.3), we can find the natural frequency as

$$\omega = \sqrt{\frac{g\,\Sigma Py}{\Sigma Py^2}}$$

The static deflections at two points are given as

$y_1 = M_1 g a_{11} + M_2 g a_{12}$

and $y_2 = M_1 g a_{21} + M_2 g a_{22}$

$$a_{11} = \frac{a^3}{3EI} = \frac{(.18)^3}{3 \times 1.96 \times 10^{11} \times 4 \times 10^{-7}} = 2.479 \times 10^{-8}$$

$$a_{22} = \frac{l^3}{3EI} = \frac{(0.3)^3}{3 \times 1.96 \times 10^{11} \times 4 \times 10^{-7}} = 1.147 \times 10^{-7}$$

$$a_{12} = a_{21} = \frac{a^2(3l - a)}{6EI} = \frac{(.18)^2(3 \times .30 - .18)}{6 \times 1.96 \times 10^{11} \times 4 \times 10^{-7}}$$

$= 4.959 \times 10^{-8}$

So $y_1 = M_1 g a_{11} + M_2 g a_{12}$

$= 100 \times 9.8 \times 2.479 \times 10^{-8} + 50 \times 9.8 \times 4.959 \times 10^{-8}$

$= .486 \times 10^{-4}$ m

$y_2 = M_1 g a_{21} + M_2 g a_{22}$

$= 100 \times 9.8 \times 4.959 \times 10^{-8} + 50 \times 9.8 \times 1.147 \times 10^{-7}$

$= 10.48 \times 10^{-5}$ m

Thus

$$\omega = \sqrt{\frac{9.8[P_1y_1 + P_2y_2]}{(P_1y_1^2 + P_2y_2^2)}}$$

$$= \sqrt{\frac{9.8\,[100 \times 4.86 \times 10^{-5} + 50 \times 10.48 \times 10^{-5}]}{100 \times (4.86 \times 10^{-5})^2 + 50 \times (10.48 \times 10^{-5})^2}}$$

$\omega = 354.96$ rad/sec

6.9. HOLZER'S METHOD

This is trial and error method used to find the natural frequency and mode shape of multimass lumped parameter system. This can be applied to both free and forced vibrations. This method can be used for the analysis of damped, undamped, semidefinite systems with fixed ends having linear and angular motions. First of all, a trial frequency of the system is assumed. A solution is found when the trial frequency satisfies the constraints of the system. Figure 6.7 shows a four-disc semi-

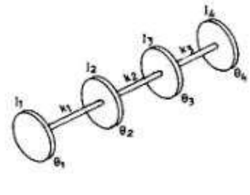


Fig. 6.7.

$$I_1\ddot{\theta}_1 + k_1(\theta_1 - \theta_2) = 0 \qquad ...(6.9.1)$$

$$I_2\ddot{\theta}_2 + k_1(\theta_2 - \theta_1) + k_2(\theta_2 - \theta_3) = 0 \qquad ...(6.9.2)$$

$$I_3\ddot{\theta}_3 + k_2(\theta_3 - \theta_2) + k_3(\theta_3 - \theta_4) = 0 \qquad ...(6.9.3)$$

$$I_4\ddot{\theta}_4 + k_3(\theta_4 - \theta_3) = 0 \qquad ...(6.9.4)$$

The motions are harmonic at a principal mode of vibration. Assuming $\theta_i = \phi_i \cdot \sin \omega t$ and substituting it in the above equations, we get

$$\omega^2 I_1\,\phi_1 = k_1(\phi_1 - \phi_2) \qquad ...(6.9.5)$$

$$\omega^2 I_2\,\phi_2 = k_1(\phi_2 - \phi_1) + k_2(\phi_2 - \phi_3) \qquad ...(6.9.6)$$

$$\omega^2 I_3\,\phi_3 = k_2(\phi_3 - \phi_2) + k_3(\phi_3 - \phi_4) \qquad ...(6.9.7)$$

$$\omega^2 I_4\,\phi_4 = k_3(\phi_4 - \phi_3) \qquad ...(6.9.8)$$

Summing the various terms of the above equations, we get

$$\sum_{i=1}^{4} \omega^2 I_i\,\phi_i = 0 \qquad ...(6.9.9)$$

For a set of n discs equation (6.9.9) can be written as

$$\sum_{i=1}^{n} \omega^2 I_i\,\phi_i = 0 \qquad ...(6.9.10)$$

In the above equation it is explained that the sum of the inertia orques $k_1(\phi_1 - \phi_2)$, $k_2(\phi_3 - \phi_2)$, etc., must be zero and the assumed trial requency ω must satisfy this equation.

Procedure

1. Assume a trial frequency ω
2. Take ϕ_1 as unity arbitrarily.
3. Calculate ϕ_2 from equation (6.9.5) as

$$\phi_2 = \phi_1 - \frac{I_1\,\omega^2\,\phi_1}{k_1} \qquad ...(6.9.11)$$

or $$= \left(1 - \frac{I_1\,\omega^2}{k_1}\right)\phi_1$$

Similarly, ϕ_3 and ϕ_4 can be computed from equations (6.9.6) and (6.9.7) as

$$\phi_3 = \phi_2 - \frac{\omega^2(I_1\phi_1 + I_2\phi_2)}{k_2} \qquad ...(6.9.12)$$

$$\phi_4 = \phi_3 - \frac{(I_1\phi_1 + I_2\phi_2 + I_3\phi_3)\,\omega^2}{k_3} \qquad ...(6.9.13)$$

4. The values of ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are put in equation (6.9.9) to check whether the equation is satisfied or not. If equation (6.9.9) is not satisfied, a new trial value of ω is assumed and the whole process is repeated.

5. Prepare a table showing various terms as :

Row	I	ϕ	$I\,\phi\,\omega^2$	$\Sigma I\,\phi\,\omega^2$	k	$\frac{1}{k}\Sigma I\,\phi\omega^2$
1.	I_1	ϕ_1	$I_1\,\phi_1\,\omega^2$		k_1	
2.	I_2	ϕ_2	$I_2\,\phi_2\,\omega^2$	Summation	k_2	Summation
3.	I_3	ϕ_3	$I_3\,\phi_3\,\omega^2$		k_3	

and so on.

6.10. STODOLA METHOD

It is an iterative process for the calculation of the fundamental natural frequency of the system. The steps of the method are :

(*i*) Assume a suitable deflection curve. Say for a three degree of freedom system it is assumed as

$$y_1 = 1,\ y_2 = 1,\ y_3 = 1 \qquad ...(6.10.1)$$

(*ii*) Find out inertia loading of the system for the deflection as assumed in section (*i*). This will be in terms of ω^2 as

$$F_1 = m_1\,\omega^2\,y_1 = m_1\omega^2$$

$$F_2 = m_2\,\omega^2\,y_2 = m_2\,\omega^2$$

$$F_3 = m_3\,\omega^2\,y_3 = m_3\,\omega^2 \qquad ...(6.10.2)$$

(*iii*) From the inertia loading as obtained in section (*ii*), find the corresponding new deflection curve. This will be in terms of ω^2.

$$y'_1 = F_1a_{11} + F_2a_{12} + F_3a_{13}$$

$$y'_2 = F_1a_{21} + F_2a_{22} + F_3a_{23}$$

$$y'_3 = F_1a_{31} + F_2a_{32} + F_3a_{33} \qquad ...(6.10.3)$$

where a_{11}, a_{12}, a_{21}, a_{22}, etc. are influence coefficients

(iv) If the assumed deflection curve y_1, y_2 and y_3 of section (i) is equal to the derived deflection curve of section (iii) y'_1, y'_2 and y'_3, then the shape of assumed curve of section (i) is correct. Then with the help of equations (6.10.1 and 6.10.3) we can find ω^2.

(v) If $y_1 \neq y'_1, y_2 \neq y'_2$ and $y_3 \neq y'_3$, then the derived deflection curve y'_1, y'_2 and y'_3 may be used as new starting points in place of y_1, y_2 and y_3 in the process.

Repeat the process till the assumed and derived deflection curves are equal.

6.11. EIGENVALUES AND EIGENVECTORS

Let us again consider the system shown in figure 6.2. The equations of motion can be written as

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$
$$m_2\ddot{x}_2 - k_1x_1 + (k_1 + k_2)x_2 - k_2x_3 = 0$$
$$m_3\ddot{x}_3 - k_3x_2 + k_3x_3 = 0$$

In matrix form the equations of motion can be written as

$$[M]\{\ddot{x}\} + [k]\{x\} = 0 \qquad ...(6.11.1)$$

Multiplying equation (6.11.1) by $[M]^{-1}$, we get

$$[M]^{-1}[M]\{\ddot{x}\} + [k][M]^{-1}\{x\} = 0$$
$$[I]\{\ddot{x}\} + [C]\{x\} = 0 \qquad ...(6.11.2)$$

where $[M]\ [M]^{-1} = [I]$, a unit matrix

$[k]\ [M]^{-1} = [C]$, a dynamic matrix

Let us assume the solution of the form

$$x = A \sin \omega t$$

So
$$\ddot{x} = -\omega^2 A \sin \omega t$$
$$= -\lambda A \sin \omega t$$

where $\omega^2 = \lambda$, eigen value

Since, we are considering the three degrees of freedom system, so there will be three values of ω^2. Hence, there will be three eigenvalues for a three degree of freedom system.

Equation (6.11.2) can be written as

$$-\omega^2[I]\{A\} + [C]\{A\} = 0 \qquad ...(6.11.3)$$

Here, the column A has three values which are known as eigenvectors. In general, we can say that a n degree of freedom system has n eigenvalues which are real and corresponding n eigenvectors. In the above expression ω is known as the natural frequency of the system and its lowest value is called the fundamental or first natural frequency. Equation (6.11.3) can be written as

$$\omega^2[I]\{A\} = [C]\{A\}$$
$$\lambda[I]\{A\} = [C]\{A\}$$
$$\{\lambda\ [I] - [C]\}\{A\} = 0 \qquad ...(6.11.4)$$

The solution of the above equation is obtained by putting its determinant equal to zero *i.e.*

$$|\lambda I - C| = 0 \qquad ...(6.11.5)$$

This equation gives the values of natural frequencies. Once we know the values of natural frequencies, the mode shapes or eigenvectors can be determined with the help of equation (6.11.4).

Now let us put some numerical problem for solution. Suppose the problem in figure 6.2 is considered. See article (6.4) for the determination of natural frequencies.

We get
$$\lambda_1 = 0.198\,\frac{k}{m}$$
$$\lambda_2 = 1.555\,\frac{k}{m}$$
and
$$\lambda_3 = 3.247\,\frac{k}{m}$$

The natural frequencies can be determined by many methods. One of the methods to find the natural frequencies which has already been used in two degrees of freedom system is again presented here.

The equation of motion can be put in determinant form as

$$\begin{vmatrix} -m\,\omega^2 + 2k & -k & 0 \\ -k & -m\omega^2 + 2k & -k \\ 0 & -k & -m\,\omega^2 + k \end{vmatrix} = 0$$

The frequency equation is obtained by expanding the above determinant as

$$-m^3\omega^6 + 5m^2k\omega^4 - 6mk^2\omega^2 + k^3 = 0$$
$$\omega^6 - 5\,\frac{k}{m}\,\omega^4 + 6\,\frac{k^2}{m^2}\,\omega^2 - \frac{k^3}{m^3} = 0$$

This is the same equation as obtained is section 6.4. Anyway, now try to find the mode shapes. Mode shapes or eigenvectors can be calculated with the help of equation (6.11.4) as

$$\{\lambda\ [I] - [C]\}\{A\} = 0$$

Mode shapes for first mode :

$$\left[\frac{.198k}{m}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{k}{m}\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\right]\begin{Bmatrix}A_1\\A_2\\A_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

$$\frac{k}{m}\begin{bmatrix}-1.802&1&0\\1&-1.802&1\\0&1&-.802\end{bmatrix}\begin{Bmatrix}A_1\\A_2\\A_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

We can have three homogeneous linear equations in three unknowns A_1, A_2 and A_3.

$$-1.802\,A_1 + A_2 = 0$$
$$A_1 - 1.802\,A_2 + A_3 = 0$$
$$A_2 - .802\,A_3 = 0$$

The mode shape will be $\begin{Bmatrix}A_1\\ \frac{A_2}{A_1}\\ \frac{A_3}{A_1}\end{Bmatrix}$

We get $\frac{A_2}{A_1} = 1.802$ and $\frac{A_3}{A_1} = 2.247$

Now the first mode shape is given by

$$A_{i1} = A_1^1\begin{Bmatrix}1.0\\1.802\\2.247\end{Bmatrix}$$

Second mode shape :

$$\lambda_2 = 1.555\ \text{k/m}$$
$$\{\lambda[I] - [C]\}\{A\} = 0 \qquad \text{(Using equation 6.11.4)}$$

$$\left[1.555\,\frac{k}{m}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{k}{m}\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\right]\begin{Bmatrix}A_1\\A_2\\A_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

$$\left[\begin{bmatrix}1.555&0&0\\0&1.555&0\\0&0&1.555\end{bmatrix} + \begin{bmatrix}-2&1&0\\1&-2&1\\0&1&-1\end{bmatrix}\right]\begin{Bmatrix}A_1\\A_2\\A_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

They can be put in equations as

$$-.445\,A_1 + A_2 = 0$$
$$A_1 - .445A_2 + A_3 = 0$$
$$A_2 + .555A_3 = 0$$
$$\frac{A_2}{A_1} = .455$$

and
$$A_1 - .445 \times .455\,A_1 = -A_3$$
$$\frac{A_3}{A_1} = -.8019$$

Thus the second mode shape can be expressed as

$$A_{2i} = A_2^1\begin{Bmatrix}1.0\\0.445\\-0.8019\end{Bmatrix}$$

Third mode shape :

We take $\lambda = 3.247\,\frac{k}{m}$

$$\left[3.247\,\frac{k}{m}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{k}{m}\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\right]\begin{Bmatrix}A_1\\A_2\\A_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

$$\begin{bmatrix}3.247-2&1&0\\1&3.247-2&1\\0&1&3.247-1\end{bmatrix}\begin{Bmatrix}A_1\\A_2\\A_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

$$1.247A_1 + A_2 = 0$$
$$A_1 + 1.247\,A_2 + A_3 = 0$$
$$A_2 + 2.247\,A_3 = 0$$

From the above equations, we get

$$\frac{A_2}{A_1} = -1.247$$
$$A_1 + 1.247(-1.247)A_1 = -A_3$$

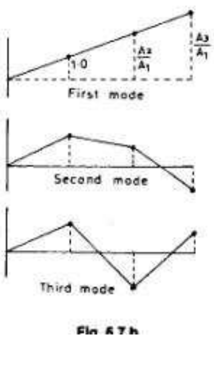


Fig. 6.7 b

$$\frac{A_3}{A_1} = 0.555$$

Thus the third mode shape can be expressed as

$$A_{i3} = A_3^1 \begin{Bmatrix} 1.0 \\ -1.247 \\ 0.555 \end{Bmatrix}$$

The values of A_1^1, A_2^1 and A_3^1 are usually taken as unity.

The mode shapes are shown in figure 6.7 (*b*).

6.12 TORSIONAL VIBRATIONS OF TWO ROTOR SYSTEM

Refer Fig. 6.8. If the shaft is of varying diameter it can be converted into a torsionally equivalent shaft of length l. The shaft carries two rotors A and B at its ends. The two rotors rotate in the opposite direction but their frequency is equal. A particular section of the shaft between A and B *i.e.* C remains unaffected. This section is termed as node. At node the amplitude of vibration is zero.

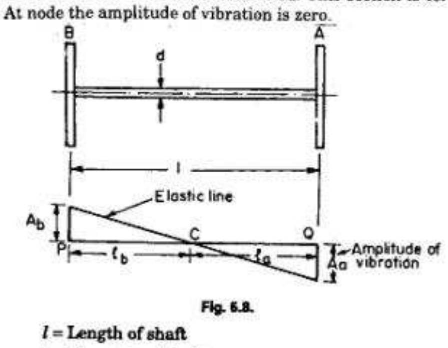


Fig. 6.8.

Let l = Length of shaft

d = Diameter of shaft

$l_b = PC$, distance of node from rotor B

$l_a = CQ$, distance of node from rotor A

J = Polar moment of inertia of shaft

C = Modulus of rigidity of shaft material

I_b, I_a = Mass moment of inertia of rotors B and A respectively.

Natural frequency of rotor A

$$f_{nA} = \frac{1}{2\pi}\sqrt{\frac{CJ}{l_a \cdot I_a}} \qquad ...(i)$$

Natural frequency of rotor B

$$f_{nB} = \frac{1}{2\pi}\sqrt{\frac{CJ}{l_b \cdot I_b}} \qquad ...(ii)$$

But $f_{nA} = f_{nB}$, so from equations (*i*) & (*ii*)

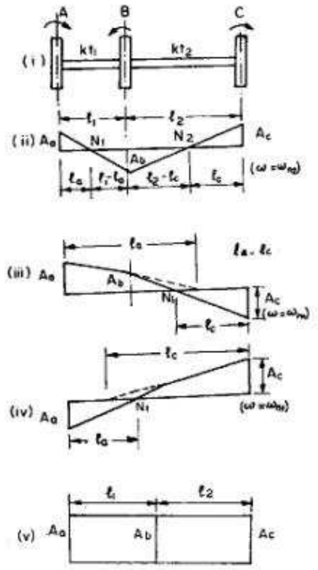
$$l_a \cdot I_a = l_b \cdot I_b \quad \text{or} \quad l_a = \frac{l_b \cdot I_b}{I_a}$$

We know that $l_a + l_b = l$

Thus we can locate the position of node with the help of above equations.

6.13 TORSIONAL VIBRATIONS OF THREE ROTOR SYSTEM

A three rotor system is shown in Fig. 6.9. The rotors A and C are attached to the ends of the shaft while rotor B is attached in between A



and C. The vibrations are torsional in nature. There are four possibilities of nodes.

Case I

Let rotors A and C rotate in the same direction and rotor B in the opposite direction. The node points N_1 and N_2 will occur as shown in Fig. 6.9 (*ii*).

Let l_1 = Distance between rotors A and B

l_2 = Distance between rotors B and C

l_a = Distance between rotor A and node N_1

l_c = Distance between rotor C and node N_2

J = Polar moment of inertia of shaft

C = Modulus of rigidity for shaft material

I_a = Mass moment of inertia of rotor A

I_b = Mass moment of inertia of rotor B

I_c = Mass moment of inertia of rotor C

The natural frequencies of rotors are torsional.

Natural frequency of rotor A is given by

$$f_{nA} = \frac{1}{2\pi}\sqrt{\frac{CJ}{l_a I_a}} \qquad ...(6.13.1)$$

Natural frequency of rotor B is given by

$$f_{nB} = \frac{1}{2\pi}\sqrt{\frac{CJ}{I_b}\left(\frac{1}{l_1 - l_a} + \frac{1}{l_2 - l_c}\right)} \qquad ...(6.13.2)$$

and natural frequency of torsional vibrations of rotor C is given by

$$f_{nC} = \frac{1}{2\pi}\sqrt{\frac{CJ}{I_c l_c}} \qquad ...(6.13.3)$$

But the natural frequency of all the rotors are equal, thus

$$f_{nA} = f_{nB} = f_{nC}$$

From equations (6.13.2) & (6.13.3), we get

$$\frac{1}{I_b}\left(\frac{1}{l_1 - l_a} + \frac{1}{l_2 - l_c}\right) = \frac{1}{I_c l_c} \qquad ...(6.13.4)$$

and from equations (6.13.1) & (6.13.3)

$$I_a \cdot l_a = I_c \cdot l_c$$

or

$$l_a = \frac{I_c \cdot l_c}{I_a} \qquad ...(6.13.5)$$

Substituting the value of l_a from (6.13.5) into eqn. (6.13.4), we get

$$\frac{1}{I_b}\left(\frac{1}{l_1 - \frac{I_c \cdot l_c}{I_a}} + \frac{1}{l_2 - l_c}\right) = \frac{1}{I_c l_c}$$

$$\frac{I_c}{I_b}\left(\frac{l_c}{\frac{I_a l_1 - I_c l_c}{I_a}} + \frac{l_c}{l_2 - l_c}\right) - 1 = 0$$

$$\frac{I_c \cdot I_a}{I_b}\left(\frac{l_c}{I_a \cdot l_1 - I_c \cdot l_c} + \frac{l_c}{I_a (l_2 - l_c)}\right) - 1 = 0$$

$$I_c \cdot I_a \left[I_a (l_2 l_c - l_c^2) + I_a \cdot l_1 l_c - I_c l_c^2 \right] - (I_a \cdot l_1 - I_c l_c) I_a (l_2 - l_c) \cdot I_b = 0$$

This is a quadratic eqn in l_c.

There are two values of l_c and two values of l_a (from eqn. (6.13.5)). Thus there will be two values of nodes and two values of node frequency. Two node frequencies can be computed with the help of equations (6.13.1) and (6.13.3).

Case II

When rotors A and B rotate in the same direction and rotor C in the opposite direction, there will be single node for torsional vibrations. It lies between rotors B and C. Refer Fig. 6.9 (*iii*). It does not give the actual position of node. In this case $l_a > l_c$.

Case III

Again there will be single node of vibration, when rotors B and C rotate in the same direction and the rotor A in the opposite direction.

It is shown in Fig. 6.9 (*iv*) where $l_c > l_a$. Actual position of node is indicated by N_1.

For an analysis of the vibrations executed by the three rotor system, we assume that the torsional stiffness of shaft between A and B and between B and C are K_{t_1} and K_{t_2} respectively. Then, let at any instant, $\theta_a, \theta_b, \theta_c$ be the displacements (angular) of rotors A, B and C from their equilibrium positions when the system executes torsional vibrations. It is assumed that all the rotors rotate in the same direction.

This means that the twists of the respective shafts at this instant are $(\theta_1 - \theta_2)$ and $(\theta_2 - \theta_3)$.

From Newton's Second law of motion the equations of motion can be written as

$$\left.\begin{aligned} I_a \ddot{\theta}_a &= -K_{t_1}(\theta_a - \theta_b) \\ I_b \ddot{\theta}_b &= +K_{t_1}(\theta_a - \theta_b) - K_{t_2}(\theta_b - \theta_c) \\ I_c \ddot{\theta}_c &= +K_{t_2}(\theta_b - \theta_c) \end{aligned}\right\} \qquad ...(6.13.6)$$

Assuming the motion of the form $\theta_x = A_x \cos \omega t$

So we get, $\ddot{\theta}_x = -\omega^2 \theta_x = -\omega^2 A_x \cos \omega t$ ...(6.13.7)

where $x = a, b$, or c

A_x = amplitude of torsional vibrations at any of the rotors A, B or C.

So for three rotor system we have,

$$\ddot{\theta}_a = -\omega^2 A_a \cos \omega t$$

$$\ddot{\theta}_b = -\omega^2 A_b \cos \omega t$$

$$\ddot{\theta}_c = -\omega^2 A_c \cos \omega t$$

Substituting these values in the above equation and cancelling $\cos \omega t$, we get

$$\left.\begin{aligned} I_a \omega^2 A_a - K_{t_1}(A_a - A_b) &= 0 \\ I_b \omega^2 A_b + K_{t_1}(A_a - A_b) - K_{t_2}(A_b - A_c) &= 0 \\ I_c \omega^3 A_c + K_{t_2}(A_b - A_c) &= 0 \end{aligned}\right\} \quad \text{...(6.13.8)}$$

On rearranging above equations (6.13.8), we get

$$\left.\begin{aligned} (I_a \omega^2 - K_{t_1}) A_a + K_{t_1} A_b &= 0 \\ K_{t_1} A_a + (I_b \omega^2 - K_{t_1} - K_{t_2}) A_b + K_{t_2} A_c &= 0 \\ K_{t_2} A_b + (I_c \omega^2 - K_{t_2}) A_c &= 0 \end{aligned}\right\} \quad \text{...(6.13.9)}$$

This is a homogeneous set of equations in A_a, A_b, A_c, and can here be a solution only if the determinant formed with their coefficient vanishes.

Equating the coefficients of A_a, A_b, A_c in determinant form to zero, we get

$$\begin{vmatrix} (I_a \omega^2 - K_{t_1}) & K_{t_1} & 0 \\ K_{t_1} & (I_b \omega^2 - K_{t_1} - K_{t_2}) & K_{t_2} \\ 0 & K_{t_2} & (I_c \omega^2 - K_{t_2}) \end{vmatrix} = 0 \quad \text{...(6.13.10)}$$

This is the frequency equation.

On rearranging the equation (6.13.6) we get,

$$\left.\begin{aligned} I_a \ddot{\theta}_a + (+K_{t_1})\theta_a - (K_{t_1})\theta_b &= 0 \\ I_b \ddot{\theta}_b - (K_{t_1})\theta_a + (K_{t_1} + K_{t_2})\theta_b - (K_{t_2})\theta_c &= 0 \\ I_c \ddot{\theta}_c - (K_{t_2})\theta_b + (K_{t_2})\theta_c &= \theta \end{aligned}\right\} = 0 \quad \text{...(6.13.11)}$$

From equation (6.13.11) the determinant form of the problem can

$$[M] = \begin{bmatrix} I_a & 0 & 0 \\ 0 & I_b & 0 \\ 0 & 0 & I_c \end{bmatrix}$$

$$\text{Stiffness Matrix } [K] = \begin{bmatrix} K_{t_1} & -K_{t_1} & 0 \\ -K_{t_1} & (K_{t_1} + K_{t_2}) & -K_{t_2} \\ 0 & -K_{t_2} & K_{t_2} \end{bmatrix}$$

From equation (6.4.4), we get

Dynamic matrix $[C] = [M]^{-1}[K]$

$$[M]^{-1} = \begin{bmatrix} 1/I_a & 0 & 0 \\ 0 & 1/I_b & 0 \\ 0 & 0 & 1/I_c \end{bmatrix} \quad \text{...(6.13.12)}$$

$$\text{Thus } [C] = \begin{bmatrix} 1/I_a & 0 & 0 \\ 0 & 1/I_b & 0 \\ 0 & 0 & 1/I_c \end{bmatrix} \begin{bmatrix} K_{t_1} & -K_{t_1} & 0 \\ -K_{t_1} & (K_{t_1} + K_{t_2}) & -K_{t_2} \\ 0 & -K_{t_2} & K_{t_2} \end{bmatrix}$$

This reduces to,

$$[C] = \begin{vmatrix} \dfrac{K_{t_1}}{I_a} & -\dfrac{K_{t_1}}{I_b} & 0 \\ \dfrac{-K_{t_1}}{I_a} & \dfrac{K_{t_1} + K_{t_2}}{I_b} & \dfrac{-K_{t_2}}{I_c} \\ 0 & \dfrac{-K_{t_2}}{I_b} & \dfrac{K_{t_2}}{I_c} \end{vmatrix}$$

The solution of equations (6.13.11) can thus be given by

$$|\lambda I - C| = 0$$

Thus $|\lambda I - C| =$

$$\begin{vmatrix} \lambda I_a - K_{t_1} & +K_{t_1} & 0 \\ +K_{t_1} & \lambda I_b - K_{t_1} - K_{t_2} & K_{t_2} \\ 0 & +K_{t_2} & \lambda I_c - K_{t_2} \end{vmatrix} = 0 \quad \text{...(6.13.13)}$$

The above equation is the frequency equation similar to (6.13.10), where

$$\lambda = \omega^2.$$

Expanding the above determinant and putting $\lambda = \omega^2$, we get

$$\omega^2 [I_a I_b I_c \omega^4 - \{(I_a I_b + I_a I_c) K_{t_2} + (I_b I_c + I_a I_c) K_{t_1}\} \omega^2 + K_{t_1} K_{t_2}(I_a + I_b + I_c)] = 0 \quad \text{...(6.13.14)}$$

The above equation is cubic in ω^2 with one of the roots of $\omega^2 = 0$. This should have been expected as we are dealing with what is known as a semi-definite system.

The two definite frequencies in this three rotor system can be obtained from equation (6.13.14) as

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left[\left(\frac{K_{t_1}}{I_a} + \frac{K_{t_1} + K_{t_2}}{I_b} + \frac{K_{t_2}}{I_c}\right) \pm \sqrt{\left(\frac{K_{t_1}}{I_a} + \frac{K_{t_1} + K_{t_2}}{I_b} + \frac{K_{t_2}}{I_c}\right)^2 - \frac{4 K_{t_1} K_{t_2}(I_a + I_b + I_c)}{I_a I_b I_c}}\,\right] \quad \text{...(6.13.15)}$$

The mode shapes can be obtained from equations (6.13.9), as

$$\left.\begin{aligned} \frac{A_a}{A_b} &= \frac{K_{t_1}}{K_{t_1} - I_a \omega^2} \\ \frac{A_c}{A_b} &= \frac{K_{t_2}}{K_{t_2} - I_c \omega^2} \end{aligned}\right\} \quad \text{...(6.13.16)}$$

When $\omega = 0$, both the above ratios are unity, indicating that the whole system rotates rigidly. For value of $\omega = \omega_1$ (the smaller frequency), one of the ratios in eqn. (6.13.16) is positive while other is negative. And for $\omega = \omega_2$, both the ratios are negative. The mode shapes can be referred from Fig. 6.9.

6.14 TORSIONAL VIBRATION OF MULTI-ROTOR SYSTEMS (A GENERALIZATION)

Torsional vibrations are most commonly encountered in almost all the machinery. The common examples are the internal combustion engines.

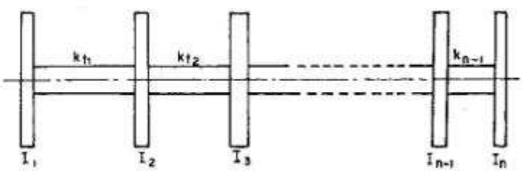


Fig. 6.10.

We consider a 'n' rotor system. The 'n' rotors are connected by $(n-1)$ shafts whose torsional stiffnesses are $K_{t_1}, K_{t_2}, \ldots K_{t_{n-1}}$ respectively. The moment of inertia of the rotors are given by $I_1, I_2, I_3 \ldots I_n$ respectively. We assume that at any instant, $\theta_1, \theta_2, \ldots \theta_n$ are the displacements of various rotors from their equilibrium positions.

This means that the twists of the respective shafts at this instant are $(\theta_1 - \theta_2), (\theta_2 - \theta_3) \ldots (\theta_{n-1} - \theta_n)$ when all the rotors rotate in same

From Newton's Second Law of motion, the equations of motion can be written as

$$\left.\begin{aligned} I_1 \ddot{\theta}_1 &= -K_{t_1}(\theta_1 - \theta_2) \\ I_2 \ddot{\theta}_2 &= +K_{t_1}(\theta_1 - \theta_2) - K_{t_2}(\theta_2 - \theta_3) \\ I_3 \ddot{\theta}_3 &= +K_{t_2}(\theta_2 - \theta_3) - K_{t_3}(\theta_3 - \theta_4) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ I_{n-1} \ddot{\theta}_{n-1} &= +K_{t_{n-2}}(\theta_{n-2} - \theta_{n-1}) - K_{t_{n-1}}(\theta_{n-1} - \theta_n) \\ I_n \ddot{\theta}_n &= +K_{t_{n-1}}(\theta_{n-1} - \theta_n) \end{aligned}\right\} \quad \text{...(6.14.1)}$$

Two cases arise from new one :

CASE A : (FREE VIBRATIONS)

Adding all the above equations (6.14.1), we get

$$\sum_{i=1}^{n} I_i \ddot{\theta}_i = 0 \quad \text{...(6.14.2)}$$

The LHS of the above eqn. (6.14.2) represents the sum of the inertia torques on all the rotors and is equal to zero since there is no external exciting torque on the system.

Assuming the motion of the form, $\theta_i = A_i \cos \omega t$, we get

$$\ddot{\theta}_i = -\omega^2 \theta_i = -\omega^2 A_i \cos \omega t \quad \text{...(6.14.3)}$$

where $i = 1, 2, 3 \ldots n$

A_i = Amplitude of torsional vibrations at any of the 'n' rotors depending on the value of 'i'.

So for a multi rotor system, we have

$$\ddot{\theta}_1 = -\omega^2 A_1 \cos \omega t$$

$$\ddot{\theta}_2 = -\omega^2 A_2 \cos \omega t$$

$$\ddot{\theta}_3 = -\omega^2 A_3 \cos \omega t$$

$$\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots$$

$$\ddot{\theta}_{n-1} = -\omega^2 A_{n-1} \cos \omega t$$

$$\ddot{\theta}_n = -\omega^2 A_n \cos \omega t$$

Substituting these values in the above equations (6.14.1), and cancelling $\cos \omega t$, we get

$$\left.\begin{aligned} I_1\omega^2 A_1 - K_{t_1}(A_1 - A_2) &= 0 \\ I_2\omega^2 A_2 + K_{t_1}(A_1 - A_2) - K_{t_2}(A_2 - A_3) &= 0 \\ I_3\omega^2 A_3 + K_{t_2}(A_2 - A_3) - K_{t_3}(A_3 - A_4) &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ I_n\omega^2 A_n + K_{t_{n-1}}(A_{n-1} - A_n) &= 0 \end{aligned}\right\} \quad \ldots(6.14.4)$$

On rearranging above equations (6.14.4), we get

$$\left.\begin{aligned} (I_1\omega^2 - K_{t_1})A_1 + K_{t_1}A_2 &= 0 \\ K_{t_1}A_1 + (I_2\omega^2 - K_{t_1} - K_{t_2})A_2 + K_{t_2}A_3 &= 0 \\ K_{t_2}A_2 + (I_3\omega^2 - K_{t_2} - K_{t_3})A_3 + K_{t_3}A_4 &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ K_{t_{n-1}}A_{n-1} + (I_n\omega^2 - K_{t_{n-1}}) &= 0 \end{aligned}\right\} \quad \ldots(6.14.5)$$

This is a homogeneous set of equations in $A_1, A_2, A_3 \ldots A_n$ and can ave a solution only if the determinant formed with their coefficient anishes. Eliminating $A_1, A_2, A_3 \ldots A_n$ from the above 'n' omogeneous set of equations in $A_1, A_2, A_3, \ldots A_n$, and the resulting nth egree equation in ω^2 would then give 'n' natural frequencies of the ystem. This would be the frequency equation.

On rearranging the equations (6.14.1), we get

$$\left.\begin{aligned} I_1\ddot{\theta}_1 + (K_{t_1})\theta_1 - (K_{t_1})\theta_2 &= 0 \\ I_2\ddot{\theta}_2 - (K_{t_1})\theta_1 + (K_{t_1} + K_{t_2})\theta_2 - (K_{t_2})\theta_3 &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ I_n\ddot{\theta}_n - (K_{t_{n-1}})\theta_{n-1} + (K_{t_{n-1}})\theta_n &= 0 \end{aligned}\right\} \quad \ldots(6.14.6)$$

From the equations (6.14.6) the determinant form of the problem an be derived. Thus the matrix expressions for the system are

$$[M] = \begin{bmatrix} I_1 & 0 & 0 \ldots\ldots\ldots\ldots 0 \\ 0 & I_2 & 0 \ldots\ldots\ldots\ldots 0 \\ 0 & 0 & I_3 \ldots\ldots\ldots\ldots 0 \\ \cdots & \cdots & \cdots\cdots\cdots\cdots \\ \cdots & \cdots & \cdots\cdots\cdots\cdots \\ 0 & 0 & 0 \ldots\ldots\ldots\ldots I_n \end{bmatrix}$$

Stiffness matrix $[K]$ =

$$[K] = \begin{bmatrix} K_{t_1} & -K_{t_1} & 0 \ldots\ldots\ldots\ldots 0 & 0 \\ -K_{t_1} & (K_{t_1} + K_{t_2}) & -K_{t_2} \ldots\ldots\ldots\ldots 0 & 0 \\ \cdots & \cdots & \cdots\cdots\cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots\cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots\cdots\cdots & \cdots \\ 0 & 0 & 0 \ldots\ldots\ldots\ldots -K_{t_{n-1}} & K_{t_{n-1}} \end{bmatrix}$$

From equation (6.4.4), we get Dynamic Matrix $[C] = [M]^{-1}[K]$

The solutions of equations (6.14.6) are given by

$$|\lambda I - C| = 0 \; ; \text{where } \lambda = \omega^2.$$

CASE B : (FORCED VIBRATIONS)

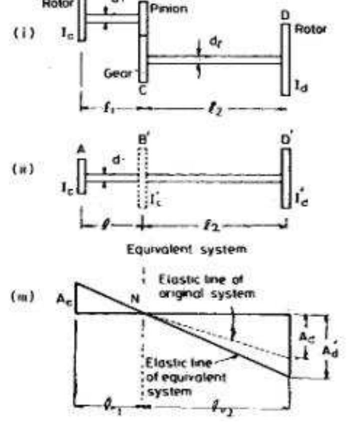
If there is an external excitation torque acting on the system at different points along the system, then under such conditions we have

$$\sum_{i=1}^{n} I_i\ddot{\theta}_i = T_e \quad \ldots(6.14.7)$$

where T_e is the sum of all the external torques on the system.

6.15 TORSIONAL VIBRATIONS OF A GEARED SYSTEM

Fig. 6.11 shows diagrammatically a geared system in which rotor A on one shaft is connected through the pinion B and the gear wheel C



to the rotor D on the second shaft. This system may be replaced by an equivalent system of uniform diameter shown at Fig. 6.11 (*ii*) and 6.11 (*iii*) by assuming that (*a*) the inertia of both the shafts and pinion B and gear C are negligible, (*b*) the drive is positive i.e. there is no slip, and (*c*) the teeth are loaded within elastic limit. With the above assumptions, the system can be treated as a two rotor system. It is a case of free vibration, the total energy at any instant is constant.

N is the position of the node, A_a and A_d being the amplitudes of vibration of rotors A and D respectively $A_{d'}$ is the amplitude of the equivalent rotor D'.

The kinetic energy of the original system is equal to the K.E. of the equivalent system. Also the strain energy of the original system is equal to the strain energy of the equivalent system.

Equating the kinetic energies

Kinetic energy of the original system = K.E. of the equivalent system

K.E. of l_1 section + K.E. of l_2 section = K.E. of l_1 section + K.E. of l'_2 section

or K.E. of l_2' section = K.E. of l_2 section

$$\frac{1}{2}I_d'\,\omega_d'^2 = \frac{1}{2}I_d\,\omega_d^2$$

or
$$I_d' = I_d\left(\frac{\omega_d}{\omega_d'}\right)^2 = I_d/G^2 \quad \ldots(6.15.1)$$

where $\omega_d' = \omega_d$ = angular speed of equivalent rotor D'

$$\frac{\omega_d'}{\omega_d} = G = \text{Gear ratio}$$

Equating the strain energies (S.E.)

S.E. of l_2' section = S.E. of l_2 section

$$\frac{1}{2}T_d'\,\theta_d' = \frac{1}{2}T_d\,\theta_d \quad \ldots(6.15.2)$$

$$\left(\because \text{S.E.} = \frac{1}{2}T\cdot\theta\right)$$

We know from Strength of Materials

$$\frac{T}{J} = \frac{C\theta}{l} \quad \text{or} \quad T = \frac{C\theta J}{l}$$

So equation (6.15.2) can be written as

$$\frac{C\theta_d' J_d'}{l_2'}\theta_d' = \frac{C\,\theta_d J_d}{l_2}\theta_d$$

$$l_2' = \left(\frac{\theta'_d}{\theta_d}\right)^2 l_2\left(\frac{J_d'}{J_d}\right) \qquad \left(\because J = \frac{\pi}{32}d^4\right)$$

$$= \left(\frac{\theta_d'}{\theta_d}\right)^2 l_2\left(\frac{d_d'}{d_2}\right)^4 \quad \ldots(6.15.3)$$

$$(\because \theta_d' = \theta_a)$$

where d_d' = diameter of equivalent shaft

d_2 = diameter of section $l_2 = d_d$

Assuming the diameter of the equivalent shaft to be equal to that of shaft 1 *i.e.*

$$d_d' = d_1$$

Then eqn. (6.15.3) can be written as

$$l_2' = \left(\frac{\theta_d'}{\theta_d}\right)^2 l_2\cdot\left(\frac{d_1}{d_2}\right)^4 = G^2 l_2\left(\frac{d_1}{d_2}\right)^4 \qquad (\because \theta = \omega t)$$

where
$$G = \frac{\theta_d'}{\theta_d} \qquad (\because \omega_d' = \omega_d)$$

So length of equivalent shaft

$$= l_1 + l_2'$$

$$= l_1 + \left(\frac{\theta_d'}{\theta_d}\right)^2 l_2\left(\frac{d_1}{d_2}\right)^4 = l_1 + G^2 l_2\left(\frac{d_1}{d_2}\right)^4$$

If the inertia of the gearing is to be considered, it is assumed that a rotor is acting at a distance l_1 from rotor A.

The mass moment of inertia of that rotor is given by

$$I_c' = I_G + \frac{I_P}{G^2}$$

where I_G and I_P are the mass moment of inertia of gear and pinion respectively and then the system will be three rotor system which have already been treated.

For vibration analysis of the geared torsional systems we refer Fig. 6.12. We assume that the gear ratio is given by G. The gear ratio is the ratio of the speed of the second shaft to the speed of first shaft i.e. the ratio of driven shaft speed to driving shaft speed.

The first step in the analysis is to convert the original geared system into an equivalent system. The original geared system can be converted into an equivalent system with respect to any of the shafts. In this analysis, we convert the original geared system to an equivalent system with respect to the first shaft (the driving shaft).

The basis for the above conversion is that the sum of the kinetic nergy and the strain energy (Potential energy) of the original and quivalent system are equal.

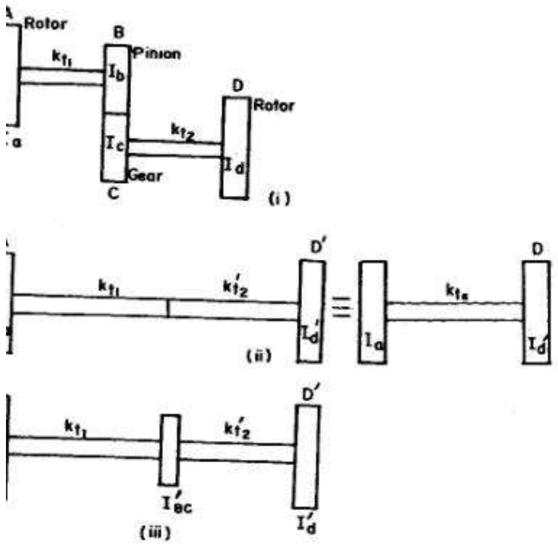


Fig. 6.12. (i) Original geared system
(ii) Equivalent system for case : A (neglecting gear inertia)
(iii) Equivalent system for case : B (considering gear inertia).

The analysis involves two cases :

CASE A : (Neglecting gear inertia)

We can neglect the inertia of the connecting gears so that the uivalent system reduces to rotors A, D' and an equivalent shaft.

Let θ_a and θ_d be the angular displacement of the rotors A and D spectively.

The kinetic energy and the potential energy of the original system are given as

$$\left.\begin{aligned}(\text{K.E.})_0 &= \frac{1}{2} I_a (\dot{\theta}_a)^2 + \frac{1}{2} I_d (\dot{\theta}_d)^2 \\ (\text{P.E.}) \text{ or } (\text{S.E.})_0 &= \frac{1}{2} T_a (\theta_a) + \frac{1}{2} T_d (\theta_d) \\ &= \frac{1}{2} K_{t_1} (\theta_a)^2 + \frac{1}{2} K_{t_2} (\theta_d)^2 \\ &\left[\because \begin{array}{l} T_a = K_{t_1} \cdot \theta_a \\ T_d = K_{t_2} \cdot \theta_d \end{array}\right]\end{aligned}\right\} \quad ...(6.15.4)$$

The Gear ratio G is given as

$$G = \frac{V_d}{V_a} = \frac{\omega_d}{\omega_a} = \frac{\theta_d}{\theta_a} \quad ...(6.15.5)$$

[**Note.** (Here it is assumed that rotor D is on the driven shaft while rotor A is on the driving shaft. The students should not confuse this with the earlier case when it was just the reverse).]

From equation (6.15.5), we get

$$\theta_d = G \cdot \theta_a \quad ...(6.15.6)$$

Substituting the value of θ_d from (6.15.6) in expressions given by (6.15.4), we get

$$(\text{K.E.})_0 = \frac{1}{2} I_a (\dot{\theta}_a)^2 + \frac{1}{2} I_d (G\dot{\theta}_a)^2$$

$$(\text{P.E.})_0 \text{ or } (\text{S.E.})_0 = \frac{1}{2} K_{t_1} (\theta_a)^2 + \frac{1}{2} K_{t_2} (G\theta_a)^2$$

or

$$\left.\begin{aligned}(\text{K.E.})_0 &= \frac{1}{2} I_a (\dot{\theta}_a)^2 + \frac{1}{2} (G^2 I_d) (\dot{\theta}_a)^2 \\ (\text{P.E.})_0 \text{ or } (\text{S.E.})_0 &= \frac{1}{2} K_{t_1} \theta_a^2 + \frac{1}{2} (G^2 K_{t_2}) (\theta_a)^2\end{aligned}\right\} \quad ...(6.15.7)$$

The above equations show that the original system can be converted into an equivalent system with respect to the first shaft. This system is shown in Fig. (*ii*). This is achieved by multiplying the inertia of the second rotor and the stiffness of the second shaft by G^2 and keeping this part of the system in series with the first part. Thus the equivalent expressions become,

$$\left.\begin{aligned}(\text{K.E.})_e &= \frac{1}{2} I_a (\dot{\theta}_a)^2 + \frac{1}{2} (I_d') (\dot{\theta}_a)^2 \\ (\text{P.E.})_e = (\text{S.E.})_e &= \frac{1}{2} K_{t_1} (\theta_a)^2 + \frac{1}{2} (K_{t_2}') (\theta_a)^2\end{aligned}\right\} \quad ...(6.15.8)$$

From the above equations (6.15.8), it applies that

$$\left.\begin{aligned}I_d' &= G^2 I_d \\ K_{t_2}' &= G^2 K_{t_2}\end{aligned}\right\} \quad ...(6.15.9)$$

From Fig. 6.12 (*ii*) it can be seen that the two torsional stiffnesses K_{t_1} and K_{t_2}' are in series. Thus the equivalent torsional stiffness is n by

$$\frac{1}{K_{t_e}} = \frac{1}{K_{t_1}} + \frac{1}{K_{t_2}'}$$

$$K_{t_e} = \frac{K_{t_2}' K_{t_1}}{(K_{t_1} + K_{t_2}')} \quad ...(6.15.10)$$

Substituting the value of K_{t_2}' from eqn. (6.15.9) in equation .10), we get

$$K_{t_e} = \frac{G^2 K_{t_2} K_{t_1}}{K_{t_1} + G^2 K_{t_2}} \quad ...(6.15.11)$$

The system now reduces to a two-rotor system. Therefore the ral frequency can be obtained using the equation (5.2.5). There- we get

$$\omega_n = \sqrt{\frac{K_{t_e} (I_a + I_d')}{I_a I_d'}} \quad ...(6.15.12)$$

Substituting the value of K_{t_e} from eqn. 6.15.10 in 6.15.12, we get

$$\omega_n = \sqrt{\frac{K_{t_2}' K_{t_1} (I_a + I_d')}{I_a I_d' (K_{t_1} + K_{t_2}')}} \quad ...(6.15.13)$$

Substituting the value of I_d' from eqn. (6.15.9) in eqn. (6.15.12), we

$$\omega_n = \sqrt{\frac{K_{t_e} (I_a + G^2 I_d)}{G^2 I_a I_d}} \quad ...(6.15.14)$$

NCIPLE OF CONVERSION

The principle for geared systems is thus quite simple and states by multiplying all the stiffness and inertias of the geared shaft by re get the equivalent system, where G is the speed ratio of the ed shaft to the reference shaft.

CASE B : (Considering the gear inertia)

This case is illustrated in Fig. 6.12 (*iii*). The equivalent system respect to the first shaft can be obtained in the same way and y we have a three rotor system which can be analysed by using n (6.13)

Here the inertia of the intermediate equivalent rotor is given by I'_{BC}.

The principle to get I'_{BC} or the intermediate equivalent rotor inertia is to add the inertia of the gear (driving gear) connected to the reference shaft (driving shaft) to G^2 times the inertia of gear (driven gear) connected to the geared shaft (driven shaft)

Thus $$I'_{BC} = I_b + G^2 I_c \quad ...(6.15.15)$$

The two definite frequencies in this three rotor equivalent system can be obtained from eqns. (6.13.14) and is given by equation (6.13.15). Thus

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left[\left(\frac{K_{t_1}}{I_a} + \frac{K_{t_1} + K_{t_2}'}{I'_{BC}} + \frac{K_{t_2}'}{I_d'}\right) \pm \sqrt{\left(\frac{K_{t_1}}{I_a} + \frac{K_{t_1} + K_{t_2}'}{I'_{BC}} + \frac{K_{t_2}'}{I_d'}\right)^2 - \frac{4K_{t_1} K_{t_2}' (I_a + I'_{BC} + I_d')}{I_a I'_{BC} I_d'}}\right] \quad ...(6.15.16)$$

where
$$K_{t_2}' = G^2 K_{t_2}$$
$$I'_{BC} = I_b + G^2 I_c$$
$$I_d' = G^2 I_d$$

6.16 TORSIONAL VIBRATIONS OF BRANCHED GEARED SYSTEMS (A SPECIAL CASE OF GEARED SYSTEMS)

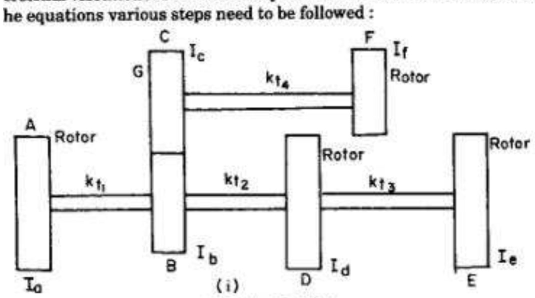
Branched geared systems are frequently encountered in dual propeller system of a marine installation and the drive shaft and differential of the automobile. Such systems can be reduced to an equivalent form with one-to-one gear, by multiplying all the inertias and stiffnesses of the branches by the squares of their speed ratios.

[We assume that at any instant, θ_a, θ_b, θ_d, θ_e, θ_f' are the displacements of various rotors from their equilibrium positions. This means that the twists of the respective shafts are $(\theta_a - \theta_{bc})$, $(\theta_{bc} - \theta_d)$, $(\theta_d - \theta_e)$ and $(\theta_{bc} - \theta_f')$.]

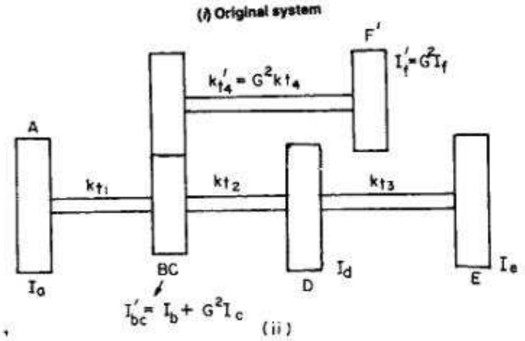
From Newton's second law of motion, the equations of motion for the equivalent system can be written as

$$\left.\begin{aligned}I_a \ddot{\theta}_a &= -Kt_1 (\theta_a - \theta_{bc}) \\ I_{bc} \ddot{\theta}_{bc} &= +Kt_1 (\theta_a - \theta_{bc}) - K_{t_2} (\theta_{bc} - \theta_d) - K_{t_4}' (\theta_{bc} - \theta_f') \\ I_d \ddot{\theta}_d &= +Kt_2 (\theta_{bc} - \theta_d) - K_{t_3} (\theta_d - \theta_e) \\ I_e \ddot{\theta}_e &= +Kt_3 (\theta_d - \theta_e) \\ I_f' \ddot{\theta}_f' &= +K_{t_4}' (\theta_{bc} - \theta_f')\end{aligned}\right\} \quad ...(6.16.1)$$

Further analysis can be carried out by using article 6.14 for orsional vibrations of multi-rotor systems. To reach the matrix form of he equations various steps need to be followed :



(i) Original system



(ii) Equivalent system considering Gear inertia.

Fig. 6.13.

Assuming the motion of the form

$$\theta_x = A_x \cos \omega t$$

$$\ddot{\theta}_x = -\omega^2 \theta_x = -\omega^2 A_x \cos \omega t \qquad ...(6.16.2)$$

So for this multi rotor branched system, we have

$$\ddot{\theta}_a = -\omega^2 A_a \cos \omega t$$

$$\ddot{\theta}_{bc} = -\omega^2 A_{bc} \cos \omega t$$

$$\ddot{\theta}_e = -\omega^2 A_e \cos \omega t$$

$$\ddot{\theta}'_f = -\omega^2 A'_f \cos \omega t$$

Substituting these values in the above equations (6.16.1) and cancelling cos ωt, we get

$$\left.\begin{aligned}
I_a \omega^2 A_a - K_{t_1}(A_a - A_{bc}) &= 0\\
I_{bc} \omega^2 A_{bc} + Kt_1 (A_a - A_{bc}) - Kt_2 (A_{bc} - A_d) - K'_{t_4}(A_{bc} - A'_f) &= 0\\
I_d \omega^2 A_d + K_{t_2}(A_{bc} - A_d) - K_{t_3}(A_d - A_e) &= 0\\
I_e \omega^2 A_e + K_{t_3}(A_d - A_e) &= 0\\
I'_f \omega^2 A'_f + K'_{t_4}(A_{bc} - A'_f) &= 0
\end{aligned}\right\} \quad ...(6.16.3)$$

On rearranging above equations, we get,

$$\left.\begin{aligned}
(I_a \omega^2 - K_{t_1}) A_a + (K_{t_1}) A_{bc} &= 0\\
(K_{t_1}) A_a + (I_{bc} \omega^2 - Kt_1 - Kt_2 - K'_{t_4}) A_{bc} + (K_{t_2}) A_d + (K'_{t_4}) A'_f &= 0\\
(K_{t_2}) A_{bc} + (I_d \omega^2 - K_{t_2} - K_{t_3}) A_d + (K_{t_3}) A_e &= 0\\
(K_{t_3}) A_d + (I_e \omega^2 - K_{t_3}) A_e &= 0\\
(K'_{t_4}) A_{bc} + (I'_f \omega^2 - K'_{t_4}) A'_f &= 0
\end{aligned}\right\} \quad ...(6.16.4)$$

This is a set of equations in A_a, A_{bc}, A_d, A_e and A'_f.

On rearranging the equations (6.16.1) further, we get

$$\left.\begin{aligned}
I_a \ddot{\theta}_a + (K_{t_1}) \theta_a - (K_{t_1}) \theta_{bc} &= 0\\
I_{bc} \ddot{\theta}_{bc} - (K_{t_1}) \theta_a + (K_{t_1} + K_{t_2} + K'_{t_4}) \theta_{bc} - (K_{t_2}) \theta_d - (K'_{t_4}) \theta'_f &= 0\\
T_d \ddot{\theta}_d - (K_{t_2}) \theta_{bc} + (K_{t_2} + Kt_3) \theta_d - (K_{t_3}) \theta_e &= 0\\
I_e \ddot{\theta}_e - (Kt_3) \theta_d + (Kt_3) \theta_e &= 0\\
I'_f \ddot{\theta}_f - (K'_{t_4}) \theta_{bc} + (K'_{t_4}) \theta'_f &= 0
\end{aligned}\right\} \quad ...(6.16.5)$$

From the equations (6.16.5) the determinant form of the problem can be derived. Thus the matrix expressions for the system are

$$[M] = \begin{bmatrix} I_a & 0 & 0 & 0 & 0\\ 0 & I_{bc} & 0 & 0 & 0\\ 0 & 0 & I_d & 0 & 0\\ 0 & 0 & 0 & I_e & 0\\ 0 & 0 & 0 & 0 & I'_f \end{bmatrix}$$

Stiffness matrix [K] =

$$[K] = \begin{bmatrix} K_{t_1} & -K_{t_1} & 0 & 0 & 0\\ -K_{t_1} & (K_{t_1} + K_{t_2} + K'_{t_4}) & -K_{t_2} & 0 & -K'_{t_4}\\ 0 & -K_{t_2} & (K_{t_2} + K_{t_3}) & -K_{t_3} & 0\\ 0 & 0 & -K_{t_3} & +K_{t_3} & 0\\ 0 & -K'_{t_4} & 0 & 0 & K'_{t_4} \end{bmatrix}$$

where $I'_f = G^2 I_f$

$K'_{t_4} = G^2 K_{t_4}$

$I_{bc} = I_b + G^2 I_c$

where $G = \text{Gear ratio} = \dfrac{\text{Speed of rotor } F}{\text{Speed of rotor } A}$

The solution of the matrix form is given by

$$|\lambda I - C| = 0$$

where $\lambda = \omega^2$

and $[C] = [M]^{-1}[K]$ = Dynamic matrix

SOLVED EXAMPLES

EXAMPLE 6.1. *Determine the value of influence coefficients for the system shown in figure 6.14.*

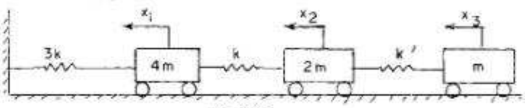


Fig. 6.14.

SOLUTION. (*i*) Here x_1, x_2 and x_3 denote the displacements of the masses 4m , 2m and m respectively. The influence coefficients of the system can be determined in terms of the stiffnesses of the spring. If we apply unit force at mass 4m and no force on the other masses *i.e.* $F_1 = 1$, $F_2 = 0 = F_3 = 0$, the deflection of the mass 4 m is given by

$$a_{11} = \text{force/stiffness} = \frac{1}{3k}$$

Since the other two masses move by the same amount of deflection, so by definition $a_{12} = a_{21} = a_{13} = a_{31} = 1/3k$

(*ii*) Now we apply a unit force at mass 2m and no force on other masses. Now two springs $3k$ and k are put to extension.

Both are connected in series, so equivalent stiffness is

$$\frac{1}{k_e} = \frac{1}{3k} + \frac{1}{k} = \frac{4}{3k}$$

So $a_{22} = \dfrac{4}{3k}$

(*iii*) The mass m covers the same displacement a_{22}.

So $a_{23} = a_{32} = \dfrac{4}{3k}$

(*iv*) At last, we apply a unit force at mass m and no force on other masses.

$$\frac{1}{k_e} = \frac{1}{3k} + \frac{1}{k} + \frac{1}{k} = \frac{7}{3k}$$

So $a_{33} = \dfrac{7}{3k}$

EXAMPLE 6.2. *Determine the flexibility influence coefficient for the system shown in figure 6.15.* $E = 2.1 \times 10^{11}\ N/m^2$.

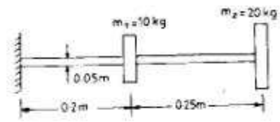


Fig. 6.15.

SOLUTION. Let us find $I = \dfrac{\pi}{64} d^4 = \dfrac{\pi}{64}(.05)^4 = 3.066 \times 10^{-7}\ \text{m}^4$

$$EI = 2.1 \times 10^{11} \times 3.066 \times 10^{-7} = 6.4 \times 10^4\ \text{Nm}^2$$

$$l = .20 + .25 = .45\ \text{m}$$

$$a_{11} = \frac{x^3}{3EI} = \frac{.2^3}{3 \times 2.1 \times 10^{11} \times 3.066 \times 10^{-7}}$$

$$= 4.1 \times 10^{-8}\ \text{m/N}$$

$$a_{12} = a_{21} = \frac{x^2(3l - x)}{6EI} = \frac{.2^2(.45 \times 3 - .2)}{6 \times 6.4 \times 10^4}$$

$$= 1.19 \times 10^{-7}\ \text{m/N}$$

$$a_{22} = \frac{l^3}{3EI} = \frac{.45^3}{3 \times 6.4 \times 10^4}$$

$$= 4.75 \times 10^{-7}\ \text{m/N}$$

EXAMPLE 6.3. *Determine the influence coefficient of the system shown in figure 6.16.*

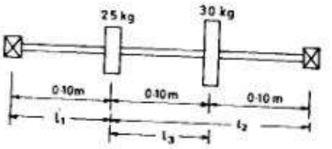


Fig. 6.16.

Take $E = 2 \times 10^{11}$ N/m², $I = 4 \times 10^{-7}$ m⁴.

SOLUTION. $a_{11} = \dfrac{l_1^2 l_2^2}{3EI(l_1 + l_2)} = \dfrac{.1^2 \times .2^2}{3 \times 2 \times 10^{11} \times 4 \times 10^{-7}(.3)}$

$$= 5.55 \times 10^{-9} \text{ m/N}$$

$$a_{12} = a_{21} = \frac{l_1 l_3 (l^2 - l_1^2 - l_3^2)}{6EIl} = \frac{.10 \times .10(.3^2 - .10^2 - .10^2)}{6EI \times .30}$$

$$= \frac{.10 \times .10(.09 - .01 - .01)}{6 \times 2 \times 10^{11} \times 4 \times 10^{-7} \times .30}$$

$$= 4.86 \times 10^{-9} \text{ m/N}$$

$$a_{22} = \frac{(l_1 + l_3)^2 (l_2 - l_3)^2}{3EIl} = \frac{(.2)^2(.10)^2}{3 \times 2 \times 10^{11} \times 4 \times 10^{-7} \times .30}$$

$$= 5.55 \times 10^{-9} \text{ m/N}$$

EXAMPLE 6.4. *Determine the influence coefficient of the spring-mass ystem shown in figure 6.17.*

SOLUTION. $a_{11} = a_{12} = a_{13} = \dfrac{1}{3k}$

$$a_{21} = a_{31} = \frac{1}{3k}$$

$$a_{22} = \frac{1}{3k} + \frac{1}{2k} = \frac{5}{6k}$$

(springs in series)

$$a_{22} = a_{32} = a_{23}$$

$$a_{33} = \frac{1}{3k} + \frac{1}{2k} + \frac{1}{k}$$

$$= \frac{2 + 3 + 6}{6k} = \frac{11}{6k}$$

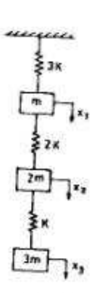


Fig. 6.17

EXAMPLE 6.5. *The vibrations of a cantilever are given by $y = y_l\left(1 - \cos\dfrac{\pi x}{2l}\right)$. Calculate the frequency with following data for the cantilever using Rayleigh's method. Modulus of elasticity of the material 2×10^{11} N/m²*

Second moment of area about bending axis 0.02 m⁴

Mass = 6×10^4 kg, Length = 30 m.

SOLUTION. According to Rayleigh's method, we have

$$\omega^2 = \frac{EI}{m} \frac{\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^l y^2 dx}$$

$$y = y_l\left(1 - \cos\frac{\pi x}{2l}\right)$$

$$\frac{d^2y}{dx^2} = y_l\left(\frac{\pi}{2l}\right)^2 \cos\frac{\pi x}{2l}$$

$$\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx = \int_0^l y_l^2 \left(\frac{\pi}{2l}\right)^4 \cos^2\frac{\pi x}{2l}\, dx$$

$$= y_l^2\left(\frac{\pi}{2l}\right)^4 \frac{1}{2}\int_0^l \left(1 + \cos\frac{\pi x}{l}\right) dx$$

$$= y_l^2\left(\frac{\pi}{2l}\right)^4 \cdot \frac{l}{2}$$

$$\int_0^l y^2 dx = \int_0^l y_l^2\left(1 - \cos\frac{\pi x}{2l}\right)^2 dx$$

$$= \int_0^l y_l^2\left(1 + \cos^2\frac{\pi x}{2l} - 2\cos\frac{\pi x}{2l}\right) dx$$

$$= y_l^2 l\left(\frac{3}{2} - \frac{4}{\pi}\right)$$

So $\omega^2 = \dfrac{EI}{m} \dfrac{y_l^2\left(\frac{\pi}{2l}\right)^4 \cdot \frac{l}{2}}{y_l^2 l\,(3/2 - 4/\pi)}$

$$= \frac{EI}{m} \frac{\pi^4}{2^5 l^4 (1.5 - 4/3.14)}$$

$$= \frac{EI}{ml^4}(13.4)$$

$$= \frac{2 \times 10^{11} \times .02}{6 \times 10^4 \times (30)^4}(13.4)$$

$$\omega^2 = 1.1028$$

$$\omega = 1.05 \text{ rad/sec}$$

$$f = \omega/2\pi = 0.167 \text{ Hz.}$$

EXAMPLE 6.6. *Determine the natural frequency of vibration for the system shown in figure 6.15.*

SOLUTION. $\omega^2 = \dfrac{g \sum_{i=1}^{2} m_i y_i}{\sum_{i=1}^{2} m_i y_i^2}$

We know $a_{11} = 4.1 \times 10^{-8}$ (From Example 6.2)

$$a_{12} = a_{21} = 1.19 \times 10^{-7} \text{ m/N}$$

$$a_{22} = 4.75 \times 10^{-7} \text{ m/N}$$

$$m_1 = 10 \text{ kg}, m_2 = 20 \text{ kg}$$

$$y_1 = m_1 g a_{11} + m_2 g a_{12}$$

$$y_2 = m_1 g a_{21} + m_2 g a_{22}$$

$$y_1 = 10 \times 9.81 \times 4.1 \times 10^{-8} + 20 \times 9.81 \times 1.19 \times 10^{-7}$$

$$= 2.737 \times 10^{-5} \text{ m}$$

$$y_2 = 10 \times 9.81 \times 1.19 \times 10^{-7} + 20 \times 9.81 \times 4.75 \times 10^{-7}$$

$$= 10.486 \times 10^{-5} \text{ m}$$

Now

$$g(m_1 y_1 + m_2 y_2) = 9.81(10 \times 2.737 \times 10^{-5} + 20 \times 10.486 \times 10^{-5})$$

$$= 2.3258 \times 10^{-2}$$

$$m_1 y_1^2 + m_2 y_2^2 = 10 \times (2.737 \times 10^{-5})^2 + 20 \times (10.486 \times 10^{-5})^2$$

$$= 2274.02 \times 10^{-10}$$

$$\omega^2 = \frac{g(m_1 y_1 + m_2 y_2)}{m_1 y_1^2 + m_2 y_2^2}$$

$$= \frac{2.3258 \times 10^{-2}}{2274.02 \times 10^{-10}}$$

$$= 1.02277 \times 10^5$$

$$\omega = 3.198 \times 10^2$$

$$= 319.8 \text{ rad/sec}$$

$$f = 50.92 \text{ Hz.}$$

EXAMPLE 6.7. *Determine the natural frequency of the system shown in figure 6.16 with the data as given in example 6.3.*

SOLUTION. $a_{11} = 5.55 \times 10^{-9}$ m/N

$$a_{12} = a_{21} = 4.86 \times 10^{-9} \text{ m/N}$$

$$a_{22} = 5.55 \times 10^{-9} \text{ m/N}$$

Given $m_1 = 25$ kg, $m_2 = 30$ kg

$$y_1 = (m_1 a_{11} + a_{12} m_2)\, g$$

$$y_2 = (m_1 a_{21} + m_2 a_{22})\, g$$

So $y_1 = (25 \times 5.55 \times 10^{-9} + 4.86 \times 10^{-9} \times 30)\, g$

$$= 28.455 \times 10^{-8}\, g$$

$$y_2 = (25 \times 4.86 \times 10^{-9} + 30 \times 5.55 \times 10^{-9})\, g$$

$$= 28.8 \times 10^{-8}\, g$$

$$\omega^2 = \frac{g(m_1 y_1 + m_2 y_2)}{m_1 y_1^2 + m_2 y_2^2}$$

$$g(m_1 y_1 + m_2 y_2) = 9.81(25 \times 28.455 \times 10^{-8} + 30 \times 28.8 \times 10^{-8})$$

$$= 1.5454 \times 10^{-4} \cdot g$$

$$m_1 y_1^2 + m_2 y_2^2 = 25 \times (28.455 \times 10^{-8})^2 + 30(28.8 \times 10^{-8})^2$$

$$= 4.5125 \times 10^{-12} \cdot g^2$$

$$\omega^2 = \frac{1.5454 \times 10^{-4}}{(4.5125 \times 10^{-12})\, g}$$

$$\omega = 1869.4 \text{ rad/sec}$$

$$f = 297.52 \text{ Hz.}$$

EXAMPLE 6.8. *Determine the natural frequencies of multi degree of freedom spring-mass system shown in figure 6.18.*

SOLUTION. Equations of motion can be written as

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_3) - k(x_1 - x_2)$$

$$m\ddot{x}_1 + 3kx_1 - kx_2 - kx_3 = 0$$

$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) - k(x_2 - x_3)$$

$$m\ddot{x}_2 + 3kx_2 - kx_3 - kx_1 = 0$$

and $$m\ddot{x}_3 = -\frac{k}{2}x_3 - k/2\,x_3 - k(x_3 - x_2) - k(x_3 - x_1)$$

$$m\ddot{x}_3 + 3kx_3 - kx_2 - kx_1 = 0$$

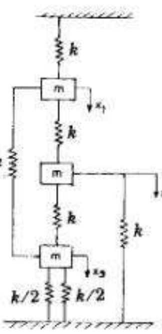


Fig. 6.18.

Assuming the motion to be harmonic, we get

$$x_1 = A_1 \sin(\omega t + \phi),\ \ddot{x}_1 = -\omega^2 A_1 \sin(\omega t + \phi)$$

$$x_2 = A_2 \sin(\omega t + \phi),\ \ddot{x}_2 = -\omega^2 A_2 \sin(\omega t + \phi)$$

$$x_3 = A_3 \sin(\omega t + \phi),\ \ddot{x}_3 = -\omega^3 A_3 \sin(\omega t + \phi)$$

Substituting the above values in the equations of motion, we get

$$(3k - m\omega^2)A_1 - kA_2 - kA_3 = 0$$

$$-kA_1 + (3k - m\omega^2)A_2 - kA_3 = 0$$

$$-kA_1 - kA_2 + (3k - m\omega^2)A_3 = 0$$

The frequency equation is obtained as

$$\begin{vmatrix} (3k - m\omega^2) & -k & -k \\ -k & (3k - m\omega^2) & -k \\ -k & -k & (3k - m\omega^2) \end{vmatrix} = 0$$

Expanding the above determinant, we get

$$\omega^6 - (9k/m)\omega^4 + (24k^2/m^2)\,\omega^2 - (16k^3/m^3) = 0$$

Solving it, we get three values of natural frequencies as

$$\omega_1 = \sqrt{\frac{k}{m}}\ \text{rad/sec}$$

$$\omega_2 = \omega_3 = 2\sqrt{k/m}\ \text{rad/sec.}$$

EXAMPLE 6.9. *A shaft of negligible weight 6 cm diameter and 5 metres long is simply supported at the ends and carries four weights 50 kg each at equal distance over the length of the shaft. Find the frequency of vibration by Dunkerley's method. Take $E = 2 \times 10^6$ kg/cm².*

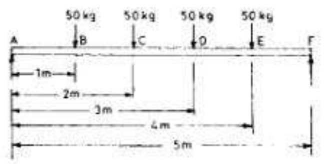


Fig. 6.19.

SOLUTION. The system is shown in figure 6.19.

$$I = \frac{\pi}{64}d^4 = \frac{\pi}{64} \times 6^4 = 63.585\ \text{cm}^4$$

The general expression for static deflection because of point load W is given by

$$y = \frac{Wl_1^2 l_2^2}{3EIl}$$

So static deflection at point B

$$y_B = \frac{50 \times 100^2 \times 400^2}{3 \times 2 \times 10^6 \times 63.585 \times 500} = 0.419\ \text{cm}$$

$$y_C = \frac{50 \times 200^2 \times 300^2}{3 \times 2 \times 10^6 \times 63.585 \times 500} = 0.943\ \text{cm}$$

$$y_D = \frac{50 \times 300 \times 300 \times 200 \times 200}{3 \times 2 \times 10^6 \times 63.585 \times 500} = 0.943\ \text{cm}$$

$$y_E = \frac{50 \times 400^2 \times 100^2}{3 \times 2 \times 10^5 \times 63.585 \times 500} = 0.419\ \text{cm}$$

General expression for natural frequency is given by

$$\omega = \sqrt{\frac{g}{y}}\ \text{rad/sec}$$

$$f = \omega/2\pi\ \text{Hz}$$

$$f_B = \frac{1}{2\pi}\sqrt{\frac{9.81}{.419 \times 10^{-2}}} = 7.7\ \text{Hz}$$

$$f_C = \frac{1}{2\pi}\sqrt{\frac{9.81}{.943 \times 10^{-2}}} = 5.13\ \text{Hz}$$

$$f_D = f_C = 5.13\ \text{Hz}$$

$$f_E = 7.7\ \text{Hz}$$

According to Dunkerley's relation, we know

$$\frac{1}{f^2} = \frac{1}{f_1^2} + \frac{1}{f_2^2} + \frac{1}{f_3^2} + \ldots$$

$$\frac{1}{f^2} = \frac{1}{(7.7)^2} + \frac{1}{(5.13)^2} + \frac{1}{(5.13)^2} + \frac{1}{(7.7)^2}$$

$$= .03373 + .07599$$

$$f^2 = 9.113$$

$$f = 3.01\ \text{Hz.}$$

EXAMPLE 6.10. *Using matrix method, termine the natural frequencies of the system own in figure 6.20.*

SOLUTION. The equations of motion are

$$2m\ddot{x}_1 + 2kx_1 + k(x_1 - x_2) = 0$$

$$2m\ddot{x}_2 + k(x_2 - x_1) + k(x_2 - x_3) = 0$$

$$m\ddot{x}_3 + k(x_3 - x_2) = 0$$

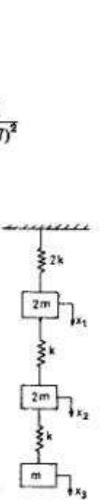


Fig. 6.20

Mass matrix $$[m] = \begin{bmatrix} 2m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}$$

$$m^{-1} = \frac{\text{adj}\ m}{|m|}$$

$$|m| = 2m \times 2m \times m = 4m^3$$

$$m^{-1} = \frac{\text{adj}\ m}{|m|} = \frac{1}{4m^3}\begin{bmatrix} 2m^2 & 0 & 0 \\ 0 & 2m^2 & 0 \\ 0 & 0 & 4m^2 \end{bmatrix}$$

We know that $[C] = [m]^{-1} \cdot [k]$

$$[C] = \frac{1}{2m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

$$[C] = \begin{bmatrix} \frac{3}{2}\frac{k}{m} & -\frac{k}{2m} & 0 \\ -\frac{k}{2m} & \frac{k}{m} & -\frac{k}{2m} \\ 0 & -\frac{k}{m} & \frac{k}{m} \end{bmatrix}$$

Let us assume $\omega^2 = \lambda$

$$|\lambda I - C| = 0$$

$$\begin{bmatrix} \lambda - \frac{3}{2}\frac{k}{m} & +\frac{k}{2m} & 0 \\ +\frac{k}{2m} & \lambda - \frac{k}{m} & +\frac{k}{2m} \\ 0 & +\frac{k}{m} & \lambda - \frac{k}{m} \end{bmatrix} = 0$$

Expanding it, we get

$$\lambda^3 - 3.5\,\lambda^2\frac{k}{m} + 3.25\lambda\frac{k^2}{m^2} - .5\frac{k^3}{m^3} = 0$$

Solving it for λ, we have

$$\lambda_1 = .2\left(\frac{k}{m}\right), \quad \omega_1 = \sqrt{.2\frac{k}{m}} = .44\sqrt{\frac{k}{m}}\ \text{rad/sec}$$

$$\lambda_2 = 1.87\left(\frac{k}{m}\right), \quad \omega_2 = 1.36\sqrt{\frac{k}{m}}\ \text{rad/sec}$$

$$\lambda_3 = 2\left(\frac{k}{m}\right), \quad \omega_3 = 1.414\sqrt{\frac{k}{m}}\ \text{rad/sec}$$

EXAMPLE 6.11. *Determine the natural frequencies and mode shapes of the system shown in figure 6.20, by matrix iteration method.*

SOLUTION. The influence coefficients are determined as

$$a_{11} = a_{12} = a_{13} = a_{21} = a_{31} = \frac{1}{2k}$$

$$a_{22} = a_{23} = a_{32} = \frac{3}{2k}$$

$$a_{33} = \frac{5}{2k}$$

The equations for the above system in terms of influence coefficients can be written as

$$x_1 = 2ma_{11}x_1\omega^2 + 2ma_{12}x_2\omega^2 + ma_{13}x_3\omega^2$$

$$x_2 = 2ma_{21}x_1\omega^2 + 2ma_{22}x_2\omega^2 + ma_{23}x_3\omega^2$$

$$x_3 = 2ma_{31}x_1\omega^2 + 2ma_{32}x_2\omega^2 + ma_{33}x_3\omega^2$$

The equation can be written in matrix form as

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = m\omega^2\begin{bmatrix} 2a_{11} & 2a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & a_{23} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$= m\omega^2 \begin{bmatrix} \frac{1}{k} & \frac{1}{k} & \frac{1}{2k} \\ \frac{1}{k} & \frac{3}{k} & \frac{3}{2k} \\ \frac{1}{k} & \frac{3}{k} & \frac{5}{2k} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 3 & 3/2 \\ 1 & 3 & 5/2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

First iteration

Let us assume $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$

$$\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{Bmatrix} 1+1+\frac{1}{2} \\ 1+3+\frac{3}{2} \\ 1+3+\frac{5}{2} \end{Bmatrix} = 2.5\,\frac{m\omega^2}{k} \begin{Bmatrix} 1 \\ 2.2 \\ 2.6 \end{Bmatrix}$$

Second iteration

$$\begin{Bmatrix} 1 \\ 2.2 \\ 2.6 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 3 & 3/2 \\ 1 & 3 & 5/2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2.2 \\ 2.6 \end{Bmatrix}$$

$$= 4.5\,\frac{m\omega^2}{k} \begin{Bmatrix} 1 \\ 2.555 \\ 3.133 \end{Bmatrix}$$

Third iteration

$$\begin{Bmatrix} 1 \\ 2.555 \\ 3.133 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 3 & 3/2 \\ 1 & 3 & 5/2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2.555 \\ 3.133 \end{Bmatrix}$$

$$= (5.12)\,\frac{m\omega^2}{k} \begin{Bmatrix} 1 \\ 2.61 \\ 3.22 \end{Bmatrix}$$

Fourth iteration

$$\begin{Bmatrix} 1 \\ 2.61 \\ 3.22 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 3 & 3/2 \\ 1 & 3 & 5/2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2.61 \\ 3.22 \end{Bmatrix}$$

$$\begin{Bmatrix} 1 \\ 2.61 \\ 3.22 \end{Bmatrix} = (5.22)\frac{m\omega^2}{k} \begin{Bmatrix} 1 \\ 2.61 \\ 3.23 \end{Bmatrix}$$

So $$1 = 5.22\,\frac{m\omega^2}{k}, \quad \omega^2 = \frac{1}{5.22}\,\frac{k}{m}$$

Thus $\omega_1 = .437\sqrt{\frac{k}{m}}$ rad/sec

To find the second principal mode, the orthogonality relation is used as

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 3 & 3/2 \\ 1 & 3 & 5/2 \end{bmatrix} \begin{bmatrix} 0 & \frac{-m_2}{m_1}\left(\frac{x_2}{x_1}\right) & \frac{-m_3}{m_1}\left(\frac{x_3}{x_1}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$= \frac{m\omega^2}{k} \begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 3 & 3/2 \\ 1 & 3 & 5/2 \end{bmatrix} \begin{bmatrix} 0 & -1\left(\frac{2.61}{1}\right) & -\frac{1}{2}(3.23) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$= \frac{m\omega^2}{k} \begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & .39 & -.11 \\ 0 & .39 & 1.89 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

First iteration

Let us say $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$

$$\begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & .39 & -.11 \\ 0 & .39 & 1.89 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

$$= \frac{m\omega^2}{k} \begin{bmatrix} 1.11 \\ .11 \\ -1.89 \end{bmatrix} = .11\left(\frac{m\omega^2}{k}\right) \begin{bmatrix} 10.9 \\ 1 \\ -17.18 \end{bmatrix}$$

Second iteration

$$\begin{Bmatrix} 10.9 \\ 1 \\ -17.18 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & .39 & -.11 \\ 0 & .39 & 1.89 \end{bmatrix} \begin{Bmatrix} 10.9 \\ 1 \\ -17.18 \end{Bmatrix}$$

$$= \frac{m\omega^2}{k} \begin{bmatrix} 17.46 \\ 2.28 \\ -32.08 \end{bmatrix} = (2.28)\,\frac{m\omega^2}{k} \begin{bmatrix} 7.65 \\ 1 \\ -14.00 \end{bmatrix}$$

Third iteration

$$\begin{Bmatrix} 7.65 \\ 1 \\ -14.0 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & .39 & -.11 \\ 0 & .39 & 1.89 \end{bmatrix} \begin{Bmatrix} 7.65 \\ 1 \\ -14 \end{Bmatrix}$$

$$= (1.93)\,\frac{m\omega^2}{k} \begin{Bmatrix} 7.2 \\ 1 \\ -13.5 \end{Bmatrix}$$

Fourth iteration

$$\begin{Bmatrix} 7.2 \\ 1 \\ -13.5 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & .39 & -.11 \\ 0 & .39 & 1.89 \end{bmatrix} \begin{Bmatrix} 7.2 \\ 1.0 \\ -13.5 \end{Bmatrix}$$

$$= 1.875\,\frac{m\omega^2}{k} \begin{bmatrix} 7.13 \\ 1 \\ -13.4 \end{bmatrix}$$

Fifth iteration

$$\begin{Bmatrix} 7.13 \\ 1 \\ -13.4 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & .39 & -.11 \\ 0 & .39 & 1.89 \end{bmatrix} \begin{Bmatrix} 7.13 \\ 1 \\ -13.4 \end{Bmatrix}$$

$$= (1.864)\,\frac{m\omega^2}{k} \begin{bmatrix} 7.11 \\ 1 \\ -13.37 \end{bmatrix}$$

So $$1 = 1.864\,\frac{m\omega^2}{k}$$

$$\omega_2 = .73\sqrt{k/m} \text{ rad/sec}$$

Similarly using orthogonality relation, we can find ω_3 which is found to be $\omega_3 = 1.41\sqrt{k/m}$

Example 6.12. *Determine the natural frequencies of the spring mass system shown in figure 6.3 by matrix method.* (P.U., 85)

Solution. The equations of motion can be written as

$$4m\ddot{x}_1 + 3kx_1 + k(x_1 - x_2) = 0$$
$$2m\ddot{x}_2 + k(x_2 - x_1) + k(x_2 - x_3) = 0$$
$$m\ddot{x}_3 + k(x_3 - x_2) = 0$$

Rearranging the above equations, we have

$$4m\ddot{x}_1 + 4kx_1 - kx_2 = 0$$
$$2m\ddot{x}_2 - kx_1 + 2kx_2 - kx_3 = 0$$
$$m\ddot{x}_3 - kx_2 + kx_3 = 0$$

The above equations can be put in matrix form as

$$\begin{bmatrix} 4m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 4k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

It is of the form

$$[M]\{\ddot{x}\} + [k]\{x\} = 0$$
$$\{\ddot{x}\} + [k][M]^{-1}\{x\} = 0$$
$$\{\ddot{x}\} + [C]\{x\} = 0$$

$$\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} \quad \text{and} \quad \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

For harmonic oscillation at frequency ω $\quad \{\ddot{x}\} = -\omega^2\{x\}$

The mass matrix [M] and the stiffness matrix [k] can be written as

$$[M] = \begin{bmatrix} 4m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}$$

$$[k] = \begin{bmatrix} 4k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

The dynamic matrix $[C] = [M]^{-1}[k]$

$$M^{-1} = \frac{\text{adj } M}{|M|}$$

$$|M| = 4m \times 2m \times m = 8m^3$$

$$\text{adj } M = \begin{bmatrix} 2m^2 & 0 & 0 \\ 0 & 4m^2 & 0 \\ 0 & 0 & 8m^2 \end{bmatrix}$$

$$M^{-1} = \frac{1}{4m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$[C] = \frac{1}{4m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 4k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

$$[C] = \begin{bmatrix} \frac{k}{m} & -\frac{k}{4m} & 0 \\ -\frac{k}{2m} & \frac{k}{m} & -\frac{k}{2m} \\ 0 & -\frac{k}{m} & \frac{k}{m} \end{bmatrix}$$

$$-\omega^2\{x\} + [C]\{x\} = 0$$

Let us assume $\omega^2 = \lambda$

$$|\lambda I - C| = 0$$

Then

$$\begin{bmatrix} \lambda - \frac{k}{m} & -\frac{k}{4m} & 0 \\ -\frac{k}{2m} & \lambda - \frac{k}{m} & -\frac{k}{2m} \\ 0 & -\frac{k}{m} & \lambda - \frac{k}{m} \end{bmatrix} = 0$$

Expanding it, we have

$$\lambda^3 - 3\lambda^2\frac{k}{m} + \frac{19}{8}\lambda\frac{k^2}{m^2} - \frac{3}{8}\frac{k^3}{m^3} = 0$$

Solving the equation for λ, we get three values which give three values of natural frequencies

$$\omega_1 = 0.46\sqrt{\frac{k}{m}}\ \text{rad/sec}$$

$$\omega_2 = \sqrt{\frac{k}{m}}\ \text{rad/sec}$$

and

$$\omega_3 = 1.32\sqrt{\frac{k}{m}}\ \text{rad/sec.}$$

EXAMPLE 6.13. *Three rail bogies are connected by two springs of stiffness 40×10^5 N/m each. The mass of each bogey is 20×10^3 kg. Determine the frequencies of vibration. Neglect friction between the wheels and rails.*

SOLUTION. Refer figure 6.21.

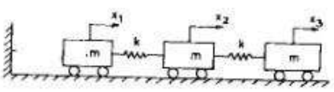


Fig. 6.21.

The equations of motion can be written as

$$m\ddot{x}_1 + k(x_1 - x_2) = 0$$

$$m\ddot{x}_2 + k(x_2 - x_1) + k(x_2 - x_3) = 0$$

$$m\ddot{x}_3 + k(x_3 - x_2) = 0$$

Let us assume the oscillation of the form $x_i = A_i \sin \omega t$ and substituting this value in the above equations, we have

$$(k - m\omega^2)A_1 - kA_2 = 0$$

$$-kA_1 + (2k - m\omega^2)A_2 - kA_3 = 0$$

$$-kA_2 + (k - m\omega^2)A_3 = 0$$

The frequency equation is obtained by putting the determinant of coefficients of A's equal to zero,

$$\begin{bmatrix} k - m\omega^2 & -k & 0 \\ -k & (2k - m\omega^2) & -k \\ 0 & -k & k - m\omega^2 \end{bmatrix} = 0$$

$$(k - m\omega^2)(m^2\omega^4 - 3km\omega^2) = 0$$

which yields

$$\omega_1 = 0, \quad \omega_2 = \sqrt{\frac{k}{m}}, \quad \omega_3 = \sqrt{\frac{3k}{m}}$$

Substituting the numerical values, we get

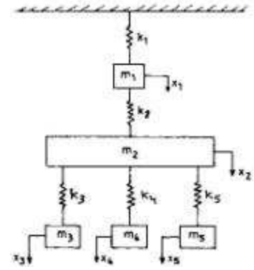
$$\omega_2 = \sqrt{\frac{400 \times 10^4}{20 \times 10^3}} = 14.14\ \text{rad/sec}$$

$$f_2 = 2.25\ \text{Hz}$$

$$\omega_3 = \sqrt{\frac{3 \times 40 \times 10^5}{20 \times 10^3}} = 24.49\ \text{rad/sec}$$

$$f_3 = \frac{\omega_3}{2\pi} = 3.9\ \text{Hz.}$$

EXAMPLE 6.14. *A five spring mass branched system is shown in figure 6.22. The masses are moving in the vertical direction only, derive the frequency equation of the system.*



SOLUTION. The equations of motion can be written as

$$m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3(x_2 - x_3) + k_4(x_2 - x_4) + k_5(x_2 - x_5) = 0$$

$$m_3\ddot{x}_3 + k_3(x_3 - x_2) = 0$$

$$m_4\ddot{x}_4 + k_4(x_4 - x_2) = 0$$

$$m_5\ddot{x}_5 + k_5(x_5 - x_2) = 0$$

Rearranging above equations, we get

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 + (k_2 + k_3 + k_4 + k_5)x_2 - k_3x_3 - k_4x_4 - k_2x_1 - k_5x_5 = 0$$

$$m_3\ddot{x}_3 + k_3x_3 - k_3x_2 = 0$$

$$m_4\ddot{x}_4 + k_4x_4 - k_4x_2 = 0$$

$$m_5\ddot{x}_5 + k_5x_5 - k_5x_2 = 0$$

Assuming the motion of the form $x_i = A_i \sin(\omega t + \phi)$ and substituting the values of $\ddot{x}_1, \ddot{x}_2$ etc. in the above equations, we get

$$(k_1 + k_2 - m_1\omega^2)A_1 - k_2A_2 = 0$$

$$-k_2A_1 + (k_2 + k_3 + k_4 + k_5 - m_2\omega^2)A_2 - k_3A_3 - k_4A_4 - k_5A_5 = 0$$

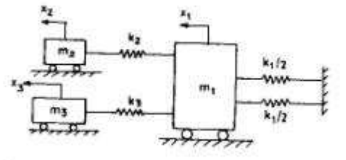
$$(k_3 - m_3\omega^2)A_3 - k_3A_2 = 0$$

$$(k_4 - m_4\omega^2)A_4 - k_4A_2 = 0$$

$$(k_5 - m_5\omega^2)A_5 - k_5A_2 = 0$$

The frequency equation is obtained by equating the coefficients of A_1, A_2, A_3, A_4 and A_5 to zero in determinant.

EXAMPLE 6.15. *Find the lowest natural frequency of the system shown in figure 6.23.*



Assume $k_1 = 7k$, $k_2 = 5k$, $k_3 = 5k$

$m_1 = 4m$, $m_2 = 3m$, $m_3 = 2m$.

SOLUTION. The equations of motion can be written as

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) - k_3(x_1 - x_3)$$

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1)$$

$$m_3\ddot{x}_3 = -k_3(x_3 - x_1)$$

Rearranging the above, we get

$$m_1\ddot{x}_1 + (k_1 + k_2 + k_3)x_1 - k_2x_2 - k_3x_3 = 0$$

$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$$

$$m_3\ddot{x}_3 - k_3x_1 + k_3x_3 = 0$$

Putting in the determinant form

$$\begin{vmatrix} (k_1 + k_2 + k_3 - m_1\omega^2) & -k_2 & -k_3 \\ -k_2 & k_2 - m_2\omega^2 & 0 \\ -k_3 & 0 & k_3 - m_3\omega^2 \end{vmatrix} = 0$$

Substituting $k_1 = 7k$, $k_2 = 5k$, $k_3 = 5k$

and $m_1 = 4$ m, $m_2 = 3$ m and $m_3 = 2$ m

$$\begin{vmatrix} (17k - 4m\omega^2) & -5k & -5k \\ -5k & 5k - 3m\omega^2 & 0 \\ -5k & 0 & 5k - 2m\omega^2 \end{vmatrix} = 0$$

Expanding the determinant, we get

$$175k^3 - 400mk^2\omega^2 + 202km^2\omega^4 - 24m^3\omega^6 = 0$$

Solving the equation for ω

$$\omega = 0.78\sqrt{\frac{k}{m}}\ \text{rad/sec}$$

EXAMPLE 6.16. *Determine the lowest natural frequency of the system shown in figure 6.23 by matrix method.*

SOLUTION. The differential equations can be written as

$$m_1\ddot{x}_1 + (k_1 + k_2 + k_3)x_1 - k_2x_2 - k_3x_3 = 0$$

$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$$

$$m_3\ddot{x}_3 - k_3x_1 + k_3x_3 = 0$$

Substituting the values of masses and stiffnesses, we get

$$4m\ddot{x}_1 + 17kx_1 - 5kx_2 - 5kx_3 = 0$$

$$3m\ddot{x}_2 - 5kx_1 + 5kx_2 = 0$$

$$2m\ddot{x}_3 - 5kx_1 + 5kx_3 = 0$$

Putting the above equations in matrix form, we get

$$\begin{bmatrix} 4m & 0 & 0 \\ 0 & 3m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 17k & -5k & -5k \\ -5k & 5k & 0 \\ -5k & 0 & 5k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$$

Dynamic matrix $[C]$ can be written as

$$[C] = [M]^{-1}[k]$$

$$[M]^{-1} = \frac{\text{adj } M}{|M|} = \frac{1}{24m^3} \begin{bmatrix} 6m^2 & 0 & 0 \\ 0 & 8m^2 & 0 \\ 0 & 0 & 12m^2 \end{bmatrix}$$

$$[C] = \frac{1}{24m^3} \begin{bmatrix} 6m^2 & 0 & 0 \\ 0 & 8m^2 & 0 \\ 0 & 0 & 12m^2 \end{bmatrix} \begin{bmatrix} 17k & -5k & -5k \\ -5k & 5k & 0 \\ -5k & 0 & 5k \end{bmatrix}$$

$$= \begin{bmatrix} \frac{17}{4}\frac{k}{m} & \frac{-5}{4}\frac{k}{m} & \frac{-5}{4}\frac{k}{m} \\ \frac{-5}{3}\frac{k}{m} & \frac{5}{3}\frac{k}{m} & 0 \\ \frac{-5}{2}\frac{k}{m} & 0 & \frac{5}{2}\frac{k}{m} \end{bmatrix} \qquad (\because \ddot{x} = -\omega^2 x)$$

$$-\omega^2\{x\} + [C]\{x\} = 0 \quad \text{or} \quad |\lambda I - C| = 0$$

where $\omega^2 = \lambda$

$$\begin{vmatrix} \lambda - \frac{17k}{4m} & \frac{-5}{4}\frac{k}{m} & \frac{-5}{4}\frac{k}{m} \\ \frac{-5k}{3m} & \lambda - \frac{5}{3}\frac{k}{m} & 0 \\ \frac{-5}{2}\frac{k}{m} & 0 & \lambda - \frac{5}{2}\frac{k}{m} \end{vmatrix} = 0$$

$$\left(\lambda - \frac{17}{4}\frac{k}{m}\right)\left(\lambda - \frac{5}{3}\frac{k}{m}\right)\left(\lambda - \frac{5}{2}\frac{k}{m}\right) + \frac{5}{4}\frac{k}{m}\left(\frac{-5}{3}\frac{k}{m}\right)\left(\lambda - \frac{5}{2}\frac{k}{m}\right)$$
$$\frac{-5}{4}\frac{k}{m}\;\frac{5}{2}\frac{k}{m}\left(\lambda - \frac{5}{3}\frac{k}{m}\right) = 0$$

$$\lambda^3 - \frac{101}{12}\lambda^2\frac{k}{m} + \frac{310}{24}\lambda\frac{k^2}{m^2} - \frac{175}{24}\frac{k^3}{m^3} = 0$$

$$\lambda = .6084\frac{k}{m}$$

So $$\omega = \sqrt{.6084\frac{k}{m}} = .78\sqrt{\frac{k}{m}} \text{ rad/sec.}$$

EXAMPLE 6.17. A steel shaft of diameter 10 cm is carrying three masses 5 kg, 7.5 kg and 14 kg respectively as shown in figure 6.24.

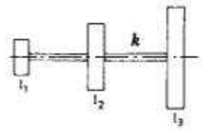


Fig. 6.24.

The distances between the rotors are 0.70 m. Determine the natural frequencies of torsional vibrations. The radii of gyration of three rotors are 0.20, 0.30 and 0.40 m respectively.

Take $G = 9 \times 10^8\ N/m^2$.

SOLUTION. Let us assume that θ_1, θ_2 and θ_3 are the angular displacements of the three rotors having moment of inertias as I_1, I_2 and I_3.

The equations of motion can be written as

$$I_1\ddot{\theta}_1 = -k(\theta_1 - \theta_2)$$
$$I_2\ddot{\theta}_2 = k(\theta_1 - \theta_2) - k(\theta_2 - \theta_3)$$
$$I_3\ddot{\theta}_3 = +k(\theta_2 - \theta_3)$$

Assuming the motion of the form $\theta_i = A_i \cos \omega t$

So $\ddot{\theta}_1 = -\omega^2 A_1$ and so on.

Substituting these values in the above equations, we get

$$(k - I_1\omega^2)A_1 - kA_2 = 0$$
$$-kA_1 + (2k - I_2\omega^2)A_2 - kA_3 = 0$$
$$-kA_2 + (k - I_3\omega^2)A_3 = 0$$

Equating the coefficients of A_1, A_2 and A_3 in determinant form to zero.

$$\begin{vmatrix} k - I_1\omega^2 & -k & 0 \\ -k & 2k - I_2\omega^2 & -k \\ 0 & -k & k - I_3\omega^2 \end{vmatrix} = 0$$

Expanding the determinant, we get

$$\omega^2\left[\omega^4 - \left(\frac{k}{I_3} + \frac{2k}{I_2} + \frac{k}{I_1}\right)\omega^2 + \frac{k^2}{I_1I_2} + \frac{k^2}{I_1I_3} + \frac{k^2}{I_2I_3}\right] = 0$$

So $\omega_1 = 0$

According to strength of materials we know

$$\frac{T}{I} = \frac{G\theta}{l}$$

$$k = \frac{T}{\theta} = \frac{GI}{l} = \text{Stiffness of shaft}$$

$$k = \frac{9 \times 10^8}{.7}\;\frac{\pi}{32} \times .10^4 = 1.2616 \times 10^4 \text{ N-m/rad}$$

$$I_1 = \frac{1}{2}m_1k_1^2 = \frac{1}{2} \times 5 \times .2^2 = .10 \text{ kg- m sec}^2$$

$$I_2 = \frac{1}{2}m_2k_2^2 = \frac{1}{2} \times 7.5 \times .3^2 = 0.3375 \text{ kg- m sec}^2$$

$$I_3 = \frac{1}{2}m_3k_3^2 = \frac{1}{2} \times 14 \times .4^2 = 1.12 \text{ kg - m sec}^2$$

$$\omega^4 - \omega^2\left(\frac{1.2616 \times 10^4}{1.12} + \frac{2 \times 1.2616 \times 10^4}{0.3375} + \frac{1.2616}{.10}\right)$$
$$+ \frac{(1.2616 \times 10^4)^2}{.1 \times .3375} + \frac{(1.2616 \times 10^4)^2}{.1 \times 1.12} + \frac{(1.2616 \times 10^4)^2}{.3375 \times 1.12}$$

$$\omega^4 - \omega^2 \times 1.2616 \times 10^4(.8928 + 5.9259 + 10) + 65.581121 \times 10^8 = 0$$

$$\omega^4 - 21.2184 \times 10^4\omega^2 + 65.581121 \times 10^8 = 0$$

$$\omega = \frac{21.2184 \times 10^4 \pm \sqrt{(21.2184 \times 10^4)^2 - 4 \times 65.5811 \times 10^8}}{2}$$

$$= \frac{21.2184 \times 10^4 \pm 13.7075 \times 10^4}{2}$$

$$\omega_1 = 0$$
$$\omega_3 = 417.88 \text{ rad/sec}$$
$$\omega_2 = 193.78 \text{ rad/sec.}$$

EXAMPLE 6.18. Find the natural frequencies and mode shapes of the system shown in figure 6.2 for $k_1 = k_2 = k_3$ and $m_1 = m_2 = m_3$ using the matrix iteration method. (P.U., 93, ME)

SOLUTION. The differential equations in terms of influence coefficients can be written as

$$x_1 = a_{11}m_1x_1\omega^2 + a_{12}m_2x_2\omega^2 + a_{13}m_3x_3\omega^2$$
$$x_2 = a_{21}m_1x_1\omega^2 + a_{22}m_2x_2\omega^2 + a_{23}m_3x_3\omega^2$$
$$x_3 = a_{31}m_1x_1\omega^2 + a_{32}m_2x_2\omega^2 + a_{33}m_3x_3\omega^2$$

The influence coefficients are

$$a_{11} = \frac{1}{k} = a_{12} = a_{13} = a_{21} = a_{31}$$
$$a_{22} = \frac{1}{k} + \frac{1}{k} = \frac{2}{k} = a_{23} = a_{32}$$
$$a_{33} = \frac{1}{k} + \frac{1}{k} + \frac{1}{k} = \frac{3}{k}$$

Substituting the values of influence coefficients in the above equations, we get

$$x_1 = \frac{m}{k}x_1\omega^2 + \frac{m}{k}x_2\omega^2 + \frac{m}{k}x_3\omega^2$$
$$x_2 = \frac{m}{k}x_1\omega^2 + \frac{2m}{k}x_2\omega^2 + \frac{2m}{k}x_3\omega^2$$
$$x_3 = \frac{m}{k}x_1\omega^3 + \frac{2m}{k}x_2\omega^2 + \frac{3m}{k}x_3\omega^2$$

This can be written in matrix form as

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{\omega^2 m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

To start with the iteration process, let us assume

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

First iteration

$$\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \frac{m\omega^2}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \frac{m\omega^2}{k}(6)\begin{bmatrix} 1 \\ 1.833 \\ 2.33 \end{bmatrix}$$

Second iteration

$$\begin{Bmatrix} 1 \\ 1.83 \\ 2.33 \end{Bmatrix} = \frac{m\omega^2}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{Bmatrix} 1 \\ 1.83 \\ 2.33 \end{Bmatrix} = \frac{m\omega^2}{k}\begin{bmatrix} 5.16 \\ 9.32 \\ 11.62 \end{bmatrix}$$

$$\begin{Bmatrix} 1 \\ 1.806 \\ 2.25 \end{Bmatrix} = \frac{m\omega^2}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{Bmatrix} 1 \\ 1.806 \\ 2.25 \end{Bmatrix}$$

$$= \frac{m\omega^2}{k}\begin{bmatrix} 5.056 \\ 9.112 \\ 11.362 \end{bmatrix} = (5.506)\frac{m\omega^2}{k}\begin{bmatrix} 1 \\ 1.80 \\ 2.24 \end{bmatrix}$$

So $$1 = 5.506\frac{m\omega^2}{k}$$

$$\omega_1 = 0.426\sqrt{\frac{k}{m}}\ \text{rad/sec}$$

To find the second mode, the orthogonality relation is written as

$$m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0$$
$$m(1)A_2 + m(1.8)B_2 + m(2.24)C_2 = 0$$
$$A_2 + 1.8B_2 + 2.24C_2 = 0$$
$$A_2 = -1.8B_2 - 2.24C_2$$
$$B_2 = B_2,\ C_2 = C_2$$

and in matrix form

$$\begin{Bmatrix} A_2 \\ B_2 \\ C_2 \end{Bmatrix} = \begin{bmatrix} 0 & -1.8 & -2.24 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \end{Bmatrix}$$

Combining it with the matrix equation for the first mode, it gives convergence to second mode.

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1.8 & -2.24 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$= \frac{m\omega^2}{k}\begin{bmatrix} 0 & -0.8 & -1.24 \\ 0 & 0.20 & -0.24 \\ 0 & 0.20 & 0.76 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

First iteration

Assume $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

$$\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{m\omega^2}{k}\begin{bmatrix} 0 & -0.8 & -1.24 \\ 0 & 0.20 & -0.24 \\ 0 & 0.20 & 0.76 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= \frac{m\omega^2}{k}\begin{bmatrix} -204 \\ -.04 \\ 0.96 \end{bmatrix} = (0.04)\frac{m\omega^2}{k}\begin{bmatrix} -51 \\ -1 \\ 24 \end{bmatrix}$$

Second iteration

$$\begin{Bmatrix} -51 \\ -1 \\ 24 \end{Bmatrix} = \frac{m\omega^2}{k}(-5.96)\begin{bmatrix} -4.86 \\ -1.0 \\ 3.02 \end{bmatrix}$$

Third iteration

$$\begin{Bmatrix} -4.86 \\ -1.0 \\ 3.02 \end{Bmatrix} = \frac{m\omega^2}{k}\begin{bmatrix} 0 & -0.8 & -1.24 \\ 0 & 0.20 & -.24 \\ 0 & 0.20 & 0.76 \end{bmatrix}\begin{Bmatrix} -4.86 \\ -1.0 \\ 3.02 \end{Bmatrix}$$

$$= \frac{m\omega^2}{k}\begin{bmatrix} .80 - 1.24\times 3.02 \\ -.20 - .24\times 3.02 \\ -.2 + .76\times 3.02 \end{bmatrix} = \frac{m\omega^2}{k}\begin{bmatrix} 2.94 \\ .92 \\ -2.09 \end{bmatrix}$$

$$= (2.94)\frac{m\omega^2}{k}\begin{bmatrix} 1.0 \\ 0.31 \\ -0.71 \end{bmatrix}$$

Fourth iteration

$$\begin{Bmatrix} 1.0 \\ 0.31 \\ -0.71 \end{Bmatrix} = \frac{m\omega^2}{k}\begin{bmatrix} 0 & -.8 & -1.24 \\ 0 & .20 & -.24 \\ 0 & .20 & 0.76 \end{bmatrix}\begin{Bmatrix} 1.0 \\ 0.31 \\ -.71 \end{Bmatrix}$$

$$= \frac{m\omega^2}{k}(.632)\begin{bmatrix} 1.0 \\ 0.36 \\ -.76 \end{bmatrix}$$

Fifth iteration

$$\begin{Bmatrix} 1.0 \\ 0.36 \\ -.76 \end{Bmatrix} = \frac{m\omega^2}{k}\begin{bmatrix} 0 & -.8 & -1.24 \\ 0 & 0.20 & -.24 \\ 0 & 0.20 & 0.76 \end{bmatrix}\begin{Bmatrix} 1.0 \\ 0.36 \\ -.76 \end{Bmatrix}$$

$$= .6544\frac{m\omega^2}{k}\begin{bmatrix} 1.0 \\ .38 \\ -.77 \end{bmatrix}$$

Sixth iteration

$$\begin{Bmatrix} 1.0 \\ .38 \\ -.77 \end{Bmatrix} = \frac{m\omega^2}{k}\begin{bmatrix} 0 & -.8 & -1.24 \\ 0 & .20 & -.24 \\ 0 & .20 & 0.76 \end{bmatrix}\begin{Bmatrix} 1.0 \\ .38 \\ -.77 \end{Bmatrix}$$

$$= .6508\frac{m\omega^2}{k}\begin{bmatrix} 1.0 \\ 0.40 \\ -.78 \end{bmatrix}$$

Seventh iteration

$$\begin{Bmatrix} 1.0 \\ 0.40 \\ -.78 \end{Bmatrix} = \frac{m\omega^2}{k}\begin{bmatrix} 0 & -.8 & -1.24 \\ 0 & .20 & -.24 \\ 0 & .20 & 0.76 \end{bmatrix}\begin{Bmatrix} 1.0 \\ .40 \\ -.78 \end{Bmatrix}$$

$$= .647\frac{m\omega^2}{k}\begin{bmatrix} 1.0 \\ .41 \\ -.79 \end{bmatrix}$$

Eighth iteration

$$\begin{Bmatrix} 1.0 \\ .41 \\ -.79 \end{Bmatrix} = \frac{m\omega^2}{k}\begin{bmatrix} 0 & -.8 & -1.24 \\ 0 & .20 & -.24 \\ 0 & .20 & .76 \end{bmatrix}\begin{Bmatrix} 1.0 \\ .41 \\ -.79 \end{Bmatrix}$$

$$= \frac{m\omega^2}{k}\begin{bmatrix} -.8\times .41 + 1.24\times .79 \\ .2\times .41 + .24\times .79 \\ .2\times .41 - .76\times .79 \end{bmatrix}$$

$$= .6516\frac{m\omega^2}{k}\begin{bmatrix} 1.0 \\ .41 \\ .79 \end{bmatrix}$$

So, $\quad 1 = .6516\dfrac{m\omega^2}{k}$

$$\omega_2 = 1.23\sqrt{\frac{k}{m}}\ \text{rad/sec}$$

Similarly, we can determine third natural frequency and mode shapes.

EXAMPLE 6.19. *Determine the natural frequency of the spring-mass system shown in figure 6.2. Take $m_1 = m_2 = m_3 = m$ and $k_1 = k_2 = k_3 = k$.*

Use Stodola's method..

SOLUTION. Influence coefficients are

$$a_{11} = a_{12} = a_{13} = a_{21} = a_{31} = \frac{1}{k}$$

$$a_{22} = a_{23} = a_{32} = \frac{2}{k}$$

$$a_{33} = \frac{3}{k}$$

First trial

Let us assume the deflections as

$$x_1 = 1,\ x_2 = 1,\ x_3 = 1$$

So $\quad \dfrac{x_2}{x_1} = 1$

$$\frac{x_3}{x_1} = 1$$

The inertia forces are given by

$$F_1 = m\,\omega^2 x_1 = m\omega^2$$
$$F_2 = m\omega^2 x_2 = m\omega^2$$
$$F_3 = m\omega^2 x_3 = m\,\omega^2$$

The corresponding deflections are given by

$$x'_1 = F_1a_{11} + F_2a_{12} + F_3a_{13} = \frac{3m\omega^2}{k}$$

$$x'_2 = F_1a_{21} + F_2a_{22} + F_3a_{23} = \frac{m\omega^2}{k} + m\,\omega^2\cdot\frac{2}{k} + \frac{m\omega^2 2}{k}$$

$$= \frac{5m\omega^2}{k}$$

$$x'_3 = F_1a_{31} + F_2a_{32} + F_3a_{33}$$

$$\frac{m\omega^2}{k} + m\omega^2\cdot\frac{2}{k} + m\omega^2\cdot\frac{3}{k} = \frac{6m\omega^2}{k}$$

The ratios are

$$x'_1 : x'_2 : x'_3,\quad 3 : 5 : 6$$
$$\text{or } 1 : 1.66 : 2$$

The ratios are much different from the starting values of ratios.

Second trial

$$F'_1 = m_1\omega^2 x'_1 = m\omega^2$$
$$F'_2 = m_2\omega^2 x'_2 = 1.66\ m\omega^2$$
$$F'_3 = m_3\,\omega^2 x'_3 = 2m\omega^2$$
$$x''_1 = F_1'a_{11} + F_2'a_{12} + F_3'a_{13}$$
$$\frac{m\omega^2}{k} + \frac{1.66\ m\omega^2}{k} + \frac{2m\omega^2}{k} = 4.66\ \frac{m\omega^2}{k}$$
$$x_2'' = F_1'a_{21} + F_2'a_{22} + F_3'a_{23}$$
$$\frac{m\omega^2}{k} + \frac{1.66\ m\omega^2\times 2}{k} + \frac{2m\omega^2}{k}.2 = \frac{8.32}{k}\ m\omega^2$$
$$x_3'' = F_1'a_{31} + F_2'a_{32} + F_3'a_{33}$$
$$\frac{m\omega^2}{k} + 1.66\ \frac{m\omega^2}{k}\times 2 + \frac{2m\omega^2}{k}\times 3 = 10.32\ \frac{m\omega^2}{k}$$

So $x''_1 : x''_2 : x''_3 = 4.66 : 8.32 : 10.32$

$$1 : 1.78 : 2.21$$

The ratios are again different from the starting ratios.

Third trial

The inertia forces are

$$F_1'' = m\,\omega^2 x_1'' = m\omega^2$$
$$F_2'' = m\omega^2 x''_2 = m\omega^2\times 1.78$$
$$F_3'' = m\omega^2 x''_3 = 2.21\ m\omega^2$$

The deflections are

$$x_1''' = F_1''a_{11} + F_2''a_{12} + F_3''a_{13}$$

$$\frac{m\omega^2}{k} + 1.78\frac{m\omega^2}{k} + 2.21\frac{m\omega^2}{k} = 4.99\frac{m\omega^2}{k}$$

$$x_2''' = F_1''a_{21} + F_2''a_{22} + F_3''a_{23}$$

$$\frac{m\omega^2}{k} + 1.78\frac{m\omega^2}{k}\times 2 + 2.21\frac{m\omega^2}{k}\times 2 = 8.98\frac{m\omega^2}{k}$$

$$x_3''' = F_1''a_{31} + F_2''a_{32} + F_3''a_{33}$$

$$= \frac{m\omega^2}{k} + 1.78\, m\omega^2\frac{2}{k} + 2.21\frac{m\omega^2}{k}\cdot\frac{3}{k} = 11.19\frac{m\omega^2}{k}$$

The ratios are 4.99 : 8.98 : 11.19

or 1 : 1.79 : 2.24

The values of ratios are approximately close to the starting values of ratios for this trial. The assumed and derived values of deflections are approximately equal.

Thus $x_1'' = x_1'''$

or $$1 = 4.99\frac{m\omega^2}{k}$$

$$\omega = 0.44\sqrt{\frac{k}{m}} \text{ rad/sec}$$

This is the same value of frequency which we got in article 6.4 by matrix method.

Example 6.20. *Use Stodola's method to find the natural frequency of the system shown in figure 6.25.*

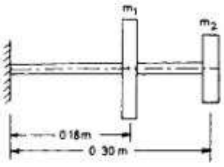


Fig. 6.25

$E = 1.96\times 10^{11}\ N/m^2$, $I = 4\times 10^{-7}\ m^2$, $m_1 = 100\ kg$, $m_2 = 50\ kg$.

(P.U., M.E. 93)

Solution. We find influence coefficients as

$$a_{11} = \frac{.18^3}{3EI} = 2.4795\times 10^{-8}$$

$$a_{12} = a_{21} = \frac{.18^2(3l - .18)}{6EI} = 4.959\times 10^{-8}$$

$$a_{22} = \frac{.3\times .3\times .3}{3EI} = \frac{(.3)^3}{3\times 1.96\times 10^{11}\times 4\times 10^{-7}} = 11.479\times 10^{-8}$$

First trial

Inertia forces are

$F_1 = m_1\,\omega^2 x_1 = 100\omega^2$ (Assuming $x_1 = 1$ and $x_2 = 1$)

$F_2 = m_2\,\omega^2 x_2 = 50\omega^2$

Deflections are calculated as

$$x_1' = F_1a_{11} + F_2a_{12}$$
$$= 100\omega^2\times 2.4795\times 10^{-8} + 50\omega^2\times 4.959\times 10^{-8}$$
$$= 495.9\times 10^{-8}\,\omega^2$$
$$x_2' = F_1a_{21} + F_2a_{22}$$
$$= 100\omega^2\times 4.959\times 10^{-8} + 50\omega^2\times 11.479\times 10^{-8}$$
$$= 1069.85\times 10^{-8}\,\omega^2$$

The ratios of calculated deflections are

$$\frac{x_2'}{x_1'} = \frac{1069.85}{495.9} = \frac{2.15}{1.0}$$

This ratio is much different from the assumed ratio.

Second trial

$$F_1' = m_1\,\omega^2 x_1' = 100\,\omega^2$$
$$F_2' = m_2\,\omega^2 x_2' = 50\,\omega^2\times 2.15 = 107.5\,\omega^2$$
$$x''_1 = F_1'a_{11} + F_2'a_{12}$$
$$= 100\omega^2\times 2.4795\times 10^{-8} + 107.5\,\omega^2\times 4.959\times 10^{-8}$$
$$= 781\times 10^{-8}\,\omega^2$$
$$x_2'' = F_1'a_{21} + F_2'a_{22}$$
$$= 100\,\omega^2\times 4.959\times 10^{-8} + 107.5\,\omega^2\times 11.479\times 10^{-8}$$
$$= 1729.89\times 10^{-8}\,\omega^2$$

So $x''_2 : x''_1 = 2.21 : 1$

The ratio is quite different from the assumed ratio in the start of this trial.

Third trial

We get $x_1''' = 795.9\times 10^{-8}\,\omega^2$

and $x_2''' = 1764.32\times 10^{-8}\,\omega^2$

The ratio of deflections is

$$\frac{x_2'''}{x_1'''} = 2.21 : 1$$

This ratio is equal to the starting value for this trial. Thus the assumed and calculated values of ratio are equal.

So $x_1''' = 1$ (assumed)

$x_1''' = 795.9\times 10^{-8}\,\omega^2$ (calculated)

or $$1 = 795.9\times 10^{-8}\,\omega^2$$

$$\omega = 354.5 \text{ rad/sec}$$

$$f = \frac{\omega}{2\pi} = 56.45 \text{ Hz.}$$

Example 6.21. *Using Holzer method to find the natural frequency of the system shown in figure 6.2. Assume $m_1 = m_2 = m_3 = 1$ kg and $k_1 = k_2 = k_3 = 1$ N/m.*

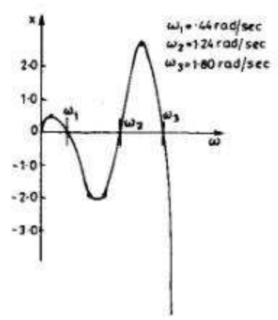
Solution. Assuming the initial displacement $x_1 = 1$ and natural frequency $\omega = .30$ rad/sec.

$$\omega^2 = .3\times .3 = .09$$

$$x_2 = x_1 - \frac{m_1x_1\,\omega^2}{k_1}$$

$$= 1 - \frac{1}{1}\times .09 = .91$$

$$x_3 = x_2 - \omega^2\frac{(m_1x_1 + m_2x_2)}{k_2}$$



$$= .91 - .09(1 + .91) = 0.74$$

$$x_4 = x_3 - \omega^2\frac{(m_1x_1 + m_2x_2 + m_3x_3)}{k_3}$$

$$= 0.74 - .09\,(1 + .91 + .74) = 0.51$$

Similarly, other deflections can be calculated and are directly put in the table 1 for different assumed frequency. The results for frequency are obtained by drawing a curve between ω and displacement x as shown in figure 6.26.

The natural frequencies are

$$\omega_1 = 0.44 \text{ rad/sec}$$
$$\omega_2 = 1.24 \text{ rad/sec}$$

and $$\omega_3 = 1.80 \text{ rad/sec.}$$

Example 6.22. *Using Holzer method find the natural frequency of the system shown in figure 6.3. Assume $k = 1$ kg/cm and $m = 1$ kg.*

Solution. Assuming $x_1 = 1$ and $\omega = .10$

so $\omega^2 = .01$

$$x_2 = 1 - \omega^2\frac{m_1x_1}{k_1}$$

$$= 1 - .01 = 0.99$$

$$x_3 = x_2 - \omega^2\frac{(m_1x_1 + m_2x_2)}{k_2}$$

$$= 0.99 - .01(1 + 2\times 0.99) = 0.96$$

$$x_4 = x_3 - \omega^2\frac{(m_1x_1 + m_2x_2 + m_3x_3)}{k_3}$$

$$= 0.96 - \frac{.01(1 + 2\times .99 + 4\times .96)}{3} = 0.93$$

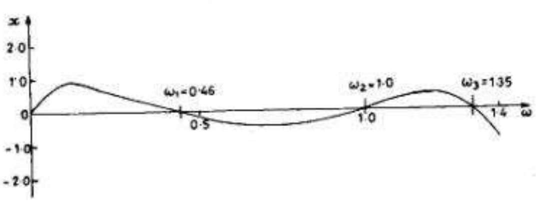


Fig. 6.27

TABLE 1

Assumed Frequency	Row	m	$m\omega^2$	x	$mx\omega^2$	$\Sigma mx\omega^2$	k	$\frac{1}{k}\Sigma mx\omega^2$
$\omega = 0.30$	1	1	.09	1.0	.09	0.09	1	.09
$\omega^2 = .09$	2	1	.09	0.91	.0819	0.17	1	.17
	3	1	.09	0.74	.0666	0.23	1	0.23
				0.51				
$\omega = 0.50$	1	1	.25	1.0	0.25	0.25	1	0.25
$\omega^2 = 0.25$	2	1	.25	0.75	0.19	0.44	1	0.44
	3	1	.25	0.31	0.07	0.51	1	0.51
				-0.20				
$\omega = 0.75$	1	1	0.56	1.0	0.56	0.56	1	0.56
$\omega^2 = 0.56$	2	1	0.56	0.44	0.24	0.80	1	0.80
	3	1	0.56	-0.36	-0.20	0.60	1	0.60
				-0.96				
$\omega = 1.0$	1	1	1	1	1	1	1	1
$\omega^2 = 1.0$	2	1	1	0	0	1	1	1
	3	1	1	-1	-1	0	1	0
				-1				

Assumed Frequency	Row	m	$m\omega^2$	x	$mx\omega^2$	$\Sigma mx\omega^2$	k	$\frac{1}{k}\Sigma mx\omega^2$
$\omega = 1.25$	1	1	1.56	1.0	1.56	1.56	1	1.56
$\omega^2 = 1.56$	2	1	1.56	-.56	-.87	0.69	1	0.69
	3	1	1.56	-1.25	-1.95	-1.26	1	-1.26
				.01				
$\omega = 1.50$	1	1	2.25	1.0	2.25	2.25	1	2.25
$\omega^2 = 2.25$	2	1	2.25	-1.25	-2.82	-0.57	1	-0.57
	3	1	2.25	-0.68	-1.53	-2.10	1	-2.10
				1.40				
$\omega = 1.75$	1	1	3.06	1.0	3.06	3.06	1	3.06
$\omega^2 = 3.06$	2	1	3.06	-2.06	-6.30	-3.24	1	-3.24
	3	1	3.06	1.18	3.60	0.36	1	0.36
				.81				
$\omega = 2.0$	1	1	4.0	1	4	4	1	4
$\omega^2 = 4.0$	2	1	4.0	-3	-12	-8	1	-8
	3	1	4.0	5	20	12	1	12
				-7				

Similarly, other deflections for assumed different frequencies can be calculated which are put in the table 2. Plotting the curve between ω and displacement (x), we get three values of natural frequency (where x is zero) *i.e.*,

$$\omega_1 = 0.46 \text{ rad/sec}$$
$$\omega_2 = 1.0 \text{ rad/sec}$$
$$\omega_3 = 1.35 \text{ rad/sec.}$$

EXAMPLE 6.23. *Use Holzer method to find the natural frequencies of the system shown in figure 6.28. Take $I_1 = I_2 = I_3 = 1$ and $k_{t1} = k_{t2} = 1$.*

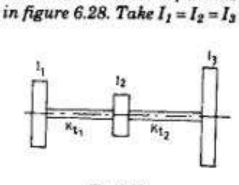


Fig. 6.28.

SOLUTION. The solution of the problem is presented in table 3 and figure 6.29.

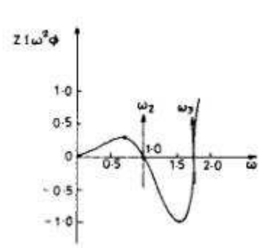


Fig. 6.29.

EXAMPLE 6.24. *Find the lowest natural frequency of transverse vibrations for the system shown in figure 6.30 by Rayleigh's method.*

$E = 1.96 \times 10^{11}\ N/m^2$, $\quad I = 10^{-6}\ m^4$. (P.U., 88)

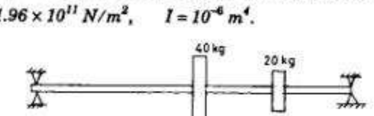


TABLE 2.

Assumed frequency	Row	m	$m\omega^2$	x	$m\omega^2 x$	$\Sigma m\omega^2 x$	k	$\frac{1}{k}\Sigma m\omega^2 x$
$\omega = 0.1$	1	1	.01	1.0	.01	.01	1	.01
$\omega^2 = .01$	2	2	.02	0.99	.0198	.0298	1	.0298
	3	4	.04	0.96	.0384	.0682	3	.0227
				0.93				
$\omega = 0.20$	1	1	.04	1	0.04	.04	1	0.04
$\omega^2 = .04$	2	2	.08	0.96	0.077	.117	1	0.117
	3	4	.16	0.84	0.1324	.249	3	0.083
				0.75				
$\omega = 0.4$	1	1	0.16	1	0.16	0.160	1	0.160
$\omega^2 = .16$	2	2	0.32	0.84	0.269	0.429	1	0.429
	3	4	0.64	0.411	0.264	0.693	3	0.231
				0.180				
$\omega = 0.50$	1	1	0.25	1.0	0.25	0.25	1	0.25
$\omega^2 = 0.25$	2	2	0.50	0.75	0.375	0.625	1	0.625
	3	4	1.0	0.125	0.125	0.750	3	0.250
				-0.125				

Assumed frequency	Row	m	$m\omega^2$	x	$m\omega^2 x$	$\Sigma m\omega^2 x$	k	$\frac{1}{k}\Sigma m\omega^2 x$
$\omega = 0.60$	1	1	0.36	1.0	0.36	0.36	1	0.360
$\omega^2 = 0.36$	2	2	0.72	0.64	0.46	0.82	1	0.820
	3	4	1.44	-1.80	-.259	0.561	3	0.187
				-.367				
$\omega = 0.80$	1	1	.64	1.0	0.64	.640	1	0.64
$\omega^2 = .64$	2	2	1.28	0.36	0.461	1.101	1	1.101
	3	4	2.56	-.741	-1.90	-.80	3	-.267
				-.474				
$\omega = 1.0$	1	1	1	1	1	1	1	1
$\omega^2 = 1.0$	2	2	2	0	0	0	1	0
	3	4	4	-1	-4	-3	3	-1
				0				
$\omega = 1.2$	1	1	1.44	1	1.44	1.44	1	1.44
$\omega^2 = 1.44$	2	2	2.88	-.44	-1.27	0.17	1	.17
	3	4	5.76	-.61	-3.51	-3.34	3	-1.11
				0.50				

Assumed frequency	Row	m	$m\omega^2$	x	$m\omega^2 x$	$\Sigma m\omega^2 x$	k	$\frac{1}{k}\Sigma m\omega^2 x$
$\omega = 1.3$	1	1	1.69	1.0	1.69	1.69	1	1.69
$\omega^2 = 1.69$	2	2	3.38	-.69	-2.33	-0.64	1	-0.64
	3	4	6.76	-.05	-.338	-0.97	3	-0.32
				0.27				
$\omega = 1.4$	1	1	1.96	1.0	1.96	1.96	1	1.96
$\omega^2 = 1.96$	2	2	3.92	-0.96	-3.76	-1.80	1	-1.80
	3	4	7.84	0.84	6.58	4.78	3	1.66
				-0.82				

TABLE 3

Assumed frequency	Row	I	$I\omega^2$	ϕ	$I\omega^2\phi$	$\Sigma I\omega^2\phi$	k	$\frac{1}{k}\Sigma I\omega^2\phi$
$\omega = 0.25$	1	1	0.0625	1.0	.0625	.0625	1	0.0625
$\omega^2 = .0625$	2	1	0.0625	0.93	0.058	0.12	1	0.12
	3	1	0.0625	0.80	0.05	0.17	–	–
$\omega = 0.30$	1	1	0.09	1.0	.09	.09	1	0.09
$\omega^2 = .09$	2	1	0.09	0.91	0.0819	0.17	1	0.17
	3	1	0.09	0.74	0.666	0.23	–	–
$\omega = 0.50$	1	1	0.25	1	0.25	0.25	1	0.25
$\omega^2 = 0.25$	2	1	0.25	0.75	0.19	0.44	1	0.44
	3	1	0.25	0.31	0.07	0.51	–	–
$\omega = 0.75$	1	1	0.56	1.0	0.56	0.56	1	0.56
$\omega^2 = 0.56$	2	1	0.56	0.44	0.24	0.80	1	0.80
	3	1	0.56	-0.36	-0.20	0.60	–	–
$\omega = 1.0$	1	1	1.0	1.0	1.0	1.0	1	1.0
$\omega^2 = 1.0$	2	1	1.0	0	0	1.0	1	1.0
	3	1	1.0	-1.0	-1.0	0	–	–

$\omega = 1.25$	1	1	1.56	1.0	1.56	**1.56**	1	1.56
$\omega^2 = 1.56$	2	1	1.56	-.56	-0.87	**0.69**	1	0.69
	3	1	1.56	-1.25	-1.95	-1.26	–	
$\omega = 1.50$	1	1	2.25	1.0	2.25	2.25	1	2.25
$\omega^2 = 2.25$	2	1	2.25	-1.25	-2.82	-.57	1	-0.57
	3	1	2.25	-0.68	-1.53	-2.10	–	
$\omega = 1.75$	1	1	3.06	1.0	3.06	3.06	1	3.06
$\omega^2 = 3.06$	2	1	3.06	-2.06	-6.30	-3.24	1	-3.24
	3	1	3.06	1.18	3.60	0.36	–	
$\omega = 1.80$	1	1	3.24	1	3.24	3.24	1	3.24
$\omega^2 = 3.24$	2	1	3.24	-2.24	-7.25	-4.01	1	-4.01
	3	1	3.24	1.77	5.77	1.76	–	–

$$\sum I\omega^2\phi = T$$

where T is the remaining torque.

Plotting the curve between T and ω, we obtain the natural frequencies as

$\omega_1 = 0.0$, $\omega_2 = 1.0$ rad/sec, $\omega_3 = 1.7$ rad/sec.

Refer Fig. 6.23.

SOLUTION. $l_1 = .16$ m

$$l_2 = 0.18 + 0.08 = 0.26 \text{ m}$$

$$l_3 = .08 \text{ m}$$

$$l = 0.42 \text{ m}$$

$$a_{11} = \frac{l_1^2 l_2^2}{3EI(l_1 + l_2)} = \frac{.16^2 \times 0.26^2}{3 \times 1.96 \times 10^{11} \times 10^{-6} \times 0.42}$$

$$= 7.007 \times 10^{-9}$$

$$a_{12} = a_{21} = \frac{l_1 l_3 (l^2 - l_1{}^2 - l_3^2)}{6EIl}$$

$$= \frac{.16 \times .08(.42^2 - .26^2 - .08^2)}{6 \times 1.96 \times 10^{11} \times 10^{-6} \times .42}$$

$$= 7.556 \times 10^{-9}$$

$$a_{22} = \frac{(l_1 + l_3)^2 (l_2 - l_3)^2}{3EIl} = \frac{(.16 + .08)^2 (.26 - .08)^2}{3 \times 1.96 \times 10^5 \times .42}$$

$$= 1.959 \times 10^{-9}$$

$$\omega = \sqrt{\frac{g(M_1 y_1 + M_2 y_2)}{M_1 y_1^2 + M_2 y_2^2}}$$

$$y_1 = M_1 g a_{11} + M_2 g a_{12} = 3477.4 \times 10^{-9} \text{ m}$$

$$y_2 = M_1 g a_{21} + M_2 g a_{22} = 2947.8 \times 10^{-9} \text{ m}$$

Putting these values in the above frequency equation, we get

$$\omega = \sqrt{\frac{9.8(40 \times 3477.4 \times 10^{-9} + 20 \times 2947.8 \times 10^{-9})}{40 \times (3477.4 \times 10^{-9})^2 + 20(2947.8 \times 10^{-9})^2}}$$

$$= 1718 \text{ rad/sec.}$$

EXAMPLE 6.25. *For a taut string having tension T and three concentrated masses as shown in figure 6.31, find the three natural frequencies.* (M.D.U., 93)

SOLUTION.

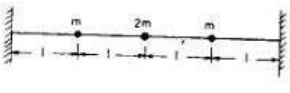


Fig. 6.31

SOLUTION.

Tension T in the string remains unchanged for small deflections. Let us assume the deflections as shown in figure. The equations of motions can be written as

$$m\ddot{x}_1 = \frac{T(x_2 - x_1)}{l} - \frac{Tx_1}{l}$$

$$2m\ddot{x}_2 = -\frac{T(x_2 - x_1)}{l} - \frac{T(x_2 - x_3)}{l}$$

$$m\ddot{x}_3 = \frac{T(x_2 - x_3)}{l} - \frac{Tx_3}{l}$$

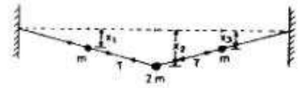


Fig. 6.32

Let us assume the motion of the form

$$x_1 = A_1 \sin \omega t,$$

$$\ddot{x}_1 = -\omega^2 A_1 \sin \omega t$$

$$x_2 = A_2 \sin \omega t,$$

$$\ddot{x}_2 = -\omega^2 A_2 \sin \omega t$$

$$x_3 = A_3 \sin \omega t,$$

$$\ddot{x}_3 = -\omega^2 A_3 \sin \omega t$$

Rearranging the above equations, we get

$$m\ddot{x}_1 + \frac{2Tx_1}{l} - \frac{Tx_2}{l} = 0$$

$$\left(-m\omega^2 + \frac{2T}{l}\right)A_1 - \frac{T}{l}A_2 = 0$$

$$2m\ddot{x}_2 + \frac{2Tx_2}{l} - \frac{Tx_1}{l} - \frac{Tx_3}{l} = 0$$

$$\left(-2m\omega^2 + \frac{2T}{l}\right)A_2 - \frac{T}{l}A_1 - \frac{T}{l}A_3 = 0$$

$$m\ddot{x}_3 + \frac{2Tx_3}{l} - \frac{Tx_2}{l} = 0$$

$$\left(-m\omega^2 + \frac{2T}{l}\right)A_3 - \frac{T}{l}A_2 = 0$$

Putting the determinants of coefficients A_1, A_2 and A_3 equal to zero, we have

$$\begin{vmatrix} -m\omega^2 + \frac{2T}{l} & -\frac{T}{l} & 0 \\ -\frac{T}{l} & -2m\omega^2 + \frac{2T}{l} & -\frac{T}{l} \\ 0 & -\frac{T}{l} & -m\omega^2 + \frac{2T}{l} \end{vmatrix} = 0$$

Expanding the determinant, the frequency equation is obtained.

$$\left(-m\omega^2 + \frac{2T}{l}\right)\left[\left(-2m\omega^2 + \frac{2T}{l}\right)\left(-m\omega^2 + \frac{2T}{l}\right) - \frac{T^2}{l^2}\right]$$

$$+ \frac{T}{l}\left[\frac{-T}{l}\left(-m\omega^2 + \frac{2T}{l}\right)\right] = 0$$

$$-2m^3\omega^6 + \frac{10T}{l}m^2\omega^4 - \frac{14mT^2}{l^2}\omega^2 + \frac{4T^3}{l^3} = 0$$

$$\omega^6 - \frac{5T}{ml}\omega^4 + \frac{7T^2}{m^2 l^2}\omega^2 - \frac{2T^3}{m^3 l^3} = 0$$

Solving it, we get

$$\omega_1^2 = \frac{2T}{ml}, \quad \omega_2^2 = \frac{.38T}{ml}, \quad \omega_3^2 = \frac{2.6T}{ml}$$

EXAMPLE 6.26. *A three rotor system shown in figure has the following physical constants :*

$$J_1 = 50 \text{ kg-cm-sec}^2$$

$$J_2 = 100 \text{ kg-cm-sec}^2$$

$$J_3 = 70 \text{ kg - cm - sec}^2$$

$$k_{t_1} = 2.2 \times 10^6 \text{ kg-cm/rad}$$

$$k_{t_2} = 0.8 \times 10^6 \text{ kg-cm/rad}$$

Find the natural frequency of the system and corresponding mode shapes. (P.U., 88)

Fig. 6.33

SOLUTION. The equations of motion can be written as

$$J_1\ddot{\theta}_1 + k_{t_1}(\theta_1 - \theta_2) = 0$$

$$J_2\ddot{\theta}_2 + k_{t_2}(\theta_2 - \theta_3) + k_{t_1}(\theta_2 - \theta_1) = 0$$

$$J_3\ddot{\theta}_3 + k_{t_2}(\theta_3 - \theta_2) = 0$$

Let us assume

$$\theta_1 = A_1 \sin \omega t$$

$$\theta_2 = A_2 \sin \omega t$$

$$\theta_3 = A_3 \sin \omega t$$

Using these relations in the above equations, we get

$$-J_1\omega^2 A_1 + k_{t_1}A_1 - k_{t_1}A_2 = 0$$

$$(-J_1\omega^2 + k_{t_1})A_1 - k_{t_1}A_2 = 0$$

$$(-J_2\omega^2 + k_{t_2} + k_{t_1})A_2 - k_{t_1}A_1 - k_{t_2}A_3 = 0$$

$$(-J_3\omega^2 + k_{t_2})A_3 - k_{t_2}A_2 = 0$$

The determinant of coefficients of A_1, A_2 and A_3 is equal to zero to get the frequency equation.

$$\begin{vmatrix} (k_{t_1} - J_1\omega^2) & -k_{t_1} & 0 \\ -k_{t_1} & k_{t_1} + k_{t_2} - J_2\omega^2 & -k_{t_2} \\ 0 & -k_{t_2} & k_{t_2} - J_3\omega^2 \end{vmatrix} = 0$$

Expanding the determinant, we get

$$(k_{t_1} - J_1\omega^2)\,[(k_{t_1} + k_{t_2} - J_2\omega^2)(k_{t_2} - J_3\omega^2) - k_{t_2}^2]$$

$$+ k_{t_1}(-k_{t_1})(k_{t_2} - J_3\omega^2) = 0$$

$$\omega^2\,[J_1J_2J_3\omega^4 - \{(J_1J_2 + J_1J_3)k_{t_2} + (J_2J_3 + J_1J_3)k_{t_1}\} \times \omega^2$$

$$+ k_{t_1}k_{t_2}(J_1 + J_2 + J_3)] = 0$$

Substituting the values of various terms

$$J_1J_2J_3 = 50 \times 100 \times 70 = 35 \times 10^4$$

$$J_1 + J_2 + J_3 = 50 + 100 + 70 = 220$$

$$J_1J_2 = 50 \times 100 = 5000$$

$$J_1J_3 = 50 \times 70 = 3500$$

$$J_2J_3 = 100 \times 70 = 7000$$

$$k_{t_1}k_{t_2} = 2.2 \times .8 \times 10^{12} = 1.76 \times 10^{12}$$

$$\omega^2[35 \times 10^4\,\omega^4 - \{(5000 + 3500).8 \times 10^6 + (7000 + 3500) \times 2.2 \times 10^6\}$$

$$+ 1.76 \times 10^{12} \times 220] = 0$$

$$35 \times 10^4\,\omega^4 - 29900 \times 10^6 + 387.2 \times 10^{12} = 0$$

$$\omega^4 - 854.28 \times 10^2 + 11.062 \times 10^8 = 0$$

$$\omega_1 = 182.4 \text{ rad/sec}$$

$$\omega_2 = 0$$

Amplitude ratios

$$\left(\frac{A_2}{A_1}\right)_{\omega=0} = \frac{-J\omega^2 + k_{t_1}}{k_{t_1}} = 1.0$$

or

$$\left(\frac{A_1}{A_2}\right) = 1.0$$

$$\left(\frac{A_3}{A_2}\right)_{\omega=0} = \frac{k_{t_2}}{-J_3\omega^2 + k_{t_2}} = 1.0$$

First mode $\begin{pmatrix} 1.0 \\ 1.0 \\ 1.0 \end{pmatrix}$

For $\omega_1 = 182.4$ rad/sec, we can find the other mode shapes.

From equations of motion, we have Second Mode :

$$\frac{A_1}{A_2} = \frac{k_{t_1}}{-J_1\,\omega^2 + k_{t_1}}$$

Putting $\omega = 182.4$, $k_{t_1} = 2.2 \times 10^6$ and

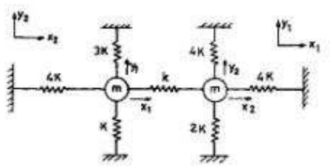
$J_1 = 50$, we get the amplitude ratio

$$= \frac{2.2 \times 10^6}{-50 \times (182.4)^2 + 2.2 \times 10^6} = 3.66$$

$$\frac{A_3}{A_2} = \frac{k_{t_2}}{k_{t_2} - J_3\,\omega^2} = \frac{.8 \times 10^6}{.8 \times 10^6 - 70 \times (182.4)^2}$$

$$= -.52 \text{ and so on.}$$

Example 6.27. *Find the frequencies of the system shown in figure 6.34.* (P.U., ME, 89)



Solution. The equations of motion for the system can be written as

$$m\ddot{x}_1 + 5kx_1 - kx_2 = 0 \qquad \text{(For horizontal movement)}$$

$$m\ddot{x}_2 + 5kx_2 - kx_1 = 0$$

$$\left.\begin{aligned} m\ddot{y}_1 + 4ky_1 = 0 \\ m\ddot{y}_2 + 6ky_2 = 0 \end{aligned}\right\} \text{ For vertical movement}$$

The above equations can be put in matrix form

$$[M]\{\ddot{x}\} + [k]\{x\} = 0$$

$[M]$ = mass or inertia matrix and $[k]$ = stiffness matrix

$$[M] = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix}$$

$$[k] = \begin{bmatrix} 5k & -k & 0 & 0 \\ -k & 5k & 0 & 0 \\ 0 & 0 & 4k & 0 \\ 0 & 0 & 0 & 6k \end{bmatrix}$$

The dynamic matrix [C] can be written as

$$[C] = [M]^{-1}[k]$$

and

$$[M]^{-1} = \frac{1}{m}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then

$$[C] = \frac{1}{m}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5k & -k & 0 & 0 \\ -k & 5k & 0 & 0 \\ 0 & 0 & 4k & 0 \\ 0 & 0 & 0 & 6k \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5k}{m} & \frac{-k}{m} & 0 & 0 \\ \frac{-k}{m} & \frac{5k}{m} & 0 & 0 \\ 0 & 0 & \frac{4k}{m} & 0 \\ 0 & 0 & 0 & \frac{6k}{m} \end{bmatrix}$$

Let us find the frequency equation by putting the determinant equal to zero i.e.

$$|\lambda I - C| = 0 \qquad \text{(equation 6.11.5)}$$

where $\lambda = \omega^2$.

$$\begin{bmatrix} \lambda - \frac{5k}{m} & \frac{k}{m} & 0 & 0 \\ \frac{k}{m} & \lambda - \frac{5k}{m} & 0 & 0 \\ 0 & 0 & \lambda - \frac{4k}{m} & 0 \\ 0 & 0 & 0 & \lambda - \frac{6k}{m} \end{bmatrix} = 0$$

Expanding the determinant, we get

$$\left(\lambda - \frac{5k}{m}\right)^2\left(\lambda - \frac{4k}{m}\right)\left(\lambda - \frac{6k}{m}\right) - \left(\frac{k}{m}\right)^2\left(\lambda - \frac{4k}{m}\right)\left(\lambda - \frac{6k}{m}\right) = 0$$

$$\left(\lambda - \frac{4k}{m}\right)\left(\lambda - \frac{6k}{m}\right)\left[\left(\lambda - \frac{5k}{m}\right)^2 - \left(\frac{k}{m}\right)^2\right] = 0$$

From the above equation, we get

$$\omega_1 = \sqrt{\frac{4k}{m}} \text{ rad/sec}$$

$$\omega_2 = \sqrt{\frac{6k}{m}} \text{ rad/sec}$$

and

$$\left(\lambda - \frac{5k}{m}\right)^2 - \frac{k^2}{m^2} = 0$$

$$\lambda^2 + 24\frac{k^2}{m^2} - 10\,\lambda\,k/m = 0$$

So

$$\omega_3 = \sqrt{\frac{6k}{m}} \text{ and } \omega_4 = \sqrt{\frac{4k}{m}} \text{ rad.}$$

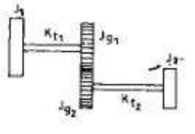
Example 6.28. *For the system shown in figure 6.35*

$$J_1 = 10 \text{ kg-cm sec}^2$$

$$J_2 = 20 \text{ kg-cm sec}^2$$

$$J_{g_1} = 0.5 \text{ kg-cm sec}^2$$

$$J_{g_2} = 2.0 \text{ kg-cm sec}^2$$



Diameter of gear 2=twice the diameter of gear 1.

$$k_{t_1} = 3.2 \times 10^5 \text{ kg . cm rad}$$

$$k_{t_2} = 0.8 \times 10^5 \text{ kg cm rad}$$

Find the natural frequency of torsional oscillations taking into account the inertias of gears. (P.U., 87)

Solution. Let us assume that θ_1 and θ_2 are the angular displacements of gears J_1 and J_2 and n is the velocity ratio.

Thus

$$\frac{\dot{\theta}_2}{\dot{\theta}_1} = n$$

or

$$\theta_2 = n\theta_1$$

The total kinetic and potential energy of the system remains constant.

$$\left.\begin{aligned} \text{K.E.} &= \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2 \\ \text{P.E.} &= \frac{1}{2}k_{t_1}\theta_1^2 + \frac{1}{2}k_{t_2}\theta_2^2 \end{aligned}\right\} \qquad ...(1)$$

$$\left.\begin{aligned} \text{K.E.} &= \frac{1}{2}J_1\,\dot{\theta}_1^2 + \frac{1}{2}\left(J_2n^2\right)\dot{\theta}_1^2 \\ \text{P.E.} &= \frac{1}{2}k_{t_1}\theta_1^2 + \frac{1}{2}(k_{t_2}n^2)\,\theta_1^2 \end{aligned}\right\} \qquad ...(2)$$

So it is clear from the above expression that the geared system can be converted into an equivalent system by multiplying its stiffness and inertia by n^2 *i.e.*

So J_2 becomes $J_2' = n^2J_2$

and k_{t_2} becomes $k'_{t_2} = n^2\,k_{t_2}$

Now the equivalent system can be shown in figure 6.36. The shafts are connected in series, so their equivalent stiffness $\frac{1}{k_{t_e}} = \frac{1}{k_{t_1}} + \frac{1}{n^2k_{t_2}}$

$$k_{te} = \frac{n^2k_{t_1}k_{t_2}}{k_{t_1} + n^2\,k_{t_2}}$$

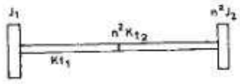


Fig. 6.36

The relation for frequency is

$$\omega = \sqrt{\frac{k(J_1 + J_2)}{J_1 J_2}}$$

$$= \sqrt{\frac{k_{te}(J_1 + n^2 J_2)}{J_1 n^2 J_2}}$$

$$= \sqrt{\frac{n^2 k_{t_1} k_{t_2} (J_1 + n^2 J_2)}{(k_{t_1} + n^2 k_{t_2}) J_1 J_2 n^2}}$$

$$= \sqrt{\frac{k_{t_1} k_{t_2} (J_1 + n^2 J_2)}{(k_{t_1} + n^2 k_{t_2}) J_1 J_2}}$$

But in the problem the inertias of gears are given, so there will be a third inertia $\left(J_{g_1} + n^2 J_{g_2}\right)$ in between J_1 and $n^2 J_2$.

Now the problem can be solved as example 6.26.

We are given

$J_1, J_2'' = J_{g_1} + n^2 J_{g_2}, J_3 = n^2 J_2, k_{t_1}$ and $n^2 k_{t_2}$

We get equation (example 6.26)

$$\omega^2 [J_1 J_2 J_3 \omega^4 - \{(J_1 J_2 + J_1 J_3) n^2 k_{t_2} + (J_2 J_3 + J_1 J_3) k_{t_1}\} \omega^2 + k_{t_1} n^2 k_{t_2} (J_1 + J_2 + J_3)] = 0$$

$$J_1 = 10$$

$$J''_2 = (J_{g_1} + n^2 J_{g_2}) = J_2 \text{ (say)}$$

$$= (.5 + 4 \times 2) = 8.5 = J_2$$

$$J_3 = 20$$

$$n^2 k_{t_2} = 4 \times .8 \times 10^5 = 3.2 \times 10^5$$

$$J_1 J_2 J_3 = 10 \times 8.5 \times 20 = 1700$$

$$J_1 J_2 = 85$$

$$J_1 J_3 = 200$$

$$J_2 J_3 = 170$$

$$n^2 k_{t_1} k_{t_2} = 4 \times 3.2 \times .8 \times 10^{10}$$

$$= 10.24 \times 10^{10}$$

$$J_1 + J_2 + J_3 = 10 + 8.5 + 20 = 38.5$$

$$\omega^2 [1700\, \omega^4 - \{(85 + 200) 3.2 \times 10^5 + (170 + 200) 3.2 \times 10^5\} \omega^2 + 10.24 \times 10^{10} \times 38.5]$$

$$\omega^4 - 1.2329 \times 10^5\, \omega^2 + 23.2 \times 10^8 = 0$$

$$\omega^2 = \frac{1.2329 \times 10^5 \pm \sqrt{(1.2329 \times 10^5)^2 - 4 \times 23.2 \times 10^8}}{2}$$

$$= \frac{(1.2329 \pm .76)}{2} \times 10^5$$

$$\omega_1 = 316.4 \text{ rad/sec}$$

and $\omega_2 = 151.4$ rad/sec

Problems

1. Determine the frequency of the system shown in figure 6.1 P by Matrix iteration method. (*P.U., 89*)

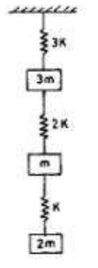


Fig. 6.1 P.

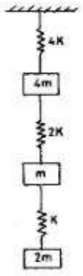


Fig. 6.2 P.

2. Using Matrix Inversion or iteration method, find the natural frequency of the system shown in figure 6.2 P. (*P.U., 92*)

3. Determine the natural frequency of the system shown in figure 6.3 P by influence coefficient method. (*P.U., 93*)

$m_1 = 100$ kg, $m_2 = 200$ kg, $m_3 = 300$ kg,

$k_1 = 2\,k$ N/m, $k_2 = 1.5\,k$ N/m, $k_3 = 2\,k$ N/m

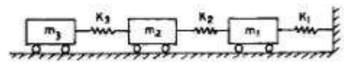


Fig. 6.3 P.

4. Use matrix iteration method to find the natural frequency of the system shown in figure 6.4 P. (*P.U., 93*)

$m_1 = 10$ kg, $m_2 = 15$ kg, $m_3 = 20$ kg

$k_1 = 1\,k$ N/mm, $k_2 = k_3 = .5\,k$ N/mm

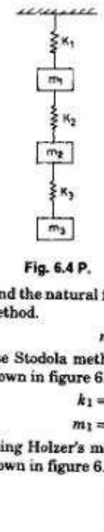


Fig. 6.4 P.

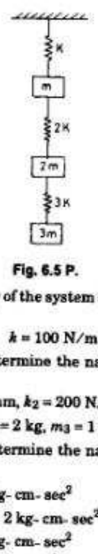


Fig. 6.5 P.

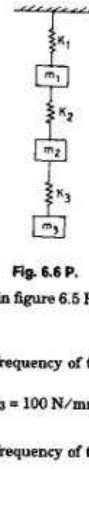


Fig. 6.6 P.

5. Find the natural frequency of the system shown in figure 6.5 P by matrix method. (*P.U., 93*)

$m = 1$ kg, $k = 100$ N/m

6. Use Stodola method to determine the natural frequency of the system shown in figure 6.6 P. (P.U., 93)

$k_1 = 300$ N/mm, $k_2 = 200$ N/mm, $k_3 = 100$ N/mm

$m_1 = 4$ kg, $m_2 = 2$ kg, $m_3 = 1$ kg.

7. Using Holzer's method, determine the natural frequency of the system shown in figure 6.7 P. (*P.U., 93*)

$J_1 = 10$ kg- cm- sec^2

$J_2 = J_4 = 2$ kg- cm- sec^2

$J_3 = 15$ kg- cm- sec^2

$k_{t_1} = k_{t_3} = 10 \times 10^6$ kg- cm/rad

$k_{t_2} = 20 \times 10^6$ kg- cm/rad.

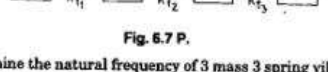


Fig. 6.7 P.

8. (a) Determine the natural frequency of 3 mass 3 spring vibrating system by matrix method. Assume $m_1 = 4$ m, $m_2 = 2$ m, $m_3 = $ m, $k_1 = 3$ k, $k_2 = k_3 = k$.

(b) Explain the coupling of co-ordinates in a multi degree freedom system. How will you obtain the type of coupling present by the help of matrices and energy expressions? (Roorkee Uni.)

9. Determine the fundamental natural frequency of transverse vibration of the system shown in Fig. 6.8P by both.
(a) Dunkerley's method (b) Rayleigh's method
The values of the influence coefficients for figure 6.8P are given below : (Roorkee Uni., 1999-2000)

$$\alpha_{11} = \alpha_{33} = \frac{3l^3}{256\,EI}\ ;\quad \alpha_{12} = \alpha_{21} = \alpha_{23} = \alpha_{32} = \frac{11\,l^3}{768\,EI}$$

$$\alpha_{13} = \alpha_{31} = \frac{7l^3}{768\,EI}\ ;\quad \alpha_{22} = \frac{l^3}{48\,EI}$$

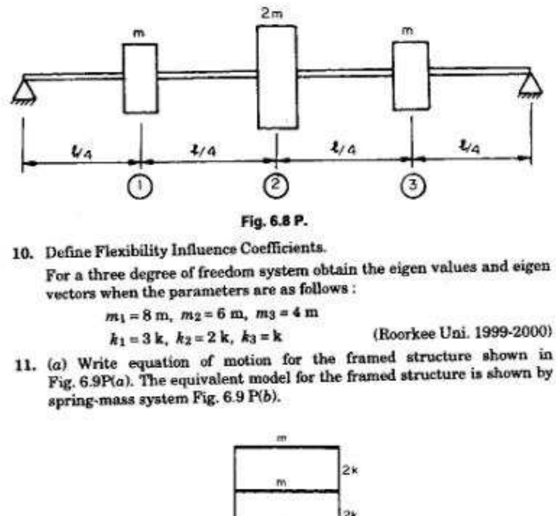


Fig. 6.8 P.

10. Define Flexibility Influence Coefficients.
For a three degree of freedom system obtain the eigen values and eigen vectors when the parameters are as follows :

$m_1 = 8$ m, $m_2 = 6$ m, $m_3 = 4$ m

$k_1 = 3$ k, $k_2 = 2$ k, $k_3 = $ k (Roorkee Uni. 1999-2000)

11. (*a*) Write equation of motion for the framed structure shown in Fig. 6.9P(*a*). The equivalent model for the framed structure is shown by spring-mass system Fig. 6.9 P(*b*).

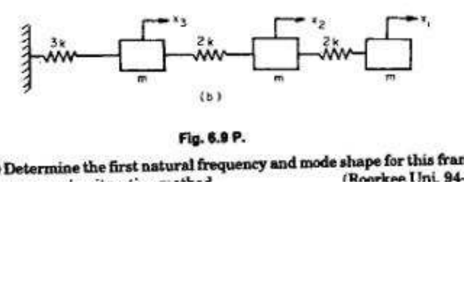


Fig. 6.9 P.

(b) Determine the first natural frequency and mode shape for this framed ... (Roorkee Uni. 94-95)

12. The turbine shown in the accompanying figure (6.10P) has an operating speed of 3600 r.p.m. and drives the generator at 1800 r.p.m. through the gear arrangement shown. The shear modulus of elasticity for the steel in both shafts is $G = 80$ gigapascals. Other parameter values of the system are as follows :

$I_1 = 3600$ kg-m^2 (turbine)

$I_2 = 200$ kg-m^2 (gear)

$I_3 = 800$ kg-m^2 (gear)

$I_4 = 4800$ kg-m^2 (generature armature)

$d_1 = 150$ mm (diameter of turbine shaft)

$l_1 = 3.5$ m (length of turbine shaft)

$d_2 = 200$ mm (diameter of generator shaft)

$l_2 = 3.0$ m (length of generator shaft)

Using the values above, determine the fundamental natural frequency of the turbine-generator system.

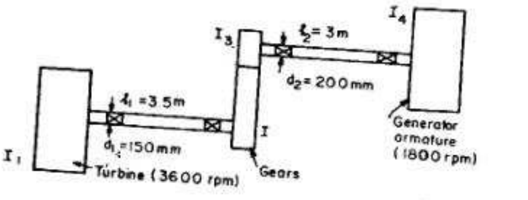
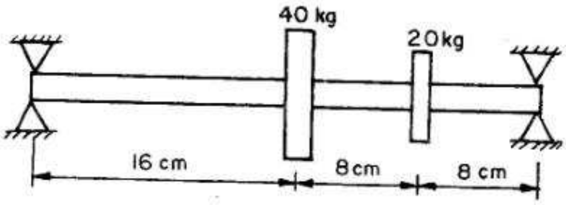


Fig. 6.10 P.

(b) Write equation of motion for the above turbine-generator drive system when subjected to torsional vibration. (Roorkee Uni., 94-95)

13. Estimate the lowest natural frequency of transverse vibrations for the system shown in Fig. 6.11P by (*i*) Rayleigh's Method (*ii*) Stodola's Method. Take $E = 2.0 \times 1011$ N/m^2, $I = 10^{-6}$ m^4 and $g = 10$ m/s^2.



14. The arrangement of the compressor-turbine and generator in a thermal power plant is shown in Fig. 6.12P. Find the natural frequencies and mode shapes of the system.

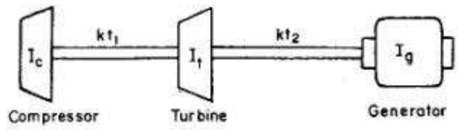


Fig. 6.12 P.

Stiffness $K_{t_1} = 6$ MN-m/rad

Stiffness $K_{t_2} = 3$ MN-m/rad

Compressor moment of Inertia, $I_c = 18$ kg-m^2

Turbine moment of Inertia, $I_t = 14$ kg-m^2

Generator moment of Inertia, $I_g = 9$ kg-m^2

7

Continuous Systems

7.1. INTRODUCTION

In the previous chapters mass, stiffness and damping of vibrating systems were assumed to be acting only at certain discrete points. (It was assumed that shafts, rotors and springs had no mass and stiffness, though practically they had. It was an ideal approach to analyse the vibrating systems). There are systems such as beams, cables, rods, etc., which have their mass and elasticity distributed continuously throughout the length. Such systems are known as continuous systems. Since such systems are supposed to be made of infinite number of particles, so they have infinite number of degrees of freedom and hence infinite natural frequencies of the system. The vibratory motions of such systems are described by space and time and partial differential equations are formulated for analysis of the systems. Partial differential equations consist of many constants which can be determined from boundary conditions and initial conditions as well. Thus the problems are boundary value based. In this chapter, we study only bodies of uniform cross-section having homogeneous and isotropic material.

Boundary conditions

The value of unknown constants in the partial differential equations can be determined by applying either geometric or natural or both boundary conditions. Geometric boundary conditions are caused because of geometric compatibility. For example, if the bar is fixed at both ends, the displacement and slope will be zero.

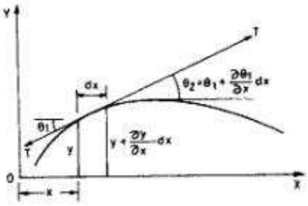
Natural boundary conditions are caused due to force and moments. For example, if the bar is hinged at one end, the bending moment at the hinged end will be zero and so on so forth. Initial conditions are related to time.

7.2. LATERAL VIBRATIONS OF A STRING

Consider a vibrating string of mass ρ per unit length having transverse vibrations under tension T as shown in figure 7.1

It is assumed that for a very small amplitude of string vibration the tension T remains constant throughout.

For very small displacements, $\tan\theta_1 = \theta_1$



String in lateral vibration
Figure 7.1

$$\tan\theta_1 = \frac{\partial y}{\partial x}$$

$$\theta_1 = \frac{\partial y}{\partial x}$$

and
$$\theta_2 = \frac{\partial y}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)dx$$

$$\theta_2 = \theta_1 + \frac{\partial\theta_1}{\partial x}dx \qquad \ldots(7.2.1)$$

Resolving the tension along y-axis

$$T\sin\left(\theta_1 + \frac{\partial\theta_1}{\partial x}dx\right) - T\sin\theta_1 = \text{mass} \times \text{acceleration}$$

$$= \rho\, dx\,\frac{\partial^2 y}{\partial t^2}$$

$$T\left(\theta_1 + \frac{\partial\theta_1}{\partial x}dx\right) - T\theta_1 = \rho\, dx\,\frac{\partial^2 y}{\partial t^2}$$

$$T\frac{\partial\theta_1}{\partial x}dx = \rho\, dx\,\frac{\partial^2 y}{\partial t^2}$$

$$T\frac{\partial\theta_1}{\partial x} = \rho\frac{\partial^2 y}{\partial t^2}$$

$$\frac{T}{\rho}\left(\frac{\partial\theta_1}{\partial x}\right) = \frac{\partial^2 y}{\partial t^2}$$

$$\frac{T}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial t^2} \qquad \left(\text{Substituting } \theta_1 = \frac{\partial y}{\partial x}\right)$$

$$\frac{T}{\rho}\left(\frac{\partial^2 y}{\partial x^2}\right) = \frac{\partial^2 y}{\partial t^2} \qquad \ldots(7.2.2)$$

Assuming $a^2 = \frac{T}{\rho}$, the above equation can be written as

$$a^2\left(\frac{\partial^2 y}{\partial x^2}\right) = \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2} \qquad ...(7.2.3)$$

This is one-dimensional wave equation for lateral vibrations of string.

Solution of wave equation

The lateral deflection y along the string is a function of the variables x and t. So it can be written as

$$y = y(x, t) \qquad ...(7.2.4)$$

Equation (7.2.3) has four arbitrary constants, it can be solved by boundary and initial conditions.

Let us assume the harmonic mode of vibration as the system is undamped. Thus solution of equation (7.2.3) can be written as

$$y(x, t) = X(x)T(t) \qquad ...(7.2.5)$$

Substituting the above solution in equation (7.2.3)

$$\frac{a^2}{X}\cdot\frac{d^2X}{dx^2} = \frac{1}{T}\cdot\frac{d^2T}{dt^2} \qquad ...(7.2.6)$$

In this equation L.H.S. is a function of x alone and R.H.S. a function of t alone. So we put it equal to some constant $-p^2$.

$$\frac{d^2X}{dx^2} + \left(\frac{p}{a}\right)^2 X = 0$$

and $$\frac{d^2T}{dt^2} + Tp^2 = 0 \qquad ...(7.2.7)$$

The solutions of the above two equations are

$$X(x) = A\cos\left(\frac{p}{a}\right)x + B\sin\left(\frac{p}{a}\right)x$$

$$T(t) = C\cos pt + D\sin pt$$

The general solution can be written as

$$y(x, t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{p}{a}\right)x + B_n\sin\left(\frac{p}{a}\right)x\right]$$

$$[C_n\cos pt + D_n\sin pt] \qquad ...(7.2.8)$$

In this equation p is the frequency of vibration. The values of arbitrary parameters A_n, B_n, C_n and D_n in the above equation can be

determined by assuming boundary and initial conditions, *e.g.*,

Boundary conditions – Let us assume the string is fixed at both ends,

i.e. $$y(0, t) = 0 \text{ and } y(l, t) = 0 \qquad ...(7.2.9)$$

Initial conditions – Assuming the initial displacement and velocity as :

at $t = 0$, $y(x, 0) = S(x)$

at $t = 0$, $\dot{y}(x, 0) = V(x) \qquad ...(7.2.10)$

Making use of equations (7.2.9) and (7.2.10) in equation (7.2.8)

(Using boundary conditions)

$$y(0, t) = A_n(C_n\cos pt + D_n\sin pt)$$

gives $$A_n = 0$$

$$y(l, t) = B_n\sin\left(\frac{p}{a}\right)l\,(C_n\cos pt + D_n\sin pt)$$

If $B_n \neq 0$ which gives

$$\sin\left(\frac{p}{a}\right)l = \sin n\pi = 0 \qquad ...(7.2.11)$$

This equation is called Frequency Equation.

$$\frac{p_n}{a}l = n\pi$$

$$p_n = \frac{n\pi a}{l} \qquad \left(\because a^2 = \frac{T}{\rho}\right)$$

So frequency $$p_n = \frac{n\pi}{l}\sqrt{\frac{T}{\rho}} \text{ rad/sec} \qquad ...(7.2.12)$$

Normal mode shape can be written as

$$X(x) = \sin\frac{n\pi x}{l}, \quad n = 1, 2, 3.... \qquad ...(7.2.13)$$

Each n represents a mode of vibration *e.g.* for $n = 1$ first mode, $n = 2$ second mode and so on.

Equation (7.2.8) can be written as

$$y(x, t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}(C_n\cos pt + D_n\sin pt) \qquad ...(7.2.14)$$

The values of constants C_n and D_n can be determined from initial conditions *i.e.*, displacement is $S(x)$ at $t = 0$ and velocity is $V(x)$ at $t = 0$

Applying initial conditions for above equation.

$$S(x) = \sum_{n=1}^{\infty} C_n\sin\frac{n\pi x}{l} \qquad ...(7.2.15)$$

and $$V(x) = \sum_{n=1}^{\infty} p_n D_n\sin\frac{n\pi x}{l} \qquad ...(7.2.16)$$

Equations (7.2.15) and (7.2.16) each are multiplied by $\sin\frac{m\pi x}{l}$, $m = 1, 2, 3....$

and integrated from $x = 0$ to l.

Thus $$\int_0^l S(x)\sin\frac{m\pi x}{l}dx = \int_0^l C_n\left(\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\right)dx$$

$\sin\frac{n\pi x}{l}$ and $\sin\frac{m\pi x}{l}$ are orthogonal functions and the value of above integral will be zero except when $m = n$

Putting $m = n$ for non-zero value of C_n, we get

$$\int_0^l S(x)\sin\frac{n\pi x}{l}dx = \int_0^l C_n\sin^2\frac{n\pi x}{l}dx$$

$$\int_0^l S(x)\sin\frac{n\pi x}{l}dx = C_n\int_0^l\frac{1}{2}\left(1 - \cos\frac{2n\pi x}{l}\right)dx$$

So $$C_n = \frac{2}{l}\int_0^l S(x)\sin\frac{n\pi x}{l}dx \qquad ...(7.2.17)$$

Similarly, considering equation (7.2.16)

$$\int_0^l V(x)\sin\frac{m\pi x}{l}dx = p_n D_n\int_0^l\sin\frac{n\pi x}{l}\cdot\sin\frac{m\pi x}{l}dx$$

For non-zero value of D_n, putting $m = n$

$$\int_0^l V(x)\sin\frac{n\pi x}{l}dx = p_n D_n\int_0^l\sin^2\frac{n\pi x}{l}dx$$

$$D_n = \frac{2}{lp_n}\int_0^l V(x)\sin\frac{n\pi x}{l}dx \qquad ...(7.2.18)$$

7.3. LONGITUDINAL VIBRATIONS OF BARS

Let us consider thin and uniform bar for longitudinal vibrations as shown in figure 7.2. The bar is subjected to axial forces. An element dx of the bar is considered here for analysis.

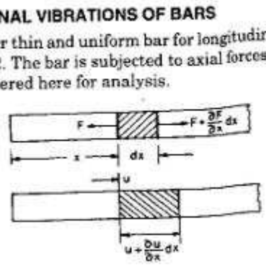


Fig. 7.2

If u is displacement at a distance x from left and it becomes $u + \frac{\partial u}{\partial x}dx$ at a distance $x + dx$. It is clear that the element dx has changed its position by an amount $\left(dx + \frac{\partial u}{\partial x}dx - dx\right) = \frac{\partial u}{\partial x}dx$

So strain of the element is given by

$$\epsilon = \frac{\frac{\partial u}{\partial x}\cdot dx}{dx} = \frac{\partial u}{\partial x} \qquad ...(7.3.1)$$

Let A = cross sectional area of the bar

ρ = density of the material

E = modulus of elasticity of the material

F = force acting axially on the bar

Net force acting on the element

$$\left(F + \frac{\partial F}{\partial x}dx\right) - F = (\text{mass}) \times (\text{acceleration of the element})$$

$$= dm \times \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial F}{\partial x}dx = (\rho\, dx\, A)\left(\frac{\partial^2 u}{\partial t^2}\right) \qquad ...(7.3.2)$$

We know that $\frac{F}{A} = \sigma$, where σ is the stress, so

$$F = \sigma A$$

$$\frac{\partial F}{\partial x} = \frac{\partial \sigma}{\partial x}A$$

$$\left(\frac{\partial F}{\partial x}\right)dx = \left(\frac{\partial \sigma}{\partial x}\right)dx\, A \qquad ...(7.3.3)$$

Equation (7.3.2) can be written with the help of above equation as

$$\left(\frac{\partial \sigma}{\partial x}\right)dx\,A = (\rho\,dx\,A)\left(\frac{\partial^2 u}{\partial t^2}\right) \qquad ...(7.3.4)$$

According to Hooke's law

$$\frac{\text{Stress}}{\text{Strain}} = E$$

$$\frac{\sigma}{\epsilon} = E$$

$$\sigma = \epsilon\,E$$

$$\frac{\partial \sigma}{\partial x} = \frac{\partial \epsilon}{\partial x}\,E$$

$$\left(\frac{\partial \sigma}{\partial x}\right)dx\,A = \left(\frac{\partial \epsilon}{\partial x}\right)dx\,AE \qquad ...(7.3.5)$$

With the help of equation (7.3.4) and (7.3.5),

We have

$$\left(\frac{\partial \epsilon}{\partial x}\right)dx\,AE = \rho\,dx\,A\left(\frac{\partial^2 u}{\partial t^2}\right)$$

$$\frac{E}{\rho}\,\frac{\partial \epsilon}{\partial x} = \frac{\partial^2 u}{\partial t^2}$$

$$\left(\frac{E}{\rho}\right)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial t^2} \qquad \left(\epsilon = \frac{\partial u}{\partial x} \text{ from } 7.3.1\right)$$

$$\frac{E}{\rho}\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\,\frac{\partial^2 u}{\partial t^2} \qquad ...(7.3.6)$$

where $a^2 = E/\rho$

This is the wave equation which is identical to equation (7.2.3). The general solution will be same as in the previous case of lateral vibrations.

A solution of the form as in equation (7.2.5)

$$u(x, t) = X(x)\,T(t)$$

So $X(x) = A \sin \frac{px}{a} + B \cos \frac{px}{a}$, $T(t) = C \sin pt + D \cos pt$ will result into the general solution as

$$u(x, t) = \sum_{n=1}^{\infty}\left(A \sin \frac{p}{a}x + B \cos \frac{p}{a}x\right)(C \sin pt + D \cos pt) \qquad ...(7.3.7)$$

7.4. TORSIONAL VIBRATION OF A UNIFORM SHAFT

An element of length dx of uniform shaft is put to torsional vibrations and it is assumed that the distortion of the shaft is very small.

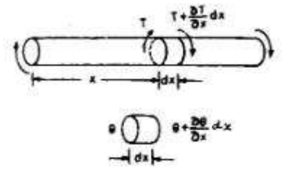


Fig. 7.3.

Refer Figure 7.3

Let θ = twist at a distance x from left side due to a moment, T

$\theta + \frac{\partial \theta}{\partial x}\,dx$ = twist at a distance $x + dx$ from left side due to a moment $T + \frac{\partial T}{\partial x}\,dx$

G = modulus of rigidity of material,

ρ = density (mass/volume) of the material

J = polar second moment of area

I = mass moment of inertia.

According to Newton's second law of rotation,

$$T = I\,.\,\alpha \qquad \text{(inertia} \times \text{angular acceleration)}$$

Net torque can be written as

$$\left(T + \frac{\partial T}{\partial x}\,dx\right) - T = I\,\frac{d^2\theta}{dt^2}$$

$$\frac{\partial T}{\partial x}\,dx = I\,\frac{d^2\theta}{dt^2} \qquad ...(7.4.1)$$

From Strength of Materials, we know that

$$\frac{T}{J} = G\,\frac{d\,\theta}{dx}$$

or

$$T = GJ\,\frac{d\,\theta}{dx}$$

So

$$\frac{\partial T}{\partial x}\,dx = GJ\,\frac{\partial}{\partial x}\left(\frac{d\theta}{dx}\right)dx \qquad ...(7.4.2)$$

From equation (7.4.1) and (7.4.2), we get

$$GJ\,\frac{\partial}{\partial x}\left(\frac{d\theta}{dx}\right)dx = I\,\frac{d^2\theta}{dt^2} \qquad ...(7.4.3)$$

For a shaft of constant cross section GJ is constant, and

$$J = \frac{\pi}{32}\,d^4$$

$$I = \frac{\pi}{32}\,d^4\,\rho\,dx \qquad ...(7.4.4)$$

Putting the values of I and J from the above equation in equation 7.4.3), we get

$$\frac{G}{\rho}\,\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2}$$

$$\frac{\partial^2\theta}{\partial x^2}(x, t) = \frac{1}{a^2}\,\frac{\partial^2\theta}{\partial t^2}(x, t) \qquad ...(7.4.5)$$

where $a^2 = G/\rho$

This is wave equation identical to equations (7.2.3) and (7.3.6)

The general solution of equation (7.4.5) can be written as

$$\theta(x, t) = \left(A \sin \frac{p_n x}{a} + B \cos \frac{p_n x}{a}\right)(C \sin p_n t + D \cos p_n t) \qquad ...(7.4.6)$$

7.5. TRANSVERSE VIBRATION OF BEAMS

If the cross-sectional dimensionals of beam are small compared to ts length, the system is known as Euler-Bernoulli beam. Only the thin eams are treated under it.

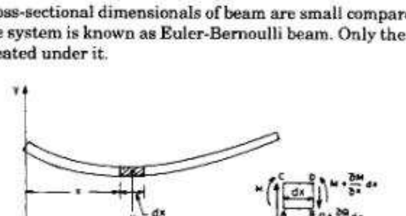


Fig. 7.4

The differential equation for transverse vibration of thin uniform eam is obtained with the help of strength of materials. The beam has ross-sectional area A, flexural rigidity EI and density of material ρ. lement dx of beam is subjected to shear force Q and bending moment M.

While deriving mathematical expression for transverse vibrations it is assumed that there are no axial forces acting on the beam and effects of shear deflection are neglected. The deformation of beam is assumed due to moment and shear force.

Net forces acting on the element

$$-Q - \left(Q + \frac{\partial Q}{\partial x}\,dx\right) = dm\,.\,\text{acceleration}$$

$$-\frac{\partial Q}{\partial x}\,dx = (\rho A dx)\,\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial Q}{\partial x} + \rho A\,\frac{\partial^2 y}{\partial t^2} = 0 \qquad ...(7.5.1)$$

Considering the moments about A, we get

$$M - \left(M + \frac{\partial M}{\partial x}\,dx\right) + \left(Q + \frac{\partial Q}{\partial x}\,dx\right)dx = 0$$

$$-\frac{\partial M}{\partial x} + Q + \frac{\partial Q}{\partial x}\,dx = 0$$

So $Q = \frac{\partial M}{\partial x}$ higher order derivatives are neglected $\left(\frac{\partial Q}{\partial x}\,dx = 0\right)$

or

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 M}{\partial x^2} \qquad ...(7.5.2)$$

From the above two equations (7.5.1) and (7.5.2), we get

$$\frac{\partial^2 M}{\partial x^2} = \rho A\,\frac{\partial^2 y}{\partial t^2} \qquad ...(7.5.3)$$

We know from beam theory that

$$M = -EI\,\frac{\partial^2 y}{\partial x^2}$$

So

$$\frac{\partial^2 M}{\partial x^2} = -EI\,\frac{\partial^4 y}{\partial x^4} \qquad ...(7.5.4)$$

Comparing equation (7.5.3) and (7.5.4), we get

$$EI\,\frac{\partial^4 y}{\partial x^4} + \rho A\,\frac{\partial^2 y}{\partial t^2} = 0$$

$$\frac{\partial^4 y}{\partial x^4} + \left(\frac{\rho A}{EI}\right)\frac{\partial^2 y}{dt^2} = 0 \qquad ...(7.5.5)$$

This is the general equation for transverse vibration which is different from wave equation.

Let us assume the solution of the form

$$y = y(x)\,\sin(\omega t + \phi) \qquad ...(7.5.6)$$

where $y(x)$ is the shape of the beam for the principal mode of vibrations.

Equation (7.5.5) can be written with the help of the above equation as

$$\frac{d^4y}{dx^4} - c^4 y = 0 \qquad ...(7.5.7)$$

where $$c^4 = \frac{\rho A}{EI}\omega^2$$

This is fourth-order differential equation. To find its solution, let us assume

$$y = e^{\lambda x}$$

So $$\frac{d^4y}{dx^4} = \lambda^4 e^{\lambda x}$$

Equation (7.5.7) can be written as

$$\lambda^4 e^{\lambda x} - c^4 e^{\lambda x} = 0$$

$$\lambda^4 - c^4 = 0 \qquad ...(7.5.8)$$

$$(\lambda + c)(\lambda - c)(\lambda^2 + c^2) = 0$$

$$\lambda_{1,2} = \pm c$$

$$\lambda_{3,4} = \pm ic \quad \text{where } i = \sqrt{-1}$$

$$e^{cx} = \cosh cx + \sinh cx$$

$$e^{-cx} = \cosh cx - \sinh cx$$

$$e^{icx} = \cos cx + i \sin cx$$

$$e^{-icx} = \cos cx - i \sin cx$$

So the solution of the differential equation can be written as

$$y = c_1(\cosh cx + \sinh cx) + c_2(\cosh cx - \sinh cx) + c_3(\cos cx + i \sin cx) + c_4(\cos cx - i \sin cx)$$

$$= (c_1 + c_2)\cosh cx + (c_1 - c_2)\sinh cx + (c_3 + c_4)\cos cx + (c_3 - c_4)\, i \sin cx$$

$$y(x, t) = A \cosh cx + B \sinh cx + C \cos cx + D \sin cx \qquad ...(7.5.9)$$

and $A = c_1 + c_2$, $B = c_1 - c_2$, $C = c_3 + c_4$ and $D = i(c_3 - c_4)$ where A, B, C and D are constants and their values can be determined from boundary conditions.

7.6. EFFECTS OF SHEAR DEFORMATION AND ROTARY INERTIA

The beams having large cross-sections are treated under thick beam theory and the effects of shear deformation and rotary inertia are taken into account. It is sometimes known as Timoshenko beam.

We know that the frequency of a system depends upon its mass and stiffness as

$$\omega = \sqrt{\frac{k}{m}}$$

By the inclusion of shear deformation and rotary inertia, the mass of the system increases and so the natural frequency of the system decreases.

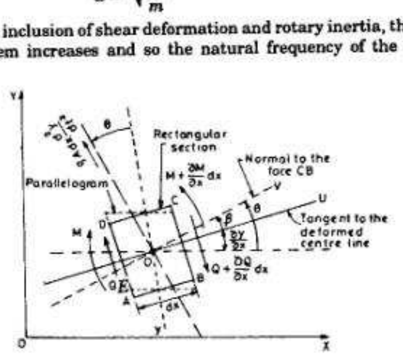


Fig 7.5

An element dx of the beam shown in figure 7.5 is discussed for analysis. The slope of the centre line of the beam is $\frac{dy}{dx}$ which means the rotary inertia effect is due to this $\frac{dy}{dx}$ rotation of the beam. Under the action of shear force Q and bending moment M the beam element dx is deformed. If shear force is neglected, OU will coincide with OV, where OV is normal to the face CB and OU tangent to the deformed centre line of the beam element dx. If the effects of shear force Q are included in the analysis the rectangular element will become a parallelogram and the tangent to the deformed centre line of the beam OU will make angle β with OV. This angle β is known as shear angle. Due to shear force deformation takes place without the rotation of the face. If θ is the slope due to bending, so net slope due to shear force is $\left(\theta - \frac{dy}{dx}\right)$ because $\frac{dy}{dx}$ is the slope due to bending and shear.

So $$\theta - \frac{dy}{dx} = \beta \qquad \text{(shear angle)}$$

According to elastic equations

$$\beta = \theta - \frac{dy}{dx} = \frac{Q}{KAG} \qquad ...(7.6.1)$$

$$\frac{d\beta}{dx} = \frac{M}{EI} \qquad ...(7.6.2)$$

From the above equations

$$Q = KAG\left(\theta - \frac{dy}{dx}\right) \qquad ...(7.6.3)$$

and $$M = EI\frac{d\beta}{dx} = EI\left(\frac{d\theta}{dx} - \frac{d^2y}{dx^2}\right)$$

where G is the shear modulus and K shape factor of the cross-section.

Considering the forces in y-direction, we get

$$Q - \left(Q + \frac{\partial Q}{\partial x}dx\right) = \rho A\frac{\partial^2 y}{\partial t^2}dx$$

$$-\frac{\partial Q}{\partial x}dx = \rho A\frac{\partial^2 y}{\partial t^2}dx \qquad \left(\because \frac{\partial Q}{\partial x}dx = dQ\right)$$

$$-\frac{\partial Q}{\partial x} = \rho A\frac{\partial^2 y}{\partial t^2}$$

Using equation (7.6.3) in the above equation, we get

$$-KAG\left(\frac{\partial\theta}{\partial x} - \frac{\partial^2 y}{\partial x^2}\right) = \rho A\frac{\partial^2 y}{dt^2} \qquad ...(7.6.4)$$

Taking moments about point E, we have

$$\left(M + \frac{\partial M}{\partial x}dx\right) - M - \left(Q + \frac{\partial Q}{\partial x}dx\right)dx = \rho I\frac{\partial^2\theta}{\partial t^2}\cdot dx$$

$$\frac{\partial M}{\partial x}dx - Q\,dx = \rho I\,dx\frac{\partial^2\theta}{\partial t^2} \qquad \left(\text{Neglecting } \frac{\partial Q}{\partial x}\cdot dx\right)$$

$$dM - Q\,dx = \rho I\,dx\frac{\partial^2\theta}{\partial t^2}$$

where I is moment of inertia and ρ is density. Making use of equation (7.6.3) in the above equation, we get

$$EI\frac{\partial^2\theta}{\partial x^2} - KAG\left(\theta - \frac{\partial y}{\partial x}\right) = \rho I\frac{\partial^2\theta}{\partial t^2} \qquad ...(7.6.5)$$

We can eliminate θ from equations (7.6.4) and (7.6.5) to obtain

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} - \rho I\left(1 + \frac{E}{KG}\right)\frac{\partial^4 y}{\partial x^2\partial t^2} + \frac{\rho^2 I}{KG}\frac{\partial^4 y}{\partial t^4} = 0 \qquad ...(7.6.6)$$

It is the general beam equation taking into account the effects of shear deformation and rotary inertia.

Boundary conditions. (i) Free end : $KAG\left(\frac{\partial y}{\partial x} - \theta\right) = EI\frac{\partial\theta}{\partial x} = 0$

(ii) Fixed end : $\theta = y = 0$

(iii) Simply supported $y = EI\frac{\partial\theta}{\partial x} = 0$

SOLVED PROBLEMS

EXAMPLE 7.1. *Derive the frequency equation of longitudinal vibrations for a free-free beam with zero initial displacement.*

SOLUTION. The general solution for longitudinal vibration can be written as

$$u(x, t) = \left(A \sin\frac{p_n}{a}x + B\cos\frac{p_n}{a}x\right)(C \sin p_n t + D\cos p_n t)$$

where $a = \sqrt{E/\rho}$ and p_n = natural frequencies

Boundary conditions are :

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0 \qquad \text{(Strains zero at both ends)}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=l} = 0$$

Applying the above two conditions, we get

$$\frac{\partial u}{\partial x} = \left(A\frac{p_n}{a}\cos\frac{p_n}{a}x - B\frac{p_n}{a}\sin\frac{p_n}{a}x\right)(C\sin p_n t + D\cos p_n t)$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = A\frac{p_n}{a}\left(C\sin\frac{p_n}{a}t + D\cos p_n t\right)$$

$$\Rightarrow \quad A = 0$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=l} = -B\frac{p_n}{a}\sin\frac{p_n}{a}l(C\sin p_n t + D\cos p_n t)$$

The values of constants C and D can be determined from initial conditions (equations 7.2.17 and 7.2.18)

So $$\sin\frac{p_n}{a}l = 0 = \sin n\pi$$

$$p_n = \frac{n\pi a}{l}, \qquad n = 1, 2, 3....$$

We know that $$p_n = 2\pi f_n$$

$$2\pi f_n = \frac{n\pi a}{l}$$

Natural frequency $$f_n = \frac{n}{2l}a = \frac{n}{2l}\sqrt{\frac{E}{\rho}}$$

where n represents the order of the mode.

Example 7.2. *Derive suitable expression for longitudinal vibrations for a rectangular uniform cross-section bar of length l fixed at one end and free at the other end.*

Solution. The general solution can be written as

$$u(x,t) = \left(A \sin \frac{p_n}{a} x + B \cos \frac{p_n}{a} x\right)(C \sin p_n t + D \cos p_n t)$$

Boundary conditions are :

$$(u)_{x=0} = 0 \qquad \text{(displacement is zero at fixed end)}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=l} = 0 \qquad \text{(strain is zero at free end)}$$

Applying these conditions to the general solution, we get

$$0 \Rightarrow B$$

$$\frac{\partial u}{\partial x} = \left(A \frac{p_n}{a} \cos \frac{p_n}{a} x - B \frac{p_n}{a} \sin \frac{p_n}{a} x\right)(C \sin p_n t + D \cos p_n t)$$

$$0 = A \frac{p_n}{a} \cos \frac{p_n}{a} l \,(C \sin p_n t + D \cos p_n t)$$

or $\quad \cos \frac{p_n}{a} l = 0 = \cos \frac{n\pi}{2} \qquad$ where $\quad n = 1, 3, 5 \ldots$

$$p_n = \frac{n\pi a}{2l}$$

$$f_n = \frac{n \cdot a}{4l} \qquad (2\pi f_n = p_n)$$

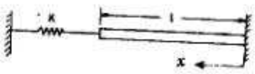
$$f_n = \frac{n}{4l} \sqrt{\frac{E}{\rho}}$$

The general solution of the equation can be written as

$$u(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \sin \frac{n\pi x}{2l} \left(C \sin \frac{n\pi a}{2l} t + D \cos \frac{n\pi a}{2l} t\right)$$

where $\quad a = \sqrt{\frac{E}{\rho}}$

Example 7.3. *A bar of length l fixed at one end and connected at the other end by a spring of stiffness k as shown in figure 7.6. Derive suitable expression of motion for longitudinal vibrations.*



Solution. The general expression for the bar fixed at one end and free at the other can be written directly (as in example 2)

$$u(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \sin \frac{p_n}{a} x \left(C \sin \frac{n\pi a}{2l} t + D \cos \frac{n\pi a}{2l} t\right) \qquad \left(\text{where } p_n = \frac{n\pi a}{2l}\right)$$

Boundary conditions are :

$$(u)_{x=0} = 0$$

$$AE \frac{\partial u}{\partial x}(l,t) = ku(l,t) \qquad \text{(Tensile force = spring force)}$$

Applying the second boundary condition, we get

$$AE \frac{p_n}{a} \cos \frac{p_n}{a} l \left(C \sin \frac{n\pi a}{2l} t + D \cos \frac{n\pi a}{2l} t\right) = k \sin \frac{p_n l}{a} \left(C \sin \frac{n\pi a}{2l} t + D \cos \frac{n\pi a}{2l} t\right)$$

$$\tan \frac{p_n l}{a} = \frac{AE}{k} \frac{p_n}{a}$$

This is the required equation.

Example 7.4. *Find the natural frequencies of a bar shown in Fig. 7.7.*

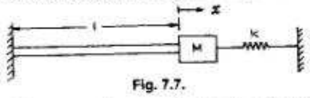


Fig. 7.7.

Solution. The general expression for longitudinal vibrations is shown as

$$u(x,t) = \left(G \sin \frac{p_n}{a} x + H \cos \frac{p_n}{a} x\right)(C \sin p_n t + D \cos p_n t)$$

Boundary conditions are :

$$(u)_{x=0} = 0 \qquad \text{(As the bar is fixed at one end, so displacement is zero)}$$

$$AE \frac{\partial u}{\partial x}(l,t) = -M \frac{\partial^2 u}{\partial t^2}(l,t) \qquad \text{(Tensile force is equal to inertia force in the bar due to mass } M)$$

$$AE \frac{\partial u}{\partial x}(l,t) = Ku(l,t) \qquad \text{(Tensile force = spring force)}$$

Applying the first boundary condition, we get

$$H = 0$$

Applying the second boundary condition, we get

$$AEG \frac{p_n}{a} \cos \frac{p_n}{a} l \,(C \sin p_n t + D \cos p_n t)$$

$$= MGp_n^2 \sin \frac{p_n}{a} l \,(C \sin p_n t + D \cos p_n t)$$

$$\frac{AE}{aMp_n} = \tan \frac{p_n}{a} l$$

Applying third boundary condition

$$AEG \frac{p_n}{a} \cos \frac{p_n}{a} l \,(C \sin p_n t + D \cos p_n t) = KG \sin \frac{p_n}{a} l \,(C \sin p_n t + D \cos p_n t)$$

$$\frac{AE}{K} \frac{p_n}{a} = \tan \frac{p_n}{a} l$$

Comparing the results of second and third boundary conditions, we get

$$\frac{AE}{aMp_n} = \frac{AEp_n}{aK}$$

$$\therefore \qquad p_n^2 = \frac{K}{M} \quad \text{or} \quad p_n = \sqrt{\frac{K}{M}}$$

Example 7.5. *A bar of uniform cross-section having length l is fixed at both ends as shown in figure 7.8. The bar is subjected to longitudinal vibrations having a constant velocity V_0 at all points. Derive suitable mathematical expression of longitudinal vibration in the bar.*

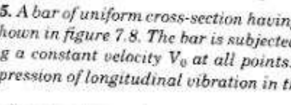


Fig. 7.8

Solution. The general expression for governing longitudinal vibrations can be written as

$$u(x,t) = \sum \left(G \sin \frac{p_n}{a} x + H \cos \frac{p_n}{a} x\right)(C \sin p_n t + D \cos p_n t)$$

Boundary conditions are :

$$u(o,t) = 0$$
$$u(l,t) = 0$$

Initial conditions are :

$$u(x,0) = 0$$
$$\dot{u}(x,0) = V_0$$

Since the bar is fixed at both ends, so with the help of two boundary conditions, we get

$$u(x,t) = \sum_{n=1,2,3\ldots}^{\infty} \sin \frac{n\pi x}{l} (C \sin p_n t + D \cos p_n t)$$

(As $H = 0$ from first B.C. and from second B.C. $\sin \frac{p_n l}{a} = 0 = \sin n\pi$, $\quad p_n = \frac{n\pi a}{l}$)

Applying first initial condition, we get

$$0 = \sum_{n=1,2,3,\ldots}^{\infty} \sin \frac{n\pi x}{l} \cdot D$$

$$\Rightarrow \qquad D = 0$$

Then expression is

$$u(x,t) = \sum_{n=1,2,3\ldots}^{\infty} \sin \frac{n\pi x}{l} \cdot C \sin p_n t$$

Applying second boundary condition

$$\dot{u}(x,t) = \sum_{n=1,2,3\ldots}^{\infty} \sin \frac{n\pi x}{l} \cdot C p_n \cos p_n t$$

$$\dot{u}(x,0) = \sum_{n=1,2,3\ldots}^{\infty} C p_n \sin \frac{n\pi x}{l} = V_0$$

or $$C = \frac{2}{n\pi a} \int_0^l V_0 \sin \frac{n\pi x}{l} dx \qquad \text{(See Equation 7.2.18)}$$

$$C = \frac{2V_0 l}{n^2\pi^2 a}(1 - \cos n\pi)$$

So $$C = \frac{4V_0 l}{n^2\pi^2 a} \qquad \text{when } n = 1, 3, 5 \ldots$$

and $$C = 0 \qquad \text{when } n = 2, 4, 6 \ldots$$

Finally, required expression can be written as

$$u(x,t) = \frac{4V_0 l}{\pi^2 a} \sum \frac{1}{n^2} \sin \frac{n\pi x}{l} \sin \frac{n\pi a}{l} t$$

Example 7.6. *Derive the frequency equation of torsional vibrations for a free-free shaft of length l.*

Solution. The general solution for equation of torsional vibration can be written as equation (7.4.6)

$$\theta(x,t)=\left(A\sin\frac{p_n}{a}x+B\cos\frac{p_n}{a}x\right)(C\sin p_nt+D\cos p_nt)$$

Boundary conditions are

$$\frac{\partial\theta}{\partial x}(0,t)=0 \qquad \text{(strains zero at both ends)}$$

$$\frac{\partial\theta}{\partial x}(l,t)=0$$

Applying the above two conditions, we get

$$\frac{\partial\theta}{\partial x}=\frac{p_n}{a}\left(A\cos\frac{p_n}{a}x-B\sin\frac{p_n}{a}x\right)(C\sin p_nt+D\cos p_nt)$$

$$\frac{\partial\theta}{\partial x}=0 \text{ at } x=0$$

$$\Rightarrow \qquad A=0$$

and $$\frac{\partial\theta}{\partial x}=0 \text{ at } x=l$$

$$0=B\frac{p_n}{a}\sin\frac{p_n}{a}l(C\sin p_nt+D\cos p_nt)$$

$$\sin\frac{p_n}{a}l=\sin n\pi$$

or $$p_n=\frac{n\pi a}{l} \text{ or } f_n=\frac{na}{2l}=\frac{n}{2l}\sqrt{\frac{G}{\rho}}$$

where $a=\sqrt{\frac{G}{\rho}}$ and $n=1,2,3....$

The general solution can be expressed as

$$\theta(x,t)=\sum_{n=1,2,3,...}^{\infty}\cos\frac{n\pi x}{l}\left(C\sin\frac{n\pi at}{l}+D\cos\frac{n\pi at}{l}\right)$$

Example 7.7. *Derive frequency equation for a beam with both ends free and having transverse vibration.* (Roorkee Uni., 84-85)

Solution. The general solution for transverse vibrations can be written as (refer equation 7.5.9)

$$y(x,t)=A\cosh cx+B\sinh cx+C\cos cx+D\sin cx$$

where $c^2=p_n\sqrt{\frac{\rho A}{EI}}$

The boundary conditions for a free-free beam are :

$$\frac{d^2y}{dx^2}(0,t)=0 \qquad \text{(Zero bending moment)}$$

$$\frac{d^2y}{dx^2}(l,t)=0 \qquad \text{(Zero bending moment)}$$

$$\frac{d^3y}{dx^3}(0,t)=0 \qquad \text{(Zero shear force)}$$

$$\frac{d^3y}{dx^3}(l,t)=0 \qquad \text{(Zero shear force)}$$

Applying the boundary conditions to the general solution, we get

$$\frac{d^2y}{dx^2}(x,t)=c^2[A\cosh cx+B\sinh cx-C\cos cx-D\sin cx]$$

$$\frac{d^3y}{dx^3}(x,t)=c^3[A\sinh cx+B\cosh cx+C\sin cx-D\cos cx]$$

$$\frac{d^2y}{dx^2}(0,t)=c^2(A-C)=0$$

$$\Rightarrow A=C$$

$$\frac{d^3y}{dx^3}(0,t)=c^3(B-D)=0$$

$$\Rightarrow B=D$$

$$\frac{d^2y}{dx^2}(l,t)=c^2[A(\cosh cl-\cos cl)+B(\sinh cl-\sin cl)]=0$$

$$\frac{d^3y}{dx^3}(l,t)=c^3[A(\sinh cl+\sin cl)+B(\cosh cl-\cos cl)]=0$$

$$A(\cosh cl-\cos cl)+B(\sinh cl-\sin cl)=0$$

$$A(\sinh cl+\sin cl)+B(\cosh cl-\cos cl)=0$$

or $$(\cosh cl-\cos cl)^2-(\sinh^2 cl-\sin^2 cl)=0$$

$$\cosh^2 cl+\cos^2 cl-2\cosh cl\cos cl-\sinh^2 cl+\sin^2 cl=0$$

as $$\cosh^2 cl-\sinh^2 cl=1 \text{ and } \cos^2 cl+\sin^2 cl=1$$

So $$\cosh cl\cos cl=1$$

Example 7.8. *Find frequency equation of a uniform beam fixed at one end and free at the other for transverse vibrations.*

(Roorkee Uni., 83-84)

Solution. The general solution for transverse vibrations can be seen as

$$y(x,t)=A\cosh cx+B\sinh cx+C\cos cx+D\sin cx$$

The boundary conditions for such a case are :

$$y(0,t)=0 \qquad \text{(zero deflection at fixed end)}$$

$$\frac{dy}{dx}(0,t)=0 \qquad \text{(zero slope)}$$

$$\frac{d^2y}{dx^2}(l,t)=0 \qquad \text{(zero bending moment)}$$

$$\frac{d^3y}{dx^3}(l,t)=0 \qquad \text{(zero shear force)}$$

Applying boundary conditions, we get

$$0=A+C, \quad A=-C$$

$$\frac{dy}{dx}(x,t)=c(A\sinh cx+B\cosh cx-C\sin cx+D\cos cx)=0$$

$$\frac{dy}{dx}(0,t)=0=B+D$$

and $$B=-D$$

$$\frac{d^2y}{dx^2}(l,t)=c^2[A(\cosh cl+\cos cl)+B(\sinh cl+\sin cl)]=0$$

$$\frac{d^3y}{dx^3}(l,t)=c^3[A(\sinh cl-\sin cl)+B(\cosh cl+\cos cl)]=0$$

or $$[\cosh cl+\cos cl]^2-(\sinh^2 cl-\sin^2 cl)=0$$

$$\cosh^2 cl+\cos^2 cl+2\cosh cl\cos cl-\sinh^2 cl+\sin^2 cl=0$$

Solving, we get

$$\cosh cl\cos cl+1=0$$

The above equation can be solved for cl to find the natural frequency of the system.

Example 7.9. ***Determine the normal functions for free longitudinal vibration of a bar of length l and uniform cross-section. One end of the bar is fixed and the other free.*** **[P.U., 89]**

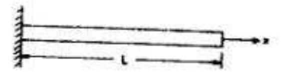


Fig. 7.9.

Solution. **The system is shown in figure 7.9.**

The general expression for longitudinal vibration of bar is shown as

$$u(x,t)=\left(A\sin\frac{p}{a}x+B\cos\frac{p}{a}x\right)(C\sin pt+D\cos pt)$$

The boundary conditions are :

(*i*) $u(0,t)=0$ (at $x=0$, the fixed end)

(*ii*) $\left(\frac{\partial u}{\partial x}\right)_{x=L}=0$

Application of boundary condition (*i*) gives

$$u(0,t)=B(C\sin pt+D\cos pt)=0$$

So $$B=0$$

Application of boundary condition (*ii*) gives

$$\left(\frac{\partial u}{\partial x}\right)_{x=L}=0=A\frac{p}{a}\cos\frac{p}{a}L$$

Thus $$\cos\frac{p}{a}L=\cos\frac{n\pi}{2} \qquad \text{where } n=1,3,5......$$

$$\frac{p}{a}L=\frac{n\pi}{2}$$

$$p=\frac{n\pi a}{2L}$$

$$u(x,t)=\sum_{n=1,3,5}^{\infty}\sin\frac{n\pi x}{2L}(C\sin pt+D\cos pt)$$

Example 7.10. *A uniform beam fixed at one end and simply supported at the other is having transverse vibrations. Derive suitable expression for frequency.*

Solution. The general solution for transverse vibration is given as

$$y(x,t)=A\cosh cx+B\sinh cx+C\cos cx+D\sin cx$$

The boundary conditions for this case are :

$$\left.\begin{array}{l} y(0,t)=0 \\ \frac{dy}{dx}(0,t)=0 \end{array}\right\} \text{for fixed end}$$

$$\left.\begin{array}{l} y(l,t)=0 \\ \frac{d^2y}{dx^2}(l,t)=0 \end{array}\right\} \text{for simply supported end}$$

Applying the boundary conditions, we get

$$y(0,t)=A+C=0$$

$$\frac{dy}{dx}(x,t)=c[A\sinh cx+B\cosh cx-C\sin cx+D\cos cx]$$

$$\frac{d^2y}{dx^2}(x,t)=c^2[A\cosh cx+B\sinh cx-C\cos cx-D\sin cx]$$

$$\frac{dy}{dx}(0, t) = B + D = 0$$

$$y(l, t) = A(\cosh cl - \cos cl) + B(\sinh cl - \sin cl) = 0$$

and $$\frac{d^2y}{dx^2}(l, t) = c^2[A \cosh cl + B \sinh cl - C \cos cl - D \sin cl] = 0$$

$$\Rightarrow A(\cosh cl + \cos cl) + B(\sinh cl + \sin cl) = 0$$

$$A(\cosh cl - \cos cl) + B(\sinh cl - \sin cl) = 0$$

$$A(\cosh cl + \cos cl) + B(\sinh cl + \sin cl) = 0$$

Eliminating A and B from the above two equations, we get

$$(\cosh cl - \cos cl)(\sinh cl + \sin cl) - (\sinh cl - \sin cl)(\cosh cl + \cos cl) = 0$$

Solving it, we get frequency equation as

$$\cos cl \sinh cl - \sin cl \cosh cl = 0$$

$$\tan cl = \tanh cl.$$

EXAMPLE 7.11. *A bar fixed at one end is pulled at the other end with a force P. The force is suddenly released. Investigate the vibration of the bar.* [P.U., 92]

SOLUTION.

Fig. 7.10.

The system is shown in figure 7.10.

One end of the bar is fixed and the other is free. The general solution for longitudinal vibrations of bar as given by equation (7.3.7) is

$$u(x, t) = \left(A \sin \frac{p}{a} x + B \cos \frac{p}{a} x\right)(C \sin pt + D \cos pt)$$

Boundary conditions

(i) At the fixed end $u(x, t) = 0$, at $x = 0$

(ii) At the free end $\left(\frac{\partial u}{\partial x}\right)_{x=L} = 0$ (assuming no force)

First of all, we derive the expression for longitudinal vibrations for the bar.

Using first boundary condition

$$0 = B(C \sin pt + D \cos pt)$$

which gives $B = 0$

So $$u(x, t) = A \sin \frac{p}{a} x (C \sin pt + D \cos pt)$$

Using second boundary condition, we get

$$\left(\frac{\partial u}{\partial x}\right)_{x=L} = 0 = A \cdot \frac{p}{a} \cos \frac{p}{a} x (C \sin pt + D \cos pt)$$

In the above expression, A cannot be equal to zero, so

$$\frac{p}{a} \cos \frac{p}{a} L = 0 = \cos \frac{n\pi}{2} \qquad \text{where } n = 1, 3, 5.....$$

$$\cos \frac{p}{a} L = \frac{n\pi}{2}$$

$$p = \frac{n\pi a}{2L}$$

Thus the general expression for longitudinal vibrations can be

$$u(x, t) = \sum_{n=1,3,5}^{\infty} \sin \frac{n\pi x}{2L}\left(C_n \sin \frac{n\pi at}{2L} + D_n \cos \frac{n\pi at}{2L}\right)$$

where $$a = \sqrt{\frac{E}{\rho}}$$

Let us assume that when force P is applied the unit elongation is ϵx at time $t = 0$.

The boundary conditions are :

(i) $u(x, t) = \epsilon x$ at $t = 0$

(ii) $\frac{\partial u}{\partial t}(x, t) = \dot{u} = 0$ at $t = 0$

Applying second boundary condition, we get

$$\dot{u} = 0 = \sin \frac{n\pi x}{2L} C_n \cos \frac{n\pi at}{2L} \cdot \frac{n\pi a}{2L}$$

which gives $C_n = 0$

Now $$u(x, t) = \sum_{n=1,3,5}^{\infty} \sin \frac{n\pi x}{2L} \cdot D_n \cos \frac{n\pi at}{2L}$$

Using first boundary condition, we have

$$\int_0^L \epsilon x \, dx = \int_0^L D_n \sin \frac{n\pi x}{2L} \, dx$$

Multiplying both sides by $\sin \frac{n\pi x}{2L}$ and integrating in the limit 0 to L, we get

$$\int_0^L D_n \sin \frac{n\pi x}{2L} \sin \frac{n\pi x}{2L} \, dx = \int_0^L \epsilon x \sin \frac{n\pi x}{2L} \, dx$$

Solving it

$$D_n = \frac{2\epsilon}{L} \int_0^L \sin \frac{n\pi x}{2L} \, dx$$

$$= \frac{8\epsilon L}{n^2\pi^2} \sin \frac{n\pi x}{2L}\Big|_0^L = \frac{8\epsilon L}{n^2\pi^2}(-1)^{(n-1)/2}$$

Finally, the expression for longitudinal vibration is

$$u(x, t) = \frac{8\epsilon L}{\pi^2} \sum_{n=1,3,5,...}^{\infty} (-1)^{(n-1)/2} \sin \frac{n\pi x}{2L} \cos \frac{n\pi at}{2L}$$

EXAMPLE 7.12. *A uniform string of length l and a large initial tension S, stretched between two supports, is displaced laterally through a distance a_0 at the centre as shown in figure 7.11, and is released at $t = 0$. Find the equation of motion for the string.* [P.U., 87, 88]

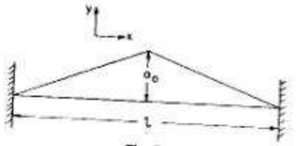


Fig. 7.11.

SOLUTION. The general equation (7.2.8) can be written as

$$y(x, t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{p}{a}\right)x + B_n \sin\left(\frac{p}{a}\right)x\right][C_n \cos(pt) + D_n \sin(pt)]$$

The boundary conditions are :

(i) $y(0, t) = 0$, at $x = 0$

(ii) $y(l, t) = 0$, at $x = l$

Applying the first boundary condition, we have

$$A_n = 0$$

The above equation can be written as

$$y(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{p}{a}\right)x (C_n \cos pt + D_n \sin pt)$$

Now we apply second boundary condition to the above equation

$$y(l, t) = 0 = \sin\left(\frac{p}{a}\right)l$$

Thus $$\sin\left(\frac{p}{a}\right)l = \sin n\pi$$

So $$p = \frac{n\pi a}{l}$$

Again the equation can be modified as

$$y(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l}\left[C_n \cos\left(\frac{n\pi a}{l}\right)t + D_n \sin\left(\frac{n\pi a}{l}\right)t\right]$$

The initial conditions for the system can be described as

(i) $y(x, 0) = \frac{2a_0 x}{l}$, $0 \le x < l/2$

(ii) $y(x, 0) = 2a_0\left(1 - \frac{x}{l}\right)$, $l/2 \le x \le l$

(iii) $\left(\frac{dy}{dt}\right)_{t=0} = 0$

Applying the initial condition (iii), we get

$$\left(\frac{dy}{dt}\right)_{t=0} = \sin \frac{n\pi x}{l}\left[\frac{n\pi a}{l} C_n \cdot 0 + \frac{n\pi a}{l} \cdot D_n\right]$$

$$= \sin \frac{n\pi x}{l} \cdot D_n = 0$$

So $$D_n = 0$$

Now the equation remains

$$y(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} C_n \cos \frac{n\pi a}{l} t$$

Initial conditions (i) and (ii) can be applied now as

$$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = \frac{2a_0 x}{l} \qquad 0 \le x \le l/2$$

$$= 2a_0(1 - x/l) \qquad l/2 \le x \le l$$

If we find C_n, we can find the solution.

Multiply both sides by $\sin \frac{n\pi x}{l}$ and integrate in the limit from $x = 0$ to $x = l$.

$$C_n \sum_{n=1}^{\infty} \sin^2 \frac{n\pi x}{l}\, dx = \int_0^{l/2} \frac{2a_0 x}{l} \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} 2a_0\left(1 - \frac{x}{l}\right)\sin \frac{n\pi x}{l}\, dx$$

$$C_n \sum_{n=1}^{\infty} \frac{1}{2}\left(1 - \cos \frac{2n\pi x}{l}\right) dx = \int_0^{l} \frac{2a_0 x}{l} \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} 2a_0\left(1 - \frac{x}{l}\right)\sin \frac{n\pi x}{l}\, dx$$

Solving the above terms, we get

$C_n = 0$, when n is even

$= (-1)^{\frac{n-1}{2}}\left(\frac{8a_0}{n^2\pi^2}\right)$, when n is odd

Now the equation of motion for the string can be written as

$$y(x, t) = \sum_{n=1, 3, 5, \ldots} (-1)^{\frac{n-1}{2}} \frac{8a_0}{n^2\pi^2} \sin \frac{n\pi x}{l} \cos \frac{n\pi a}{l} t$$

$$= \frac{8a_0}{\pi^2}\left[\sin \frac{\pi x}{l} \cos \frac{\pi a}{l} t - \frac{1}{9} \sin \frac{3\pi x}{l} \cos \frac{3\pi a t}{l} + \frac{1}{25} \sin \frac{5\pi x}{l} \cos \frac{5\pi a}{l} t\right]$$

Example 7.13. *Determine the normal functions in transverse vibration for a simply supported beam of length l and uniform cross section.* [P.U., 88, 94, Roorkee 90-91]

Solution. The equation of motion for transverse vibrations of beams of uniform cross-section is given by equation (7.5.9)

$$y(x, t) = A \cosh cx + B \sinh cx + C \cos cx + D \sin cx$$

When the beam is simply supported, the displacement and bending moment are zero at both ends.

The boundary conditions are :

(*i*) $y(0, t) = 0$, at $x = 0$ (*ii*) $y(l, t) = 0$, at $x = l$

(*iii*) $\left(\frac{d^2y}{dx^2}\right)_{x=0} = 0$ (*iv*) $\left(\frac{d^2y}{dx^2}\right)_{x=l} = 0$

There are four unknowns A, B, C and D with four boundary conditions, so they can be determined.

Application of boundary condition (*i*) gives

$$0 = A + C$$

Application of boundary condition (*iii*) gives

$$0 = -A + C$$

So both the constants A and C are zero.

Now the equation of motion remains

$$y(x, t) = B \sinh cx + D \sin cx$$

Application of boundary condition (*ii*) gives

$$0 = B \sinh cl + D \sin cl$$

Application of boundary condition (*iv*) gives

$$\left(\frac{d^2y}{dx^2}\right)_{x=l} = 0 = -Dc^2 \sin cl + Bc^2 \sinh cl$$

$$= -D \sin cl + B \sinh cl$$

Thus $B = 0$ and $\sin cl = 0 = \sin n\pi$ where $n = 1, 2, 3$......

$$C = \frac{n\pi}{l}$$

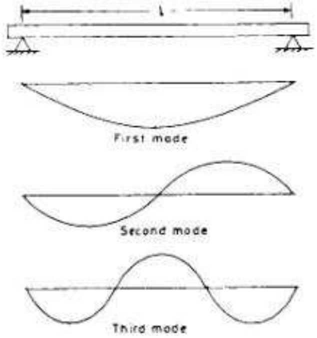


Fig. 7.12.

Thus equation of motion can be

$$y(x, t) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{l}$$

The normal functions can be written as

$$y_1 = D_1 \sin \frac{\pi x}{l}$$

$$y_2 = D_2 \sin \frac{2\pi x}{l}$$

$$y_3 = D_3 \sin \frac{3\pi x}{l}$$

and so on.

The mode shapes for a simply supported beam will be as shown in figure 7.12.

Example 7.14. *Determine the equation for the natural frequencies of a uniform rod in torsional oscillation with one end fixed and the other end free.* [P.U., ME 89]

Solution. The general solution for an equation of torsional vibration can be written as

$$\theta(x, t) = \left(A \sin \frac{p}{a} x + B \cos \frac{p}{a} x\right)(C \sin pt + D \cos pt)$$

The boundary conditions are :

(*i*) $\theta(0, t) = 0$ at $x = 0$

(*ii*) $\frac{\partial \theta}{\partial x} = 0$ at $x = l$

Application of condition (*i*) gives

$$0 = B$$

Application of boundary condition (*ii*) gives

$$\frac{\partial \theta}{\partial x} = 0 = \frac{\partial}{\partial x}\left[A \sin \frac{p}{a} x\, (C \sin pt + D \cos pt)\right]$$

$$0 = A \frac{p}{a} \cos \frac{p}{a} l$$

Here $A \neq 0$

So $\cos \frac{p}{a} l = 0 = \cos \frac{n\pi}{2}$

Thus $p = \frac{n\pi a}{2l}$ where $n = 1, 3, 5$

The torsional vibration of the shaft can be written as

$$\theta(x, t) = \sum_{n=1, 3, 5}^{\infty} \sin \frac{n\pi x}{2l}\left(C_n \sin \frac{n\pi a}{2l} t + D_n \cos \frac{n\pi a}{2l} t\right)$$

Example 7.15. *A bar is free at both ends and is initially stretched y static force P acting at the ends. The forces are released instantaneously. Derive the frequency equation/expression for natural frequencies, normal function and general series for free vibration* [P.U., 92]

Solution. Let us assume that ϵ is the unit extension at time $t = 0$.

The initial conditions are

$$u(x, 0) = \frac{\epsilon l}{2} - \epsilon x$$

and

$$\frac{du}{dt} = \dot{u} = 0 \quad \text{at } t = 0$$

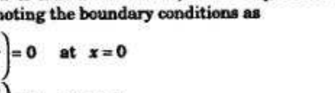


Fig. 7.13.

Refer Fig. 7.13.

For longitudinal vibration of uniform bar, the general expression is given as

$$u(x, t) = \sum_{n=1}^{\infty}\left(A \sin \frac{p}{a} x + B \cos \frac{p}{a} x\right)(C \sin pt + D \cos pt)$$

The bar is free at both ends, so the expression for that can be written by noting the boundary conditions as

(*i*) $\left(\frac{\partial u}{\partial x}\right) = 0$ at $x = 0$

(*ii*) $\left(\frac{\partial u}{\partial x}\right) = 0$ at $x = l$

So $\frac{\partial u}{\partial x} = A \frac{p}{a} \cos \frac{p}{a} x - B \frac{p}{a} \sin \frac{p}{a} x$

Applying boundary condition (*i*), we get

$$0 = A$$

Applying boundary condition (*ii*), we get

$$0 = B \frac{p}{a} \sin \frac{p}{a} l$$

Thus $\sin \frac{p}{a} l = \sin n\pi$

So $p = \frac{n\pi a}{l}$ where $n = 1, 2, 3$.....

The general expression may be written as

$$u(x, t) = \sum_{n=1}^{\infty} \cos \frac{n\pi}{l} x \,(C \sin pt + D \cos pt)$$

The initial conditions are

(iii) $u(x, 0) = \dfrac{\epsilon l}{2} - \epsilon x$

(iv) $\dot{u} = 0 \quad$ at $t = 0$

Applying condition (iii), we get

$$u(x, 0) = \frac{\epsilon l}{2} - \epsilon x = D \cos \frac{n\pi}{l} x$$

and applying condition (iv), we get

$$0 = Cp \cos pt - Dp \sin pt$$

at $\quad t = 0$

gives $\quad C = 0$

Thus $\displaystyle\int_0^l D \cos \frac{n\pi}{l} x \cos \frac{n\pi}{l} x \, dx = \int_0^l \left(\frac{\epsilon l}{2} - \epsilon x\right) \cos \frac{n\pi}{l} x \, dx$

[Multiplying both sides of the above equation by $\cos \frac{n\pi}{l} x$ and integrating from zero to l]

$$D \frac{l}{2} = \int_0^l \left(\frac{\epsilon l}{2} - \epsilon x\right) \cos \frac{n\pi}{l} x \, dx$$

or

$$D = \int_0^l \epsilon \cos \frac{n\pi}{l} x \, dx - \frac{2\epsilon}{l} \int_0^l x \cos \frac{n\pi}{l} x \, dx$$

$$= \frac{2l\epsilon}{n^2\pi^2} (1 - \cos n\pi)$$

So $\quad D = \dfrac{4\epsilon l}{\pi^2 n^2}, \quad$ for n odd values

$= 0, \quad$ for n even values

Finally, we have the general expression as

$$u(x, t) = \frac{4\epsilon l}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{\cos\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi a t}{l}\right)}{n^2}$$

Problems

1. Derive the frequency equation for longitudinal vibration of a rod of two different cross-sectional areas A_1 and A_2. Refer figure 7.1 P.

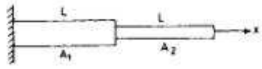


Fig. 7.1 P.

2. A uniform bar of length l is fixed at one end and the free end is stretched uniformly to l_0 and released at $t = 0$. Find the resulting longitudinal vibration.
3. Derive the orthogonality principle of normal modes for longitudinal vibration of uniform bars.
4. Determine the steady state vibration of a simply supported beam of length L acted upon by a concentrated forcing function $F_0 \sin \omega t$. Refer figure 7.2 P.

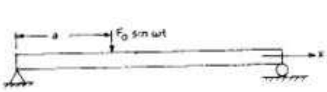


Fig. 7.2 P.

5. Show that the term $\dfrac{\gamma I}{g} \dfrac{\partial^4 y}{\partial x^2 \partial t^2}$ represents the effect of rotary inertia of the beam in the differential equation of motion for the transverse vibrations of beam.
6. Compare the fundamental natural frequencies of a round bar of steel of 200 metres length having diameter 4 cm. Assume the bar to be free at both ends.
7. A uniform string of length l fixed at its ends has a large initial tension. It is plucked at $x = l/3$ through a distance a_0 and released. Determine the subsequent motion.
8. A free bar of uniform section and length l is compressed on the two sides so as to give a total compression ϵ. The compressive forces are released suddenly, simultaneously. Derive an expression for the resultant free vibrations.
9. Show that the differential equation of motion for the torsional vibration of a circular shaft with variable diameter is

$$\rho(x)\, I_p(x) \frac{\partial^2 \theta}{dt^2} = G\left[I_p \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial I_p}{\partial x} \frac{\partial \theta}{\partial x}\right]$$

where $\rho(x)$ is the mass velocity of the shaft material.

10. Compare the frequency of longitudinal vibrations and of transverse vibrations of a copper tube.

11. Mention the conditions of Euler beam. Derive Euler's equation of motion for beam vibration. Determine the natural frequencies and mode shapes for end conditions

   (a) simply supported

   (b) cantilever. [P.U., 94]

12. Find the natural frequencies of a simply supported beam subjected to an axial compressive force. [P.U., 95]
13. Determine the effects of rotary inertia and shear deformation on the natural frequencies of a simply supported uniform beam.
14. Derive the differential equation of motion for the flexural vibration of a beam of uniform cross section and show that the frequency equation of such a cantilever beam of length L is given by

   $\cos k_i L \,.\, \cosh k_i L = -1$ where

   $k_i = (\omega/a)^{1/2}$ and $a = (EI/\rho A)^{1/2}$ (Roorkee Uni. 1999-2000)

8

Transient Vibration

8.1 INTRODUCTION

When some external excitation is applied to a system, two types of motion, namely, the steady state and transient motion are generated. The steady state motion is not dependent on time and so it persists. The transient motion is temporary and time dependent so it vanishes soon.

In many cases, we only consider steady state motion. However, the transient vibration or motion is important in both the cases when the excitation is sudden and unexpected or continuous. The system vibrates with its natural frequency and the amplitude is purely dependent on the magnitude, time and nature of the excitation.

The examples of transient vibrations are : the air pressure pulse created by gunfire, the dropping of package on hard floors, punching operations, moving of automobile on uneven surface or curbs on the road, etc.

In this chapter use of Laplace transform method is made which solves the linear differential equations and finally gives the desired form of solution.

8.2 THE LAPLACE TRANSFORM

The Laplace transform is used in solving vibration problems. It is applicable to both, the transient and steady state conditions of the system, having any number of degree of freedom. It is applied to differential equations without going through a derivation or integration. Tables are readily available to see the transform or reverse transform directly without much difficulty. This method is useful for the analysis and synthesis of dynamic systems. This method does not require the evaluation of constants of integration separately as in other methods.

The Laplace transform converts a given function $f(t)$ of real variable t into a function $F(s)$ of the complex variable s by the operation given

$$F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt = L\,[f(t)]$$

The application of Laplace transform to a simple spring mass system is shown here.

The equation of motion for simple spring-mass system with an excitation $f(t)$ can be written as

$$m\ddot{x} + c\dot{x} + kx = f(t) \qquad \text{...(8.2.1)}$$

Let $\left.\begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= \dot{x}_0 \end{aligned}\right\}$ initial conditions

The application of Laplace transform to left side of equation (8.2.1) can be written as

$$L(m\ddot{x} + c\dot{x} + kx) = mL\,\ddot{x}(t) + cL(\dot{x}) + kL\,x(t)$$

$$= m(s^2 X(s) - sx_0 - \dot{x}_0) + c(s X(s) - x_0) + kX(s)$$

$$= (ms^2 + cs + k) X(s) - (ms + c) x_0 - m\dot{x}_0$$

If $F(s)$ is the transform of excitation $f(t)$, then equation (8.2.1) can be written as

$$(ms^2 + cs + k) X(s) - (ms + c) x_0 - m\dot{x}_0 = F(s)$$

or $$X(s) = \frac{[F(s) + (ms + c) x_0 + m\dot{x}_0]}{(ms^2 + cs + k)} \qquad \text{...(8.2.2)}$$

Equation (8.2.2) is called the subsidiary equation of equation (8.2.1).

Step Input

The transient excitation acting on the system is constant in magnitude. If any how the excitation is not constant it can be broken into small sections of constant force. Then this constant excitation is supposed to act on the system to analyse the vibration for that particular time. The response of the system can be obtained till this force is over and the next force starts acting on the system. Thus the end conditions of the first force are the starting conditions of the second force.

The term $X(s)$ is known as the response transform and $(ms^2 + cs + k)$ as the characteristic function. The inverse transform of $X(s)$ gives the function $x(t)$. Table 8.1 gives various Laplace transform pairs though detailed pairs are presented in Appendix.

Table 8.1. Laplace Transform Pairs

$F(s)$	$f(t)$
$\frac{1}{s}$	$u(t)$
$\frac{1}{s^2}$	t
	t^n

$\frac{1}{s+a}$	e^{-at}
$\frac{s}{s^2+\omega^2}$	$\cos \omega t$
$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$
$\frac{a}{s^2-a^2}$	$\sin h\, at$
$\frac{s}{s^2-a^2}$	$\cos h\, at$
$\frac{e^{-as}}{s}$	$u(t-a)$
e^{-as}	$\delta(t-a)$

8.3 TRANSFORMS OF PARTICULAR FUNCTIONS

Certain type of functions which are quite useful and are obtained by the combination of other functions, are discussed here.

Unit Step Function

The unit step function may be written mathematically as

$$u(t) = \begin{cases} 0, & \text{when } t < 0 \\ 1, & \text{when } t > 0 \end{cases} \qquad \text{...(8.3.1)}$$

If $f(t) = A$, its Laplace transform can be written as

$$L\,u(t) = F(s) = LA = \int_0^\infty Ae^{-st}\,dt = -\frac{A}{s} e^{-st} \Big|_0^\infty$$

$$= \frac{A}{s}$$

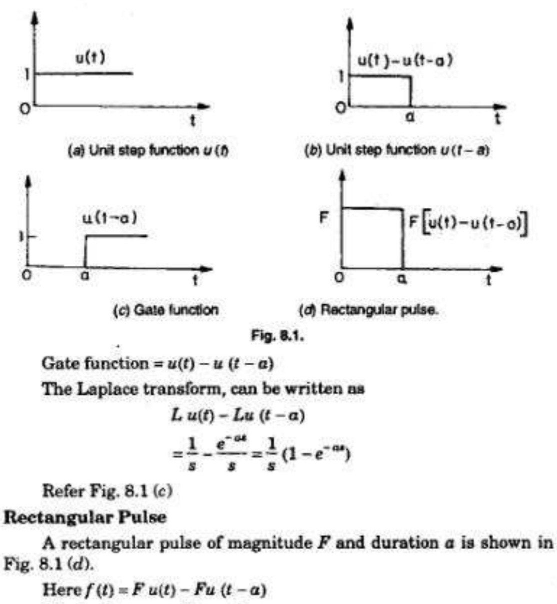
If $A = 1$, the unit step function can be as defined by equation (8.3.1). Thus $Lu(t) = \frac{1}{s}$. Refer Fig. 8.1 (*a*)

If this function is shifted to the right along axis t, as shown in Fig. 8.1 (*b*), the shifted unit step function can be defined mathematically as

$$u(t-a) = \begin{cases} 0, & \text{when } t < a \\ 1, & \text{when } t > a \end{cases} \qquad \text{...(8.3.2)}$$

Thus $Lu(t-a) = \int^\infty u(t-a)\,e^{-st}\,dt = \int^\infty e^{-st}\,dt = \frac{e^{-as}}{s}$

If one unit step function is subtracted from the other, we get a gate function. Mathematically, it can be written as



(*a*) Unit step function $u(t)$ (*b*) Unit step function $u(t-a)$

(*c*) Gate function (*d*) Rectangular pulse.

Fig. 8.1.

Gate function $= u(t) - u(t-a)$

The Laplace transform, can be written as

$$L\,u(t) - Lu(t-a)$$

$$= \frac{1}{s} - \frac{e^{-as}}{s} = \frac{1}{s}(1 - e^{-as})$$

Refer Fig. 8.1 (*c*)

Rectangular Pulse

A rectangular pulse of magnitude F and duration a is shown in Fig. 8.1 (*d*).

Here $f(t) = F\,u(t) - Fu(t-a)$

The Laplace transform gives

$$L[F\,u(t) - F\,u(t-a)]$$

$$= \frac{F}{s} - \frac{F}{s} e^{-as} = \frac{F}{s}(1 - e^{-as})$$

Thus rectangular pulse can be constructed from the difference of two step functions.

Unit Impulse

We have seen in case of rectangular pulse that the magnitude of the pulse was F and its duration was a. The impulse of force would be $F.a$. We can consider the impulse of force such that $F \,.\, a = 1$.

If the duration of the pulse is decreased, the magnitude of the pulse would increase to maintain the unit area.

In the limit, when a tends to zero, the impulse can be defined as

$$x(t) = \lim_{a \to 0} \frac{1}{a} [u(t) - u(t-a)]$$

The Laplace transform to unit impulse will be

$$L[x(t)] = \lim_{a \to 0} \frac{1}{a} \frac{(1 - e^{-as})}{s} = 1$$

SOLVED EXAMPLES

Example 8.1. *A force $F(t)$ is suddenly applied to a mass m which is supported by a spring with a constant stiffness k. After a short period of time T, the force is suddenly removed. During the time the force is active, it is a constant, F. Determine the response of the system if $t > T$. The spring and mass are initially at rest before the force $F(t)$ is applied.*

Solution. Refer Fig. 8.2.

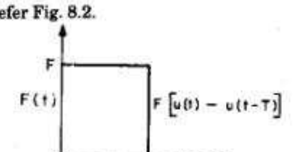


Fig. 8.2.

The equation of motion can be written as

$$m\ddot{x} + kx = F[u(t) - u(t-T)] \qquad \text{...(}i\text{)}$$

Applying Laplace transform to the differential equation

$$L[m\ddot{x}] = m[s^2 X(s) - sx_0 - \dot{x}_0]$$

$$L[kx] = k\,X(s)$$

$$LF[u(t)] = \frac{F}{s}$$

$$LF[u(t-T)] = \frac{Fe^{-sT}}{s}$$

Initial conditions : $x(0) = 0$, $\dot{x}(0) = 0$

Substituting the values in equation (*i*), we get

$$ms^2 X(s) + kX(s) = \frac{F}{s} - \frac{Fe^{-sT}}{s} = \frac{F}{s}(1 - e^{-sT})$$

$$(ms^2 + k) X(s) = \frac{F}{s}(1 - e^{-sT})$$

or $$X(s) = \frac{F}{s}(1 - e^{-sT}) / (ms^2 + k)$$

Defining $\omega_n^2 = \frac{K}{m}$ in the above equation

$$X(s) = \frac{F}{m}\left[\frac{1-e^{-sT}}{s\,(s^2+\omega_n^2)}\right]$$

From the table of Laplace transforms, the inverse

$$L^{-1}\left[\frac{1}{s\,(s^2+\omega_n^2)}\right] = \frac{1}{\omega_n^2}\,(1-\cos\omega_n t)$$

and
$$L^{-1}\left[\frac{e^{-sT}}{s\,(s^2+\omega_n^2)}\right] = \frac{1}{\omega_n^2}\,[1-\cos\omega_n (t-T)]\;u\,(t-T)$$

for $0 < t < T$, the solution is

$$x\,(t) = \frac{F}{m\omega_n^2}\,(1-\cos\omega_n t)$$

and for $t > T$, the solution is given by

$$x\,(t) = \frac{F}{m\omega_n^2}\,[(1-\cos\omega_n t) - 1 - \cos\omega_n t\,(t-T)]$$

Note. $u\,(t-T) = \begin{cases} 0, & \text{if } t<T \\ 1, & \text{if } t>T \end{cases}$ and $u\,(t) = \begin{cases} 0, & \text{if } t<0 \\ 1, & \text{if } t>T \end{cases}$

Example 8.2. *An apparatus of mass m is shipped in a container as shown in Fig. 8.3. In the process of unloading, the container is dropped from a height h to a hard floor. Find the response of the system.*

Solution. When the container hits the hard floor, its velocity is

$$\dot{x} = \sqrt{2gh}$$

The equation of motion when the container is in contact with the hard floor

$$m\ddot{x} + kx = 0$$

The initial conditions are

$$x\,(0) = 0$$
$$\dot{x}\,(0) = \sqrt{2gh}$$

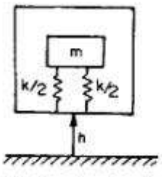


Fig. 8.3. Dropping of package on hard floor.

Applying the Laplace transform to the differential equation of motion

$$L\,[m\ddot{x} + kx] = 0$$
$$m\,[s^2 X(s) - s\,x_0 - \dot{x}_0] + k\,X(s) = 0$$
$$[s^2 X(s) - \sqrt{2gh}] + \frac{k}{m}\,X\,(s) = 0$$

$$(\because\ x(0) = 0 \text{ and } \dot{x}\,(0) = \sqrt{2gh})$$

$$[s^2 X(s) - \sqrt{2gh}] + \omega_n^2\,X\,(s) = 0$$
$$X\,(s)\,(s^2+\omega_n^2) = \sqrt{2gh}$$

or
$$X\,(s) = \frac{\sqrt{2gh}}{s^2+\omega_n^2}$$

Taking the Laplace transform, inverse

$$x\,(t) = L^{-1}X(s) = L^{-1}\,\frac{\sqrt{2gh}}{\omega_n}\,\frac{1\,.\,\omega_n}{s^2+\omega_n^2} = \sqrt{2gh}\;L^{-1}\,\frac{\omega_n}{s^2+\omega_n^2}$$

$$x\,(t) = \frac{\sqrt{2gh}}{\omega_n}\,\sin\omega_n t$$

Maximum acceleration of the mass

$$\ddot{x}_{max} = \frac{\sqrt{2gh}}{\omega_n}\,.\,\omega_n^2 = \omega_n\sqrt{2gh}$$

Example 8.3. *A spring-mass system is shown in Fig. 8.4. If the system is initially relaxed and a step-function excitation is applied to the mass, find the response of the system.*

Solution. F_0 is constant when $t \geq 0$ and it is zero when $t < 0$.

The boundary conditions are :

$$x\,(0) = 0$$
$$\dot{x}\,(0) = 0$$

The differential equation of motion for this system can be written as

$$m\ddot{x} + c\dot{x} + kx = F_0\,u(t)$$

or
$$\ddot{x} + \frac{c\dot{x}}{m} + \frac{k}{m}\,x = \frac{F_0}{m}\,u(t) \qquad \ldots(i)$$

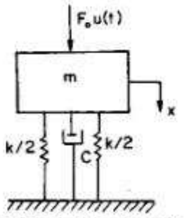


Fig. 8.4. Mechanical system with step function excitation.

Defining $\frac{c}{m} = 2\xi\,\omega_n$, $\omega_n^2 = \frac{k}{m}$ and substituting these values in equation (*i*), we get

$$\ddot{x} + 2\xi\,\omega_n\,\dot{x} + \omega_n^2\,x = \frac{F_0}{m}\,u(t) \qquad \ldots(ii)$$

Taking the Laplace transform of the above equation, we have

$$L\,[\ddot{x} + 2\,\xi\,\omega_n\,\dot{x} + \omega_n^2\,x] = \frac{F_0}{m}\,Lu(t) \qquad (t>0)$$

$$s^2 X(s) - sx(0) - \dot{x}\,(0) + 2\xi\,\omega_n\,[s\,X(s) - x(0)]$$
$$+\,\omega_n^2\,X(s) = \frac{F_0}{m}\,.\,\frac{1}{s}$$

$$(s^2 + 2\xi\,\omega_n\,s + \omega_n^2)\,X\,(s) = \frac{F_0}{m}\,.\,\frac{1}{s} \qquad (\because\ x(0) = 0\,;\ \dot{x}\,(0) = 0)$$

r
$$X(s) = \frac{F_0}{m}\left[\frac{1}{s\,(s^2 + 2\xi\,\omega_n\,s + \omega_n^2)}\right]$$

The right hand side of the above equation can be broken into artial fractions

$$X(s) = \frac{F_0}{m\,.\,\omega_n^2}\left[\frac{1}{s} - \frac{s + 2\xi\,\omega_n}{s^2 + 2\,\xi\,\omega_n\,s + \omega_n^2}\right]$$

The inverse transform of the above equation still cannot be obtained. : is assumed that $\xi < 1$ and the above equation can be written as

$$X\,(s) = \frac{F_0}{s\,.\,\omega_n^2}\left[\frac{1}{m} - \frac{s+\xi\,\omega_n}{(s+\xi\,\omega_n)^2 + (\sqrt{1-\xi}\,.\,\omega_n)^2} - \frac{\frac{\xi}{\sqrt{1-\xi^2}}\,\sqrt{1-\xi^2}\,\omega_n}{(s+\xi\,\omega_n)^2 + (\sqrt{1-\xi^2}\,.\,\omega_n)^2}\right]$$

The inverse Laplace transform of the above equation can easily be otained as

$$x\,(t) = \frac{F_0}{m\omega_n^2}\left[1 - e^{-\xi\omega_n t}\cos\sqrt{1-\xi^2}\,\omega_n t - \frac{\xi}{\sqrt{1-\xi^2}}\,e^{-\xi\,\omega_n t}\sin\sqrt{1-\xi^2}\,\omega_n t\right]$$

$$= \frac{F_0}{k}\left[1 - e^{-\xi\,\omega_n t}\,(\cos\sqrt{1-\xi^2}\,\omega_n t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\,\omega_n t\right]$$

When the system is undamped i.e. $\xi = 0$, the response equation can e written as

$$x\,(t) = \frac{F_0}{k}\,[1 - \cos\omega_n t]$$

The first term of the above equation represents steady state brations and the second term transient.

Example 8.4. *A spring mass-system shown in Fig. 8.5 which is subjected to harmonic force F cos ωt. Determine the sponse of the system.*

Given : $x\,(0) = .01\ m$
$\dot{x}\,(0) = .04\ m/sec$
$\omega = 30\ rad/sec$
$F = 1000\ N$
$m = 10\ kg$
$k = 500\ N/m$

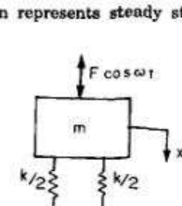


Fig. 8.5. Single degree freedom-spring mass system.

Solution. The differential equation of motion of the system, can be written as

$$m\ddot{x} + kx = F\cos\omega t$$

Applying Laplace transform, we get

$$mL\,(\ddot{x}) + kL\,(x) = FL\cos\omega t$$

$$m\,[s^2 X(s) - s\,x(0) - \dot{x}\,(0)] + k\,X(s) = F\,\frac{s}{s^2+\omega^2}$$

Substituting the values of various terms in the above equation, we get

$$10\,[s^2 X\,(s) - 0.01\,s - 0.04] + 500\ X(s) = \frac{1000\,s}{s^2 + (30)^2}$$

Solving the above equation for $X\,(s)$, we get

$$X\,(s) = \frac{0.1\,s^3 + 0.4\,s^2 + 1090\,s + 360}{(10\,s^2+500)\,(s^2+900)} \qquad \ldots(i)$$

It can be solved by inserting certain constants A, B, C and D.

$$X\,(s) = \frac{As+B}{10\,s^2+500} + \frac{Cs+D}{s^2+900} \qquad \ldots(ii)$$

$$= \frac{(As+B)\,(s^2+900) + (Cs+D)\,(10\,s^2+500)}{(10\,s^2+500)\,(s^2+900)}$$

$$= \frac{A\,s^3 + 900\,A\,s + Bs^2 + 900\,B + 10\,Cs^3 + 500\,Cs + 10\,Ds^2 + 500\,D}{(10\,s^2+500)\,(s^2+900)}$$

$$= \frac{(A+10\,C)\,s^3 + (B+10\,D)\,s^2 + (900\,A + 500\,C)\,s + 900\,B + 500\,D}{(10\,s^2+500)\,(s^2+900)}$$

Equating the like powers of s both sides of eqn. (*i*)

$$\left.\begin{aligned} A + 10\,C &= 0.1, \\ B + 10\,D &= 0.4, \\ 900\,A + 500\,C &= 1090, \\ 900\,B + 500\,D &= 360 \end{aligned}\right\} \qquad \ldots(iii)$$

By solving equation (*iii*), we have

$$A = \frac{21.7}{17} = 1.276$$
$$B = 0.40$$
$$C = \frac{-2}{17} = -\,0.117$$
$$D = 0$$

Equation (*ii*) can be written as

$$X(s) = \frac{1.276\, s}{10\, s^2 + 500} + \frac{0.40}{10\, s^2 + 500} - \frac{0.117\, s}{s^2 + 900}$$

$$= \frac{1.276\, s}{10\, (s^2 + 50)} + \frac{0.4}{10\, (s^2 + 50)} - \frac{0.117\, s}{s^2 + (30)^2}$$

$$X(s) = 0.1276 \left(\frac{s}{s^2 + (5\sqrt{2})^2} \right) + \frac{0.04.5\sqrt{2}}{5\sqrt{2}\, [s^2 + (5\sqrt{2})^2]} - \frac{0.117\, s}{s^2 + (3a)^2}$$

Taking inverse of the Laplace transform, we have

$$x(t) = 0.1276 \cos 5\sqrt{2}\, t + \frac{0.04}{5\sqrt{2}} \sin 5\sqrt{2}\, t - 0.117 \cos 30\, t$$

$$= 0.1276 \cos 5\sqrt{2}\, t + 0.0056 \sin 5\sqrt{2}\, t - 0.117 \cos 30\, t$$

This is the required response of the mass.

EXAMPLE 8.5. *Determine the equation of motion of the mass for free vibrations as shown in Fig. 8.6.*

Given : $m = 10\ kg$, $k = 1000\ N/m$,

$C = 100\ N \cdot sec/m$

$x(0) = 0.001\ m$,

$\dot{x}(0) = 0.10\ m/sec$

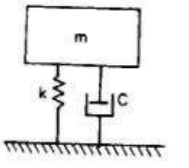


Fig. 8.6.

Solution. The differential equation for the system shown can be written as

$$m\ddot{x} + c\dot{x} + kx = 0$$

Applying the Laplace transform to the above equation, we get

$$m\, [s^2 X(s) - sx(0) - \dot{x}(0)] + C\, [s\, X(s) - x(0)] + kX(s) = 0$$

Substituting the values of the given terms in the above equation, we get

$$10\, [s^2 X(s) - s \times .001 - 0.10] + 100\, [s\, X(s) - .001] + 1000\, X(s) = 0$$

$$s^2 X(s) - .001\, s - 0.10 + 10\, s\, X(s) - .01 + 100\, X(s) = 0 \quad ...(i)$$

The equation (*i*) for $X(s)$ can be written as

$$X(s) = \frac{0.001\, s + 0.20}{s^2 + 10\, s + 100} = \frac{0.001\, s}{s^2 + 10\, s + 100} + \frac{0.20}{s^2 + 10\, s + 100} \quad ...(ii)$$

First term of the above equation can be manipulated as

$$\frac{0.001\, s}{s^2 + 10\, s + 100} = \frac{1000\,(.001\, s + .005) - 5}{1000\, (s^2 + 10\, s + 100)} = \frac{(s + 5) - 5}{1000\, [(s + 5)^2 + (5\sqrt{3})^2]}$$

Let us have first term only $\dfrac{s + 5}{1000\, [(s + 5)^2 + (5\sqrt{3})^2]}$

The second term

$$\frac{0.20 - \dfrac{5}{1000}}{s^2 + 10\, s + 100} = \frac{0.195}{s^2 + 10\, s + 100} = \frac{\dfrac{0.195 \times 5\sqrt{3}}{5\sqrt{3}}}{(s + 5)^2 + (5\sqrt{3})^2}$$

$$= \frac{0.195\,.\, 5\sqrt{3}}{5\sqrt{3}\, [(s + 5)^2 + (5\sqrt{3})^2]} = .0226 \left[\frac{5\sqrt{3}}{(s + 5)^2 + (5\sqrt{3})^2} \right]$$

Equation (*ii*) can be written in the manipulated form as

$$X(s) = \frac{(s + 5)}{1000\, [(s + 5)^2 + (5\sqrt{3})^2]} + 0.0226 \left[\frac{5\sqrt{3}}{(s + 5)^2 + (5\sqrt{3})^2} \right] \quad ...(iii)$$

Taking inverse transform of eqn. (*iii*), we get

$$X(t) = \frac{1}{1000} e^{-5t} \cos 5\sqrt{3}\, t + 0.0226\, e^{-5t} \sin 5\sqrt{3}\, t$$

$$= e^{-5t}\, (.001 \cos 5\sqrt{3} t + 0.0226 \sin 5\sqrt{3}\, t)$$

8.4 DUHAMEL'S INTEGRAL METHOD

Equation (2.2.3A) represents the motion of undamped free vibrations of single degree of freedom system. The equation (2.2.3A) is rewritten as

$$x = x_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t \quad ...(i)$$

In the above equation $x(0) = x_0$ and the initial velocity $\dot{x}(0) = v_0$. This is the general expression of simple harmonic motion. Let us consider a forcing function $F(t')$ for the spring mass system as shown in Fig. 8.7. The function $F(t')$ can be considered to be made up of series of infinitesimally small impulses, each of duration dt'. Each and every impulse influences the motion of the system. At any instant t', the

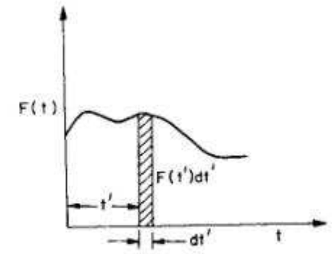


Fig. 8.7.

increment in the impulse is $F(t') \cdot dt'$. This impulse causes instantaneous change in the velocity of mass.

We know that $F(t') = \dfrac{m\, d\dot{x}}{dt'}$

or $\quad d\dot{x} = \dfrac{F(t') \cdot dt'}{m} \quad ...(8.4.1)$

It shows that there is change in momentum and mass is constant. So it is the only velocity ($d\dot{x}$) which is changing. Thus the change in displacement at any time, t due to the change in momentum at some other time t' depends on the time interval $(t - t')$. The change in displacement due to the impulse $F(t)\, dt'$ can be written as

$$dx = \frac{d\dot{x}}{\omega_n} \sin \omega_n (t - t') \quad ...(8.4.2)$$

(Taken from eqn. (*i*) as the initial displacement $x_0 = 0$)

The change in the velocity & displacement will be $d_{\dot{x}}$ and dx in time interval $(t - t')$ respectively.

Substituting the value of $d\dot{x}$ in eqn. (8.4.2) from eqn. (8.4.1), we get

$$dx = \frac{F(t')\, dt'}{m} \cdot \frac{\sin \omega_n}{\omega_n} (t - t') = \frac{F(t')}{m\omega_n} \sin \omega_n (t - t')\, dt' \quad ...(8.4.3)$$

This is the change in displacement due to one small impulse of magnitude $F(t')\, dt'$. For total change in displacement, equation (8.4.3) can be integrated in the time limit from $t' = 0$ to $t' = t$ which will be the sum of all individual impulses, so

$$x = \frac{1}{m\omega_n} \int_0^t F(t') \sin \omega_n (t - t')\, dt' \quad ...(8.4.4)$$

The above equation is known as Duhamel's integral. This method needs integration, so it cannot be applied for mathematically complicated forcing functions. It is very useful method for simple expressions and permits the use of Laplace transform for solving transient problems.

EXAMPLE 8.6. *A spring-mass system consisting of two masses and a coupling spring are shown in Fig. 8.8. If the system is initially at rest on a frictionless horizontal surface and F ($\delta(t)$ is the applied impact on mass m_1, find the motions of the masses.*

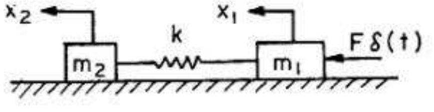


Fig. 8.8.

SOLUTION. The differential equations of motion can be written as

$$m_1 \ddot{x}_1 + kx_1 - kx_2 = F\, \delta(t) \quad ...(i)$$

$$-kx_1 + m_2 \ddot{x}_2 + kx_2 = 0 \quad ...(ii)$$

It is given that the system is initially at rest, so

$$x(0) = 0,\ \dot{x}(0) = 0$$

Applying the Laplace transform to the above equations, we get

$$m_1\, L(\ddot{x}_1) + kL(x_1) - kL(x_2) = FL\, \delta(t)$$

$$m_1\, [s^2 X_1(s) - s\, x(0) - \dot{x}(0)] + kX_1(s) - kX_2(s) = F$$

or $\quad m_1 s^2 + k)\, X_1(s) - kX_2(s) = F \quad ...(iii)$

and $\quad -kL(x_1) + m_2\, L\, \ddot{x}_2 + kL(x_2) = 0$

$$-kX_1(s) + m_2\, [s^2 X_2(s) - sx(0) - \dot{x}(0)] + kX_2(s) = 0$$

$$(k + m_2 s^2)\, X_2(s)) - kX_1(s) = 0 \quad ...(iv)$$

From equations (*iii*) & (*iv*) $X_1(s)$ and $X_2(s)$ can be determined by applying Cramer's rule

$$X_1(s) = \frac{\begin{vmatrix} F & -k \\ 0 & m_2 s^2 + k \end{vmatrix}}{\text{Value of determinant}} = \frac{F\, (m_2 s^2 + k)}{\Delta}$$

$$X_2 = \frac{\begin{vmatrix} m_1 s^2 + k & F \\ -k & 0 \end{vmatrix}}{\text{Value of determinant}} = \frac{Fk}{\Delta}$$

Value of determinant,

$$\Delta = \begin{vmatrix} m_1 s^2 + k & -k \\ -k & m_2 s^2 + k \end{vmatrix}$$

$$= [m_1 m_2 s^2 + (m_2 + m_1)\, k]\, s^2$$

Now $\quad X_1(s) = \dfrac{F \cdot (m_2 s^2 + k)}{[m_1 m_2 s^2 + (m_1 + m_2)\, k]\, s^2} \quad ...(v)$

$$X_2(s) = \frac{F\, k}{[m_1 m_2 s^2 + (m_1 + m_2)\, k]\, s^2} \quad ...(vi)$$

Defining that, $\omega^2 = \dfrac{k\, (m_1 + m_2)}{m_1 m_2}$

Solving equation (*v*)

$$X_1(s) = \frac{F\, (k + m_2 s^2)}{m_1 m_2\, [s^2 + \omega^2]\, s^2} \quad ...(vii)$$

Similarly, solving equation (*vi*)

$$X_2(s) = \frac{F\,k}{m_1\, m_2\, [s^2 + \omega^2]\, s^2} \qquad \text{...}(viii)$$

Taking the inverse Laplace transform of equations (*vii*) & (*viii*), we get the response of the masses $x_1(t)$ and $x_2(t)$.

8.5 PHASE PLANE METHOD

The phase plane method is a graphical method to solve transient vibration problems. We know that displacement and velocity describe the motion of a single degree of freedom system completely. If the displacement and velocity are taken as coordinate axes, the resulting graphical representation is known as phase plane representation. Any point P in this coordinate plane which is known as the phase plane of motion, indicates the dynamic state of the system. The locus traced by P is known as phase trajectory. The motion of the system is represented by the motion of point P in the phase plane. The state of the system depends on time. As the time varies the solution of the system changes.

We know that general expression for simple harmonic motion is given by

$$x = A \sin(\omega_n t + \phi) \qquad \text{...}(i)$$

Differentiating the above equation for velocity, we get

$$\dot{x} = A\,\omega_n \cos(\omega_n t + \phi)$$

or

$$\frac{\dot{x}}{\omega_n} = A \cos(\omega_n t + \phi) \qquad \text{...}(ii)$$

Squaring and adding equations (*i*) and (*ii*), we have

$$x^2 + \left(\frac{\dot{x}}{\omega_n}\right)^2 = A^2 \qquad \text{...}(iii)$$

The above equation is a circle with coordinate x and $\frac{\dot{x}}{\omega_n}$ and radius and centre at the origin. Ref. figure 8.9. There is any point P_1 initially aving the displacement $x(0)$ and velocity $v(0)$. After time t_1 the isplacement and velocity of the system are represented by point P_2. P_2 makes an angle $\omega_n t_1$ with OP_1. OP_1 and OP_2 are the rotating ectors. The displacement of the system at an instant is given by the orizontal projection of the rotating vector on the y-axis. The horizontal rojections of the rotating vector on a time base gives the displacement-me plot of the motion of the system as shown in Fig. 8.9 (*b*). The ertical projection on the time base, gives the velocity-time plot. The entre of the rotating vector 'O' moves up and down on the y-axis. The entre 'O' of the rotating vector will be shifted up by the amount F

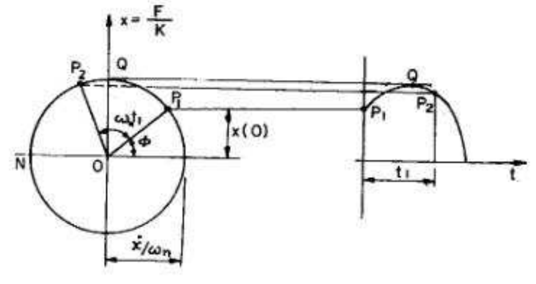


Fig. 8.9.

When $\frac{F}{k}$ is negative, the centre will be shifted downward by the amount $\frac{F}{k}$. In the present case, the static equilibrium displacement is zero and so the centre of circle is located at origin. The angle of the vector depends on time (as $\theta = \omega_n \cdot t$). The end conditions of one era (interval) must be the starting conditions of the next era.

EXAMPLE 8.7. *A spring-mass system initially at rest is subjected to excitation as shown in Fig. 8.10.*

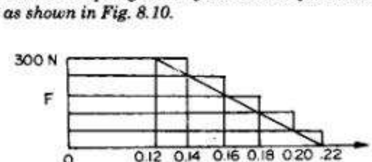


Fig. 8.10.

Given : $m = 10$ kg, $k = 10000$ N/m

SOLUTION. $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10000}{10}} = 31.62$ rad/sec

The angle of the starting point

$$= \omega_n \cdot t = 31.62 \times \frac{180}{\pi} \times .12 = 217.4°$$

The time interval from 0.12 to 0.22 sec is divided into five equal

217.4

The ramp force is approximated as a constant force at time 0.12, 0.14,..........0.22. The time interval is equal i.e. .14 – .12 = .02 sec each. There are six intervals. At these intervals the value of constant forces will be 300 N, 240 N, 180 N, 120 N, 60 N and zero. With the help of Table 8.2 phase plane diagram as shown in Fig. 8.11 can be drawn. Let O be the origin.

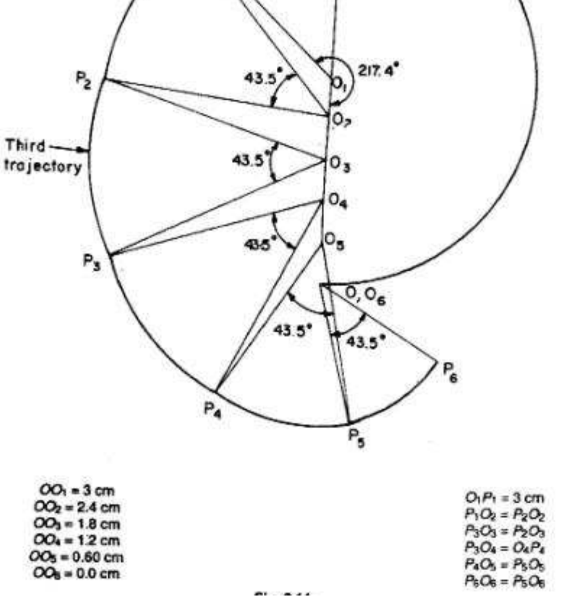


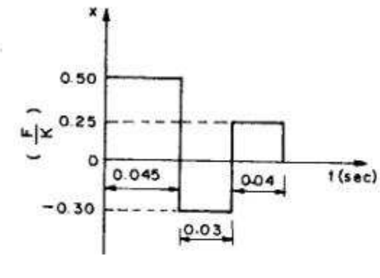
Table 8.2.

Interval	1	2	3	4	5	6
Time Interval (Δt, sec)	0.12	0.02	0.02	0.02	0.02	0.02
Change of angle ($\Delta\theta$, degrees)	217	43.5	43.5	43.5	43.5	43.5
Force (N)	300	240	180	120	60	0
$\frac{\text{Force}}{k}$, $\left(\frac{F}{k}\right)$ in m, which is the shift of origin	$\frac{300}{10000} = 0.03$ m	0.024	0.018	0.012	0.006	0
Corresponding origins	O_1 $OO_1 = 3$ cm	O_2 $OO_2 = 2.4$ cm	O_3 $OO_3 = 1.8$ cm	O_4 $OO_4 = 1.2$ cm	O_5 $OO_5 = 0.6$ cm	O_6 $OO_6 = 0$

Since the value of $\frac{F}{k}$ goes on decreasing, so the origin shifts downwards as can be seen from the table.

The phase trajectory for the first interval is the arc of a circle with included angle 217.4°. OO_1 is the radius of this circle. The centre of the arc for trajectory in the second era (interval) is located at O_2 and the radius of this arc is O_2P_1. The motion conditions represented by point P_1 for the end conditions of first era (i.e. P_1O_2) are the starting conditions for the second era as can be seen from the figure.

EXAMPLE 8.8. *A system having a natural frequency of 15 Hz is allowed to explosive type of input which has been changed to equivalent approximate steps shown in Fig. 8.12. Determine the phase plane plot*



and displacement-time plot. Find the maximum displacement of the system.

SOLUTION. $f = 15$ Hz

we know that $\omega_n = 2\pi f = 2\pi \times 15 = 30\,\pi$ rad/sec

Angle for the first era $= \omega_n \cdot \frac{180}{\pi} \times 0.045 = 243°$

Angle for the second era $= \omega_n \cdot \frac{180}{\pi} \times 0.03 = 162°$

Angle for the third era $= \omega_n \cdot \frac{180}{\pi} \times 0.04 = 216°$

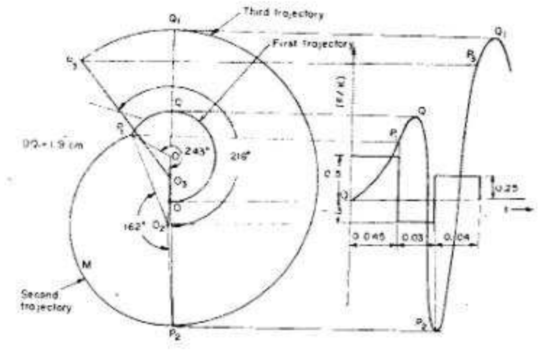
Angle for the third era $= \omega_n \cdot \frac{180}{\pi} \times 0.04 = 216°$

The relevant values are presented in Table 8.3.

Table 8.3.

Era	1	2	3
Time interval (Δt, sec)	0.045	0.03	0.04
Angular difference ($\Delta\theta$, degree)	243	162	216
$\frac{F}{k}$ (in cm) shift of origin	$0.50 = OO_1$	$-0.30 = OO_2$	$0.25 = OO_3$
Origin	O_1	O_2	O_3

Refer Fig. 8.13.



Initially origin is taken as 'O'. The maximum displacement is $OQ_1 = 1.9$ cm. $OO_1 = 0.50$ cm is the radius of first trajectory and the $\angle OQP_1 = 243°$. O_2 is the centre of second trajectory which is below the origin by 0.30 cm distance. Join O_2 to P_1. Now O_2P_1 is the radius of second trajectory with angle $P_1MP_2 = 162°$. O_3 the centre of third trajectory is above origin O by 0.25 cm. Join O_3 to P_2.

Now O_3P_2 is the radius of third trajectory and the included angle $P_2O_3P_3 = 216°$.

Problems

1. A spring mass system has weight 40 kg and spring constant = 100 kg/cm. It is subjected to excitation as shown in Fig. 8.1P.

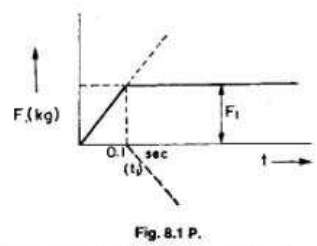


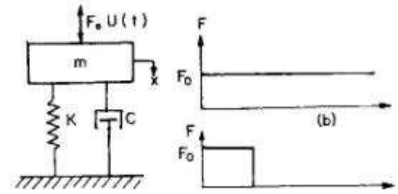
Fig. 8.1 P.

Determine the response for $t > 0.1$ second. Solve by Duhamel's integral method. (P.U., 98)

2. Analyse problem 1 by Laplace transform.

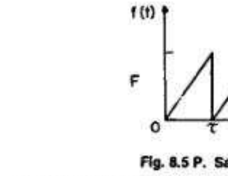
$$\left(\textbf{Hint. } m\ddot{x} + kx = \frac{F_1}{0.1} + F_1 \right)$$

3. (*a*) Using Laplace transform method, obtain the response of a second-order damped system, Fig. 8.2P (*a*), to a step impulsive force, Fig. 8.2P(*b*). Where $U(t)$ is the unit function.



(*b*) Also find the response of this system for a rectangular pulse of magnitude, Fo, and duration, T. (Roorkee Uni.)

4. Find the transform of the single pulse shown in Fig. 8.3P.



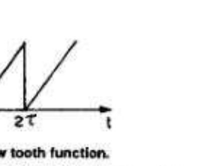


Fig. 8.3P. **Fig. 8.4P. Half-sine wave.**

5. Determine the Laplace transform of the single half-sine wave as shown in Fig. 8.4P.

6. Determine the Laplace transform of the saw tooth periodic function as shown in Fig. 8.5P.

Fig. 8.5 P. Saw tooth function.

7. An arbitrary force $f(t)$ is applied to an oscillator without damping having non-zero initial conditions, prove that the solution of the problem is of the form

$$x(t) = x_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t + \frac{1}{m\omega_n} \int_0^t f(\xi) \sin \omega_n (t - \xi_1)\, d\xi$$

9

Non-Linear Vibrations

9.1 INTRODUCTION

In the preceding chapters, the equations of motion were linear. By the term linear we mean that the equation of motion contained displacement or its derivative only to the first degree, and no squares of higher powers of displacement or velocity are involved. Linear analysis of a system explains much about the oscillatory systems. But there are a number of phenomena which cannot be predicted by Linear analysis.

Linear analysis of problems in vibration can be justified only in case of small displacements. For the purpose of convenience, we model most of the systems as linear, but in actual practice real systems are more often non-linear. The non-linear analysis becomes necessary whenever finite amplitudes of motion are encountered. One of the main reasons for modelling a physical system as a non-linear is that, totally unexpected phenomena occur sometimes in non-linear systems. These phenomena are not predicted or even hinted by linear theory.

Several methods are available for the solution of non-linear vibration problems. Some elementary methods for the analysis of non-linear vibrations are presented in this chapter.

9.2 DIFFERENCE BETWEEN LINEAR AND NON-LINEAR VIBRATIONS

In the linear systems, the cause and effect are linearly related i.e. the cause and effect change in such a manner that their relation with respect to each other is the same and is given by a linear plot. In the non-linear systems, this relationship between cause and effect is no longer proportional. The properties of a non-linear system depend on dependent variables.

The so called linear systems tend to become non-linear with larger vibration amplitudes. The analysis of non-linear systems is however comparatively difficult. Sometimes there is no exact solution available. One major difference between the linear and non-linear systems is that the superposition principle does not hold good for non-linear systems.

Another characteristic of non-linear vibrations is that the response of the system contains Sub-Harmonics. By Sub-Harmonics we mean that when the system is acted upon by a forcing frequency 'ω', the esponse may contain components of frequency lower than 'ω'. Stability roblems are also encountered with non-linear systems. Analytical rocedures for the treatment of non-linear systems are difficult and equire extensive mathematical study.

A general equation of the non-linear system is given by

$$m\ddot{x} + \beta(\dot{x}) + f(x) = F(t) \qquad ...(9.2.1)$$

ı which the second and third terms on the L.H.S. of the equation are ot linear functions of $(\dot{x})$ and (x) as used to be the case in linear ystems.

A general equation of the linear system as in earlier chapters is iven as

$$m\ddot{x} + c\dot{x} + kx = F(t) \qquad ...(9.2.2)$$

ı which all the three terms on the LHS of the equation are linear ınctions of $\ddot{x}$, $\dot{x}$ and x respectively.

3 APPLICATION OF SUPERPOSITION PRINCIPLE TO LINEAR AND NON-LINEAR SYSTEMS

As already discussed, one of the major difficulties of the non-linear ıalysis arises because superposition principle which is so useful in ıear analysis does not hold true here. We can prove this.

Let us first assume a linear system whose general equation of otion is given by equation (9.2.2) as

$$m\ddot{x} + c\dot{x} + kx = F(t) \qquad ...(9.3.1)$$

Let x_1 be the solution to the above equation

This means

$$m\ddot{x}_1 + c\dot{x}_1 + kx_1 = F_1(t) \qquad ...(9.3.2)$$

Let x_2 also be a solution to the equation (9.3.1). By substituting is value in equation (9.3.1), we get

$$m\ddot{x}_2 + c\dot{x}_2 + kx_2 = F_2(t) \qquad ...(9.3.3)$$

Now adding equation (9.3.2) and (9.3.3)

$$m(\ddot{x}_1 + \ddot{x}_2) + c(\dot{x}_1 + \dot{x}_2) + k(x_1 + x_2) = [F_1(t) + F_2(t)] \qquad ...(9.3.4)$$

From equation (9.3.4) it can be deciphered that $(x_1 + x_2)$ is also a lution to equation (9.3.1). Thus superposition principle holds good for inear system.

Let us now consider a non-linear system for which the general uation of motion is given as

$$m\ddot{x} + kx^3 = 0 \qquad ...(9.3.5)$$

Let $x_1 = \phi_1(t)$ and $x_2 = \phi_2(t)$ be the solutions satisfying the equation (9.3.5). Substituting these values in equation (9.3.5), we obtain

$$m\ddot{\phi}_1 + k\phi_1^3 = 0 \qquad ...(9.3.6)$$

$$m\ddot{\phi}_2 + k\phi_2^3 = 0 \qquad ...(9.3.7)$$

Adding the above two equations (9.3.6) and (9.3.7), we get

$$m(\ddot{\phi}_1 + \ddot{\phi}_2) + k(\phi_1^3 + \phi_2^3) = 0 \qquad ...(9.3.8)$$

Now we assume that $x = \phi_1 + \phi_2$ is a solution to equation (9.3.5) and substituting its value in equation (9.3.5), we get

$$m(\ddot{\phi}_1 + \ddot{\phi}_2) + k(\phi_1 + \phi_2)^3 = 0 \qquad ...(9.3.9)$$

Comparing equations (9.3.8) and (9.3.9) we can see that they are different thereby showing that the principle of super-position does not apply to non-linear vibrations.

9.4 EXAMPLES OF NON-LINEAR VIBRATION SYSTEM

The following examples of non-linear vibration systems are given to illustrate the nature of non-linearity in some physical systems.

9.4.1 SIMPLE PENDULUM

Consider a simple pendulum of length 'l' having a bob of mass 'm' as shown in Fig. 9.1. The differential equation governing the free vibration of the pendulum can be derived as

$$I\ddot{\theta} = -mg\sin\theta \times l \qquad ...(9.4.1.1)$$

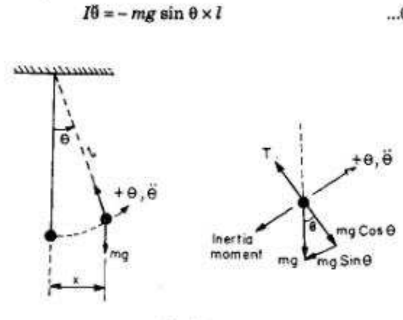


Fig. 9.1.

We know that $I = ml^2$ and for small angles $\sin\theta$ may be approximated by θ. Thus equation (9.4.1.1) reduces to

$$ml^2\ddot{\theta} + mg\,l\theta = 0 \qquad ...(9.4.1.2)$$

This reduces to

$$\ddot{\theta} + \omega^2\theta = 0 \qquad ...(9.4.1.3)$$

where

$$\omega = \sqrt{\frac{g}{l}} \qquad ...(9.4.1.4)$$

The solution of equation (9.4.1.3) is expressed as

$$\theta(t) = A\sin(\omega t + \phi) \qquad ...(9.4.1.5)$$

where A is the amplitude of oscillation, ϕ is the phase angle, and ω is angular frequency

The values of 'A' and 'φ' are determined by the initial conditions and the angular frequency 'ω' is independent of the amplitude 'A'.

Equation (9.4.1.5) denotes an approximate solution of the simple pendulum problem.

A better approximate solution can be obtained by expanding sin θ to a two term equivalent approximation. Thus for sin θ near θ = 0° we can have

$$\sin\theta = \theta - \frac{\theta^3}{6} \qquad ...(9.4.1.6)$$

Substituting the value of sin θ from the above equation in equation (9.4.1.1), we get

$$ml^2\ddot{\theta} + mgl\left(\theta - \frac{\theta^3}{6}\right) = 0$$

or

$$\ddot{\theta} + \omega^2\left(\theta - \frac{\theta^3}{6}\right) = 0 \qquad ...(9.4.1.7)$$

It can be seen that the above equation is non-linear because it contains a term involving θ^3. Thus a pendulum with large amplitude is considered as a system with non-linearity.

The equation of motion for a simple pendulum is similar to a spring-mass system with a non-linear spring. The pendulum thus possesses soft spring characteristics i.e. stiffness decreases with increase in displacement. If displacement θ is taken small, then the term containing θ^3 can be neglected and the system reduces to a linear one.

The frequency in case of a non-linear system is not constant where as in a linear system it always remains constant. Thus depending on the variation of frequency (ω) with displacement (x), there are two types of spring mass systems showing non-linear vibration characteristics. One of them is the soft spring system in which the frequency decreases with the displacement $\left(\frac{d\omega}{dx}\text{ is a decreasing function of } x\right)$. This is due to the fact that in a soft spring system the equivalent stiffness of the spring-mass system decreases with increase of frequency.

9.4.2 VIBRATION OF A STRING

A point mass m is attached to the mid point of a stretched string having an initial tension T. A and E are the area of cross-section of the

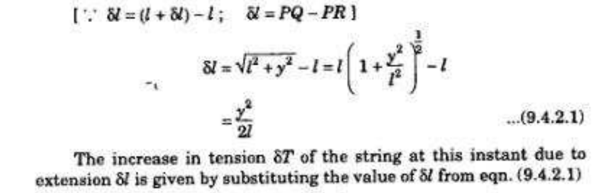


Fig. 9.2.

string and modulus of elasticity respectively. At any instant when the mass is displaced through a distance 'y', each half of the string has been extended through a distance δl, such that

$$[\because\ \delta l = (l + \delta l) - l;\quad \delta l = PQ - PR\,]$$

$$\delta l = \sqrt{l^2 + y^2} - l = l\left(1 + \frac{y^2}{l^2}\right)^{\frac{1}{2}} - l$$

$$= \frac{y^2}{2l} \qquad ...(9.4.2.1)$$

The increase in tension δT of the string at this instant due to extension δl is given by substituting the value of δl from eqn. (9.4.2.1)

$$\delta T = AE\,\frac{\delta l}{l} = \frac{AEy^2}{2l^2} \qquad ...(9.4.2.2)$$

$$\left(\text{Since } \frac{\text{Stress}}{\text{Strain}} = E\right)$$

Tensile force in the string which is the total tension in the string at this instant is $T + \delta T$ and is given by

$$T + \delta T = T + \frac{AE}{2l^2}y^2 \qquad ...(9.4.2.3)$$

Total Restoring force which is the sum of the restoring forces acting on each half of the string along QR, on the mass m is then given by :

$$F_{res} = 2(T + \delta T)\sin\theta$$

$$= 2(T + \delta T)\frac{y}{\sqrt{y^2 + l^2}} \qquad ...(9.4.2.4)$$

$$\left(\text{Since } \sin\theta = \frac{QR}{PQ}\right)$$

Substituting for δT from equation (9.4.2.2) in equation (9.4.2.4), we get

$$F_{res} = 2\left(T + \frac{AEy^2}{2l^2}\right)\frac{y}{\sqrt{y^2 + l^2}}$$

$$= 2\left(T + \frac{AEy^2}{2l^2}\right)\frac{y}{l}\left(1 + \frac{y^2}{l^2}\right)^{-1/2}$$

$$= 2\left(T + \frac{AEy^2}{2l^2}\right)\frac{y}{l}\left(1 - \frac{y^2}{2l^2} \dots\right) \text{(By Binomial expansion)}$$

$$= \frac{2y}{l}\left[T + \frac{1}{2}(AE - T)\frac{y^2}{l^2} \dots\right]$$

Now 'T' is very small as compared to 'AE' and so it is neglected. Also we neglect higher powers of y. So we get

$$F_{res} = \frac{2y}{l}\left[T + \frac{AEy^2}{2l^2}\right]$$

$$= \frac{2Ty}{l} + \frac{AEy^3}{l^3} \qquad \dots(9.4.2.5)$$

From equation (9.4.2.5) it can be seen that if y is small y^3 can be neglected reducing the restoring force to a linear expression. If y is large, however, the higher powers cannot be neglected. So the equation (9.4.2.5) shows the hardening spring non-linearity. The spring stiffness in this case increases with increase in displacement since the effective stiffness increases.

Equation (9.4.2.5) becomes

$$m\ddot{y} + \frac{2Ty}{l} + \frac{AEy^3}{l^3} = 0 \qquad \dots(9.4.2.6)$$

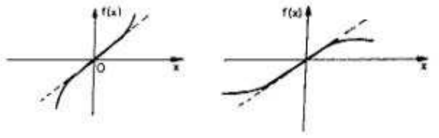
9.4.3 HARD AND SOFT SPRING

It can be seen from equation (9.4.1.7) that it is similar to the equation of motion of a spring-mass system with a non-linear spring. If the spring is non-linear (due to non-linearity of material), the restoring force can be expressed as a function of 'x' i.e. $f(x)$ where 'x' is the deformation of the spring and the equation of motion of the spring-mass system becomes

$$m\ddot{x} + f(x) = 0 \qquad \dots(9.4.3.1)$$

If $\frac{d\,f(x)}{dx} = k =$ a constant, the spring is linear. If $\frac{df(x)}{dx}$ is strictly increasing function of x, the spring is called a hard spring, and if $\frac{df(x)}{dx}$ a strictly decreasing function of x, the spring is called a soft spring.

Due to similarity of equation (9.4.1.7) and (9.4.3.1), a pendulum with a large amplitude is considered as a system with non-linear elastic component. A sketch of equation (9.4.2.5) i.e hard spring and a sketch of equation (9.4.1.7) i.e. soft spring are shown in Fig. 9.3. The dotted lines in Fig. 9.3 represent the behaviour of a linear spring mass-system.



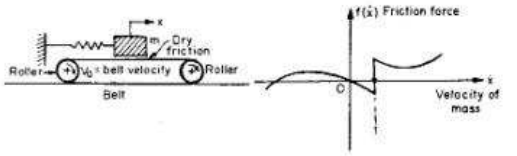
(a) Hard Spring **(b) Soft Spring**

Fig. 9.3.

9.4.4 BELT FRICTION SYSTEM

For the system shown in figure 9.4, where the belt moves with a constant velocity V_0 the equation of motion is

$$m\ddot{x} + f(\dot{x}) + kx = 0 \qquad \dots(9.4.4.1)$$



(a) (b)

Fig. 9.4.

In case of linear systems, it is assumed that the friction existing between two bodies moving relative to each other, is negligible. In case of a system having non-linear characteristics, this friction can not be neglected. Thus due to the dry friction existing between the block of mass 'm' and the moving belt, the system shown in figure 9.4 (a) behaves non-linearly. This system possesses two coefficients of friction namely, the static and the kinetic coefficients of friction. The static coefficient of friction comes into action when the body (block of mass m) is static and a certain amount of force is required to bring the body into motion. The body remains static because the force of dry friction is greater than the force which is trying to bring the body into motion. When this force exceeds the force of dry friction, the body comes into motion and possesses the kinetic coefficient of friction. This kinetic coefficient of friction remains in action as long as the body is in motion and it always acts in a reverse direction to that of motion.

In the belt mass system under study, the motion of the system with the variation in the velocity of mass, is shown in figure 9.4 (b). Initially, the mass is at rest on the belt and when the belt starts moving, the mass is under static conditions of friction.

As kinetic friction takes over and the mass starts moving along with the belt, the spring attached to the mass starts elongating. The force on the spring trying to restore it to its normal position also goes on increasing with the elongation of spring. Thus a point comes when this restoring force overcomes the effect of the force of static friction. At this point, the mass slides rapidly towards left. Thus the restoring force is relieved up to the point where kinetic friction takes over and stops the further relieving of the spring force. Then the spring again tries to build up the spring force and the above sequence of motion is repeated which is shown in figure 9.4. (b).

For large values of the displacement and thus for large values of $\dot{x}$, the damping force is positive. Thus the curve has a positive slope and removes energy from the system. For smaller values of displacement and $\dot{x}$ the damping force is negative. Thus the curve has a negative slope and puts energy into the system. Although there is no external stimulus, the system can have an oscillatory motion corresponding to a non-linear self-excited system. This phenomenon of self-excited vibration is called Mechanical Chatter.

9.4.5 VARIABLE MASS SYSTEM

In the earlier examples, it was seen that a system possessed non-linear characteristics due to its variable frequency. The same system may exhibit non-linear behaviour due to variation in mass. In figure 9.5, the mass on the spring varies with the displacement x of the

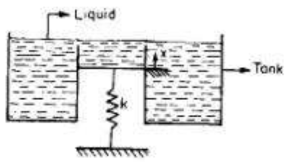


Fig. 9.5.

piston connected to the spring. The mass of the liquid column acting on the spring shows a significant variation with large deflections and thus possesses non-linear characteristics.

The equation of motion for the system shown in figure 9.5 is given as

$$\frac{d}{dt}(m\dot{x}) + kx = 0 \qquad \dots(9.4.5.1)$$

Here the first term showing variable mass, is non-linear which imparts non-linearity to the differential equation.

9.4.6 ABRUPT NON-LINEARITY

In the previous examples, it was seen that a system possessed non-linear characteristics due to variation in either frequency or mass.

Abrupt non-linearity presents the case wherein non-linearity is possessed due to variation in amplitude of vibration.

Figure 9.6 (a) shows a system with an abrupt non-linearity in spring. As long as the amplitude of vibration of the mass is less than or

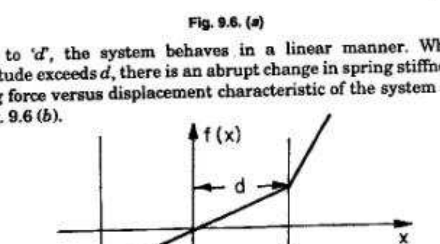
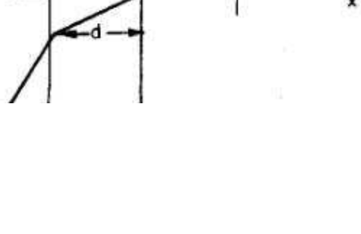


Fig. 9.6. (a)

equal to 'd', the system behaves in a linear manner. When the amplitude exceeds d, there is an abrupt change in spring stiffness. The spring force versus displacement characteristic of the system is given in Fig. 9.6 (b).



The system differential equation is $m\ddot{x} + f(x) = 0$
where restoring force $f(x) = 2k_1 x$ for $|x| \le d$

$$= 2(k_1 + k_2)x - 2k_2 d \text{ for } |x| > d$$

or

$$= 2k_1 x + 2k_2(x - d) \text{ for } |x| > d \qquad ...(9.4.6.1)$$

In the system shown in Fig. 9.6 (*a*), it is seen that when the mass vibrates on either side (left or right) with an amplitude which is less than or equal to the distance between the mass and the springs k_2 in normal position i.e. d, only spring k_1 comes into action and is compressed on one side equal to the amount of elongation of the same spring k_1 on the other side. Thus the system behaves in a linear manner. When the amplitude of vibration is large enough such that it exceeds the distance d, then the springs k_2 also come into action. As the stiffness of springs k_1 and k_2 is different, thus their rate of change with any displacement x ($x > d$) is also different. As a result there is an abrupt change in stiffness of the two springs giving rise to variation in the amplitude of vibration. This causes non-linearity in the system.

.4.7 OTHER EXAMPLES

(*a*) A rotating shaft carrying rotors when the shaft is not circular ut elliptical in section.

(*b*) A cantilever with curved guide is an example of variable tiffness. Larger the deflection of the cantilever, shorter is the effective ength. Due to the decrease in effective length of the cantilever, the tiffness of the cantilever increases as the stiffness is inversely proporonal to l^3 (For a cantilever having a load on the tip, stiffness $k = \frac{3EI}{l^3}$)

(*c*) A pendulum of variable length. It can be obtained if the chord simple pendulum is wrapped round a fixed cylinder so that its length aries during vibrations.

5 ESTIMATION AND DETERMINATION OF NON-LINEAR VIBRATIONS

There are many methods which may be employed to deal with n-linear vibration. Some of them are discussed below :

5.1 PHASE PLANE TRAJECTORIES (GRAPHICAL METHOD)

One of the methods to study non-linear vibrations is to construct ajectories' i.e. graphs between velocity and displacement of vibrating

PHASE PLANE

For a single degree of freedom system, two parameters are usually taken as the displacement and velocity of the system. When these parameters are used as co-ordinate axes, the resulting graphical representation of the motion is called the Phase Plane Representation. Each point in the phase plane represents a possible state of system. As time changes, the state of the system changes.

A typical or representative point in the phase plane (such as a point representing the state of system at time $t = 0$) moves and traces a curve known as trajectory. The variation of the solution of the system with time is demonstrated by these trajectories.

CHARACTERISTICS OF TRAJECTORIES

For autonomous systems, the phase plane representation offers one useful approach for the study of the non-linear system (A differential equation is said to be out autonomous if the properties of the system are not affected by the independent variable. For vibration problems, the system is autonomous if the independent variable '*t*' appears only as a differential in motion equations. The behaviour of the system can be represented graphically without any change by changing the origin of time or the scale on which time is measured because time is an independent variable. Consider a single degree of freedom non-linear oscillatory system whose governing equation is of the form

$$\ddot{x} + f(\dot{x}, x) = 0 \qquad ...(9.5.1.1)$$

Now

$$\frac{dx}{dt} = \dot{x} = y \qquad ...(9.5.1.2)$$

and

$$\frac{dy}{dt} = \dot{y} \qquad ...(9.5.1.3)$$

From equation (9.5.1.2), we get

$$dt = \frac{dx}{\dot{x}} \qquad ...(9.5.1.4)$$

Substituting the value of dt from (9.5.1.4) in (9.5.1.1), we get

$$\dot{x}\frac{d\dot{x}}{dx} + f(\dot{x}, x) = 0 \quad \left(\text{since } \ddot{x} = \frac{d}{dt}(\dot{x})\right) \qquad ...(9.5.1.5)$$

$$\frac{d\dot{x}}{dx} = \phi(\dot{x}, x) \qquad ...(9.5.1.6)$$

Thus if we use $\dot{x}$ and x as coordinates, it can be seen that for any point $(\dot{x}, x)$, the slope is given by $\frac{d\dot{x}}{dx}$. This can be represented by a short line. With a sufficient number of points, the entire plane can be filled by such short lines as shown in figure 9.7 (*a*) which represents a family

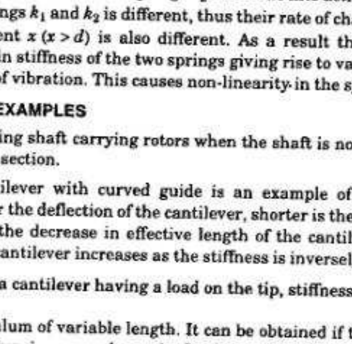


Fig. 9.8.

9.5.2 DIRECT INTEGRATION METHOD (ANALYTICAL METHOD)

Consider the conservative system defined by the equation

$$\ddot{x} + f(x) = 0 \qquad ...(9.5.2.1)$$

Now acceleration $\ddot{x} = V\frac{dv}{dx}$ ...(9.5.2.2)

Substituting (9.5.2.2) in (9.5.2.1) we get

$$Vdv = -f(x)\,dx$$

where $\dot{x} = V$. If $x = X$ when $V = 0$, its integral is

$$|V| = \left|\frac{dx}{dt}\right| = \sqrt{\int_x^X f(\lambda)\,d\lambda} \qquad ...(9.5.2.4)$$

The second integral yields

$$t - t_0 = \int_0^X \frac{d\eta}{\sqrt{\int_n^X f(\lambda)\,d\lambda}} \qquad ...(9.5.2.5)$$

where t_0 is the time corresponding to $x = 0$ the eqn. (9.5.2.5) expresses time as a function of displacement, and its inverse is the displacement—time relationship.

If the motion is periodic with period T, the time corresponding to the motion from $x = 0$ ($t = t_0$) to $x = X$ ($v = 0$) represents a quarter period. Hence we have

$$T = 4\int_0^X \frac{d\eta}{\sqrt{\int_n^X f(\lambda)\,d\lambda}} \qquad ...(9.5.2.6)$$

Thus the period becomes a function of the amplitude X.

The application of the method can be seen in the solved example at the end of this chapter.

9.5.3 METHOD OF PERTURBATION

This method is used for obtaining solutions of non-linear systems to any degree of accuracy by successive approximations.

Let us consider the non-linear system consisting of spring and mass. The equation of motion is given by

$$\ddot{x} + \omega_n^2 x + \lambda x^3 = 0 \qquad ...(9.5.3.1)$$

where non linear parameter λ can have any arbitrary value and ω_n is natural frequency of the system. The factor λ is also called PERTURBATION PARAMETER. Writing the solution in the form of a Taylor series in terms of parameter λ, we get

$$x = x_0 + \lambda x_1 + \lambda^2 x_2 + \ldots.. \qquad ...(9.5.3.2)$$

where all x's are functions of time t. Here λ is a small quantity.

Since frequency of vibrations ω which is dependent on the amplitude of vibration is also unknown, we have

$$\omega^2 = \omega_n^2 + \lambda\mu_1 + \lambda^2\mu_2 + \ldots. \qquad (9.5.3.3)$$

where all μ's are function of x (amplitude of vibration)

Substituting Eqns. (9.5.3.2) and (9.5.3.3) in eqn. (9.5.3.1) we get

$$(\ddot{x}_0 + \lambda\ddot{x}_1 + \lambda^2\ddot{x}_2 + \ldots) + (\omega^2 - \lambda\mu_1 - \lambda^2\mu_2 \ldots)$$
$$(x_0 + \lambda x_1 + \lambda^2 x_2 + \ldots.) + \lambda(x_0 + \lambda x_1 + \lambda^2 x_2 + \ldots)^3 = 0 \ldots(9.5.3.4)$$

The above eqn. after expanding and neglecting higher power of λ, can be written as

$$(\ddot{x}_0 + \omega^2 x_0) + (\ddot{x}_1 + \omega^2 x_1 - \mu_1 x_0 + x_0^3)\lambda$$
$$+ (\ddot{x}_2 + \omega^2 x_2 - \mu_1 x_1 - \mu_2 x_0 + 3x_0^2 x_1)\lambda^2 + \ldots = 0 \qquad ...(9.5.3.5)$$

Since λ is arbitrary, each turn in the parentheses must individually be zero, therefore,

$$\left.\begin{aligned}\ddot{x}_0 + \omega^2 x_0 &= 0\\ \ddot{x}_1 + \omega^2 x_1 &= \mu_1 x_0 - x_0^3\\ \ddot{x}_2 + \omega^2 x_2 &= \mu_1 x_1 + \mu_2 x_0 - 3x_0^2 x_1\\ \vdots \quad \vdots \quad &\vdots \quad \vdots \quad \vdots\end{aligned}\right\} \qquad ...(9.5.3.6)$$

Let the initial conditions be

$$\left.\begin{aligned}x &= X\\ \dot{x} &= 0\end{aligned}\right\} \text{ at } t = 0$$

Substituting above in eqn. (9.5.3.2) and its derivative, we get

$$\left.\begin{aligned}X &= x_0(0) + \lambda x_1(0) + \lambda^2 x_2(0) + ...\\ 0 &= \dot{x}_0(0) + \lambda \dot{x}_1(0) + \lambda^2 \dot{x}_2(0) + ...\end{aligned}\right\} \qquad ...(9.5.3.7)$$

Again, since λ is arbitrary, these equations must be satisfied for any value of λ, we have

$$\begin{aligned}x_0(0) &= X\\ x_1(0) &= 0\\ x_2(0) &= 0\\ \vdots \quad &\vdots \quad \vdots\end{aligned} \quad \& \quad \left.\begin{aligned}\dot{x}_0(0) &= 0\\ \dot{x}_1(0) &= 0\\ \dot{x}_2(0) &= 0\\ \vdots \quad &\vdots \quad \vdots\end{aligned}\right\} \qquad ...(9.5.3.8)$$

With the first set of initial conditions substituted in eqn. (9.5.3.8), the solution of first differential equation in eqn. (9.5.3.6) is

$$x_0 = X \cos \omega t \qquad ...(9.5.3.9)$$

Substituting the above in the right side of the second differential equation in eqn. (9.5.3.6), we get

$$\ddot{x}_1 + \omega^2 x_1 = \mu_1 X \cos \omega t - X^3 \cos^3 \omega t \qquad ...(9.5.3.10)$$

Since $\quad \cos^3 \omega t = 3/4 \cos \omega t + 1/4 \cos \omega t \qquad ...(9.5.3.10)$

becomes

$$\ddot{x}_1 + \omega^2 x_1 = \left(\mu_1 X - \frac{3}{4}X^3\right)\cos \omega t - \frac{1}{4}X^3 \cos 3\,\omega t \quad ...(9.5.3.11)$$

In above equation, the forcing function $\left(\mu_1 \times \frac{3}{4}X^3\right)\cos \omega t$ causes resonance since the left hand side shows the natural frequency of system as ω which is same as that of first part of excitation. To avoid this, we have

$$\mu_1 X - \frac{3}{4}X^3 = 0 \qquad ...(9.5.3.12)$$

$$\therefore \qquad \ddot{x}_1 + \omega^2 x_1 = -\frac{1}{4}X^3 \cos 3\,\omega t \qquad ...(9.5.3.13)$$

The solution of above equation becomes

$$x_1 = A_1 \cos \omega t + A_2 \sin \omega t + \frac{X^3}{32\,\omega^2}\cos 3\omega t \qquad ...(9.5.3.14)$$

Applying initial conditions from eqn. (9.5.3.8), we get the constants as

$$A_1 = -\frac{X^3}{32\,\omega^2} \text{ and } A_2 = 0$$

Eqn. (9.5.3.14) then becomes

$$x_1 = -\frac{X^3}{32\,\omega^2}[\cos \omega t - \cos 3\,\omega t] \qquad ...(9.5.3.15)$$

Substituting eqn. (9.5.3.9) and (9.5.3.15) in the first two terms of eqn. (9.5.3.2), the solution upto first order is given by

$$\begin{aligned}x &= x_0 + \lambda x_1\\ &= a \cos \omega t - \frac{\lambda x^3}{32\,\omega^2}[cost\ \omega t - \cos 3\omega t] \qquad ...(9.5.3.16)\end{aligned}$$

With ω^2 given by eqn. (9.5.3.3) upto first order as

$$\begin{aligned}\omega^2 &= \omega_0^2 + \lambda\mu_1\\ &= \omega_0^2 + \lambda\left(\frac{3}{4}a^2\right) \qquad \text{[From eqn. (9.5.3.12)]} \quad ...(9.5.3.17)\end{aligned}$$

Final solution having higher accuracy can be obtained by continuing the above procedure of successive approximations.

9.5.4 METHOD OF ITERATION

The method of iteration as the name suggests is one of the obtaining solutions with improved accuracy by repeatedly substituting it into the differential equation till the solution having desired accuracy is achieved.

An assumed solution $x_0(t)$ is substituted into the differential equation to obtain a solution of improved accuracy.

Let us consider an equation

$$\ddot{x} + f(x, t) = 0 \qquad ...(9.5.4.1)$$

It can be written in terms of assumed solution

$$\ddot{x} = -f[x_0(t), t] \qquad ...(9.5.4.2)$$

Let the initial conditions be

$$\left(\begin{aligned}x(0) &= A\\ \dot{x}(0) &= 0\end{aligned}\right) \text{ at } t = 0$$

Eqn. (9.5.4.2) can be integrated to obtain the improved solution $x_1(t)$

$$\text{Thus } x_1(t) = -\int_{t'}^{t}\left\{\int_0^{\eta} f[x_0(\lambda), \lambda]\, d\lambda\right\} d\eta \qquad ...(9.5.4.3)$$

here $x_1(t') = 0$. Since $x_1(0) = A$, the above equation leads to the llowing relationship :

$$A = \int_0^{t'}\left\{\int_0^{\eta} f[x_0(\lambda), \lambda]\, d\lambda\right\} d\eta \qquad ...(9.5.4.4)$$

The procedure is now repeated with $x_1(t)$ to obtain

$$x_2(t) = \int_t^{t'}\left\{\int_0^{\eta} f[x_1(\lambda), \lambda]\, d\lambda\right\} d\eta \qquad ...(9.5.4.5)$$

Putting the initial conditions again result in the equation

$$A = \int_0^{t'}\left\{\int_0^{\eta} f[x_1(\lambda), \lambda]\, d\lambda\right\} d\eta \qquad ...(9.5.4.6)$$

[**Note.** If the free vibrations take place with initial displacement ., t' will correspond to one-fourth of the time period.]

From eqn. (9.5.4.6), t' may be obtained. The time period T of the ibration may then be obtained as

$$T = 4t' \qquad ...(9.5.4.7)$$

.5.5 FOURIER SERIES METHOD

We can solve non-linear equations by assuming the solution to be n the form of Fourier series.

Let us consider a non-linear motion given by equation

$$\ddot{x} + \mu x + \lambda x^3 = 0 \qquad ...(9.5.5.1)$$

With initial conditions as

$$\left.\begin{aligned}x &= X\\ \dot{x} &= 0\end{aligned}\right\} \text{ at } t = 0$$

Let the solution be

$$x = a_0 + \sum_{n=1}^{\infty} A_n \sin \omega t + \sum_{n=1}^{\infty} B_n \cos \omega t \qquad ...(9.5.5.2)$$

Using initial conditions leads to result that

$$x = \sum^{\infty} B_n \cos n\,\omega t \qquad ...(9.5.5.3)$$

Further analysis leads to the result that the displacement in each quarter period is identical except in sign. This leads to conclusion that

$$x = \sum_{n=0}^{\infty} B_{2n+1} \cos(2n+1)\,\omega t \qquad ...(9.5.5.4)$$

As a first approximation let us consider

$$x = B_1 \cos \omega t + B_3 \cos 3\omega t \qquad ...(9.5.5.5)$$

Substituting eqn. (9.5.5.5) in eqn. (9.5.5.1), we get

$$\begin{aligned}&-\omega^2(B_1 \cos \omega t + 9B_3 \cos 3\omega t) + \mu(B_1 \cos \omega t + B_3 \cos 3\omega t)\\ &+ \frac{\lambda}{4}[B_1^3(3\cos \omega t + \cos 3\,\omega t) + 6B_1^2 B_3 \cos 3\,\omega t\\ &+ 3B_1^2 B_3(\cos \omega t + \cos 5\,\omega t) + 6B_1 B_3^2 \cos \omega t\\ &+ 3B_1 B_3^2(\cos 5\,\omega t + \cos 7\,\omega t)\\ &+ B_3^3(3\cos 3\,\omega t + \cos 9\,\omega t)] = 0 \qquad ...(9.5.5.6)\end{aligned}$$

We now assume that $|B_3| <<< |B_1|$. We also neglect coefficient of $\cos 5\omega t$, $\cos 7\omega t$, $\cos 9\,\omega t$ in the above eqn. Comparing (9.5.5.6) with (9.5.5.5), we can equate coefficients of $\cos \omega t$ to zero. Thus

$$-\omega^2 B_1 + \mu B_1 + \left(\frac{3\lambda B_1^2}{4}\right)(B_1 + B_2) = 0 \qquad ...(9.5.5.7)$$

$$\text{or} \qquad \omega^2 = \mu + \frac{3\lambda B_1 X}{4} \qquad ...(9.5.5.8)$$

$$\text{as} \qquad X = B_1 + B_3$$

From above analysis it becomes clear that the frequency is a function of the amplitude of vibration. It increases with X.

By equating the coefficients of $\cos 3\omega t$ to zero, we can determine B_1 and B_3, as

$$-9\omega^2 B_3 + \mu B_3 + B_1^2(B_1 + 6B_3)/4 = 0 \qquad ...(9.5.5.9)$$

Omitting B_3^2 and substituting ω^2 from eqn. (9.5.5.8), we get

$$\begin{aligned}\frac{B_3}{B_1} &= \frac{1}{21 + \dfrac{32\,\mu}{\lambda B_1^2}}\\ &= \rho <<< 1 \qquad ...(9.5.5.10)\end{aligned}$$

This proves that our assumption

$$|B_3| <<< |B_1| \text{ is true}$$

Now eqn. (9.5.5.10) and the fact that $X = B_1 + B_3$ can be used to get different values of B_1 and B_3.

.5.6 LINEARIZATION METHOD

The expression for energy dissipated in a system with viscous amping is given by eq. (3.2.5) i.e.

$$\Delta E = C\omega\pi A^2 \qquad ...(9.5.6.1)$$

Let's assume that $\Delta E'$ is energy dissipated due to non-viscous amping, then

$$C_{eq} = \frac{\Delta E'}{\omega\pi A^2} \qquad ...(9.5.6.2)$$

The assumption made is that steady state of actual system is armonic in case of harmonic excitation.

Now Damping force $= C\dot{x}$

Now as damping coefficient is proportional to velocity, we get

Damping force $F = a\dot{x}^2$ ...(9.5.6.3)

is proportionality constant in the above equation

$$\therefore \quad \Delta E' = 2\int_{-T/4}^{T/4} (a\dot{x}^2)\frac{dx}{dt}\,dt \quad \text{[From eqn. 3.2.4]} \qquad ...(9.5.6.4)$$

here $\quad T = \text{Time period} = \frac{2\pi}{\omega}$

Let $x = A\sin\omega t$ and $\therefore \dot{x} = \omega A\cos\omega t$

After substituting the above assumptions in eqn. (9.5.6.4) we get

$$\Delta E' = \frac{8\,a\omega^2 A^3}{3} \qquad ...(9.5.6.3)$$

Substituting eqn. (9.5.6.3) in eqn. (9.5.6.2), we get

$$C_{eq} = \frac{8\,a\omega A}{3\pi} \qquad ...(9.5.6.6)$$

Hence equation of motion of the system becomes

$$m\ddot{x} + C_{eq}\dot{x} + kx = F\sin\omega t \qquad ...(9.5.6.7)$$

The solution for above equation is provided by eqn. (4.2.9) as

$$A = \frac{F/k}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{C_{eq}\,\omega}{k}\right)^2}} \qquad ...(9.5.6.8)$$

$$\left[\begin{array}{l} \text{as } \varepsilon = \dfrac{C}{C_c} \\ \text{and } 2\varepsilon\dfrac{\omega}{\omega_n} = \dfrac{C.\omega}{k} \end{array}\right]$$

Putting $\omega/\omega_n = r$ and substituting for C_{eq} from eqn. (9.5.6.6) in eqn. (9.5.6.6) and solving for A, we get

$$A = \frac{3\pi m}{8ar^2}\sqrt{-\frac{(1-r^2)^2}{2} + \sqrt{\frac{(1-r^2)^4}{4} + \left(\frac{8ar^2 F_0}{3\pi\,km}\right)^2}} \qquad ...(9.5.6.9)$$

[**Note.** The derivation of eqn. (9.5.6.9) from above conditions is left to the students as an exercise]

The above expression is the amplitude of steady forced vibration with damping proportional to square of velocity

Also from eqn. (4.2.7), we get

$$\tan\phi = \frac{C_{eq}\,\omega}{k - m\omega^2}$$

Substituting for C_{eq} in above expression, we get

$$\tan^2\phi = -\frac{1}{2} + \sqrt{\frac{1}{4} + \left(\frac{8ar^2 F}{3\pi\,km\,(1-r^2)^2}\right)^2} \qquad ...(9.5.6.10)$$

Thus the amplitude and the phase angle for a non-linear system can be conveniently found by using the above equation, similar to that of the method for the linear systems.

9.7 FORCED VIBRATIONS WITH NON-LINEAR SPRING (DUFFING'S EQUATION)

G. Duffing made an exhaustive study of the equation

$$\ddot{x} + \omega_n^2 x \pm \lambda x^3 = \frac{F}{m}\left\{\begin{array}{l}\sin\omega t\\ \cos\omega t\end{array}\right\} \qquad ...(9.7.1)$$

The above equation shows the harmonic excitation of the mass placed on a non-linear cubic spring. The spring may be hardening or softening spring and accordingly the +ve or –ve sign is considered in the above equation. The condition for obtaining the steady state solution is that the sum of the frequencies of excitation and harmonics must be equal to that of the oscillations.

The perturbation method can be used over here as the frequency of excitation plus harmonics always lies in close proximity of natural frequency of oscillations i.e. $\omega_n^2 = k/m$. Thus the perturbation method gives the frequency of oscillation in terms of ω_n by a relation given below :

$$\omega^2 = \omega_n^2 + \lambda u_1 + \lambda^2 u_2 \qquad ...(9.7.2)$$

[From eqn. (9.5.33)]

The usage of perturbation method to find the solution near the conditions of resonance demands the combination of λ with forcing

function (the right side of the equation) and hence seek the solution for the equation given below :

$$\ddot{x} + \omega_n^2 x + \lambda x^3 = \lambda F\left\{\begin{array}{l}\sin\omega t\\ \cos\omega t\end{array}\right\} \qquad ...(9.7.3)$$

From eqn. (9.5.7.2), we get x in a series of successively increasing powers of λ

$$x = x_0 + \lambda x_1 + \lambda^2 x_2 \qquad ...(9.7.4)$$

We consider the equation,

$$\ddot{x} + \omega_n^2 x - \lambda x^3 = \lambda F\cos\omega t \qquad ...(9.7.5)$$

for investigation corresponding to softening spring.

Now taking the series of eqn. (9.7.2) and (9.7.4) upto two terms and substituting in eqn. (9.7.5), we get

$$\ddot{x}_0 + \mu\ddot{x}_1 + (\omega^2 - \lambda u_1)(x_0 + \lambda x_1)$$
$$-\lambda(x_0^3 + 3\lambda x_0^2 x_1 + \ldots) = \lambda F\cos\omega t \qquad ...(9.7.6)$$

The equations obtained from the coefficients of λ^0 and λ^1 are then given as

$$\left.\begin{array}{l}\ddot{x}_0 + \omega^2 x_0 = 0\\ \ddot{x}_1 + \omega^2 x_1 = \mu_1 x_0 + x_0^3 + F\cos\omega t\end{array}\right\} \qquad ...(9.7.7)$$

Let initial conditions be

$$\left.\begin{array}{l}x_0(0) = X\\ \dot{x}_0(0) = 0\end{array}\right\} \text{ at } t = 0$$

The generating solution is therefore given by

$$x_0 = X\cos\omega t \qquad ...(9.7.8)$$

Substituting this into the second eqn. of (9.7.7) for x_1, leads to

$$\ddot{x}_1 + \omega^2 x_1 = \left(\mu_1 + \frac{3}{4}X^2 + \frac{F}{X}\right)X\cos\omega t + \frac{1}{4}X\cos^3 3\omega t \qquad ...(9.7.9)$$

We must again impose the condition

$$\left(\mu_1 + \frac{3}{4}X^2 + \frac{F}{X}\right) = 0 \qquad ...(9.7.10)$$

to suppress the secular term in solution for x_1 since it causes resonance. It can be seen seen from eqn. (9.7.9) that the secular term causes resonance in the system which is undesirable. This term is unbounded and approaches infinity as time $t \to \infty$. Thus to obtain an exact bounded solution, this term should be equated to zero.

Thus the solution of equation (9.7.9) becomes

$$\ddot{x}_1 + \omega^2 x_1 = \frac{1}{4}X^3\cos 3\omega t \qquad ...(9.7.11)$$

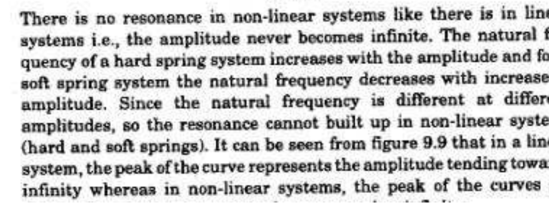
The solution for initial conditions $x_1(0) = \dot{x}_1(0) = 0$ is

$$x_1 = A_1\cos\omega t + A_2\sin\omega t + A_3\cos 3\omega t$$
$$= \frac{X^3}{32\,\omega^2}(\cos\omega t - \cos 3\omega t) \qquad ...(9.7.12)$$

Substituting this eqn. into eqn. (9.7.11), the first order solution to the forced vibration problem in the vicinity of resonance is

$$x = X\cos\omega t + \frac{\lambda X^3}{32\,\omega^2}(\cos\omega t - \cos 3\omega t) \qquad ...(9.7.13)$$

The variation of amplitude of vibration against frequency as governed by eqn. (9.7.13) is shown in figure 9.9 (*a*, *b*, *c*) for a linear, hard and soft spring systems respectively. This equation is cubic in X and therefore for any value of ω, there are in general three values of X, one is always real, the other two may be real or imaginary (complex conjugate). This is shown in figure (*b*) and (*c*) for non-linear systems.



(*a*) Linear (*b*) Hard (*c*) Soft

Fig. 9.9.

There is no resonance in non-linear systems like there is in linear systems i.e., the amplitude never becomes infinite. The natural frequency of a hard spring system increases with the amplitude and for a soft spring system the natural frequency decreases with increase in amplitude. Since the natural frequency is different at different amplitudes, so the resonance cannot built up in non-linear systems (hard and soft springs). It can be seen from figure 9.9 that in a linear system, the peak of the curve represents the amplitude tending towards infinity whereas in non-linear systems, the peak of the curves get

9.8 AMPLITUDE FREQUENCY CURVES

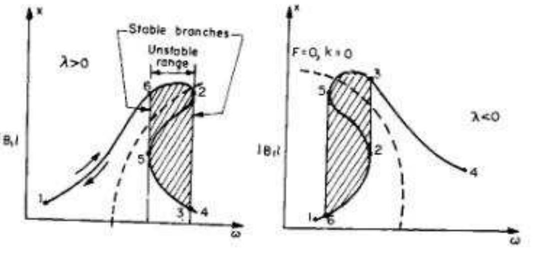
It was shown in section 9.5 that frequency in case of non-linear systems is a function of amplitude. Let us consider the equation

$$\ddot{x} + c\dot{x} + \omega_0^2 x \pm \lambda x^3 = F \cos(\omega t + \phi) \qquad ...(9.8.1)$$

Let us assume the solution to be

$$x = B_1 \sin \omega t + B_3 \sin \omega t \qquad \text{[From eqn. 9.5.5.5]}$$

The solution to eqn. (9.5.5.8) has been plotted in figure 9.10.



(a) Hard Spring (b) Soft Spring

Fig. 9.10.

When frequency is increased slowly, the response follows the curve along 1, 2, 3, 4. There is a jump from 2 to 3. This is called the Jump Phenomenon. On slowly decreasing the frequency the response follows the curve along 4, 5, 6, 1 jumping from 5 to 6.

The shaded regions of the curves show two amplitudes of vibrations for a given excitation frequency. This phenomenon is a characteristic of non-linear systems.

It is worthwhile to note that the portion of the response curve 2, 3, is never traced when frequency of excitation is gradually increased and the portion of curve 5, 2, 6 is never traced when the frequency of excitation is gradually decreased from a large value.

A stability analysis shows the middle branch to be stable. The shaded region is unstable, its width being dependent on a number of factors, such as amount of damping, rate of change of the exciting frequency; etc.

9.9 EXCITATION PROPORTIONAL TO VELOCITY

The equation of motion of a single degree of freedom system which is considered here for analysis is given as

$$m\ddot{x} + c\dot{x} + kx = F\dot{x} \qquad ...(9.9.1)$$

or

$$\ddot{x} + \left(\frac{c-F}{m}\right)\dot{x} + \frac{k}{m}x = 0 \qquad ...(9.9.2)$$

If $F > c$, damping will be negative

Assuming the solution to be $x = e^{st}$, we get

$$s^2 + \left(\frac{c-F}{m}\right)s + \frac{k}{m} = 0 \qquad ...(9.9.3)$$

$$\therefore \quad s = -\left(\frac{c-F}{2m}\right) \pm i\sqrt{\frac{k}{m} - \left(\frac{c-F}{2m}\right)^2} \qquad ...(9.9.4)$$

When $F > c$, diverging oscillation is obtained with frequency

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{c-F}{2m}\right)^2} \qquad ...(9.9.5)$$

If $\quad \frac{k}{m} > \left(\frac{c-F}{2m}\right)^2$

Non-oscillating diverging motion results from

$$\frac{k}{m} < \left(\frac{c-F}{2m}\right)^2$$

Negative Damping results in unstability of systems.

From eqn. (9.9.5), it can be seen that the velocity ω is always proportional to the excitation F. As we go on increasing the excitation, the term $\left(\frac{c-F}{2m}\right)^2$ goes on decreasing and hence the velocity also increases.

9.10 SUBHARMONIC AND SUPERHARMONIC RESONANCE

There is a possibility of resonance occurring at a frequency which is some multiple of excitation force frequency in case of linear systems. Converse to this can happen in Non-linear systems i.e., resonance can occur at a frequency which is a fraction of excitation frequency. This is called Subharmonic Resonance.

In subharmonic response, oscillations have frequency (ω_n) which is related to forcing or excitation frequency as

$$\omega_n = \frac{\omega}{n} \qquad ...(9.10.1)$$

where n = integer i.e., 2, 3, 4

Non-linear systems also exhibit superharmonic response where oscillations have frequencies which are multiples of excitation frequency i.e.,

$$\omega_n = n\omega \qquad ...(9.10.2)$$

where $\quad n = 2, 3, 4....$

In the above expressions, the value of integer n can not be taken as unity to avoid the occurance of resonance.

Example 1. *A non-linear spring for a single degree of freedom system is given by $k(x) = 10x + 2000x^3$. c for viscous damping is 1.5 kg sec/cm. A harmonic force 5 kg amplitude acts on the mass = 1 kg. Find the steady state response using the Direct Integration Method.*

Solution. The differential equation of motion is of the form

$$M\ddot{x} + c\dot{x} + k(x)\,x = F_0 \cos \omega t$$

Here $\quad m = 1;\ c = 1.5$ kg sec/cm

$$k(x) = 10x + 2000x^3,\ F_0 = 5$$

$$\therefore \quad \ddot{x} + 1.5\dot{x} + (10x + 2000x^3)x = 5\cos\omega t$$

Let $x_1 = A\cos\omega t$ be the first approximate steady state solution.

Then

$$\ddot{x}_2 = -1.5A\omega\sin\omega t - 10A^2\cos^2\omega t - 2000A^4\cos^4\omega t + 5\cos\omega t$$

Now we know that $\cos^2\omega t = \frac{1+\cos 2\omega t}{2}$

and

$$\cos^4\omega t = \left(\frac{1+\cos 2\omega t}{2}\right)\left(\frac{1+\cos 2\omega t}{2}\right)$$

$$= \frac{1}{4}\left\{1 + 2\cos 2\omega t + \frac{1+\cos 4\omega t}{2}\right\}$$

$$= \frac{1}{3}(4\cos 2\omega t + \cos 4\omega t + 3)$$

Substituting this result, we get

$$\ddot{x}_2 = -1.5A\omega\sin\omega t - 10A^2\left[\frac{1+\cos 2\omega t}{2}\right]$$

$$- 2000A^4[4\cos 2\omega t + \cos 4\omega t + 3] + 5\cos\omega t$$

$$= 5\cos\omega t - 1.5A\omega\sin\omega t - (5A^2 + 8000A^4)\cos 2\omega t$$

$$- 2000A^4\cos 4\omega t - (5A^2 + 6000A^4)$$

Integrating the above we get,

$$\dot{x}_2 = \frac{5\sin\omega t}{\omega} + \frac{1.5A\omega\cos\omega t}{\omega} - \frac{(5A^2 + 8000A^4)\sin 2\omega t}{2\omega}$$

$$- \frac{2000A^4\sin 4\omega t}{4\omega} - (5A^2 + 6000A^4)t + C_1$$

Integrating once again, we get

$$x_2 = -\frac{5\cos\omega t}{\omega^2} + \frac{1.5A\sin\omega t}{\omega} + \frac{(5A^2 + 8000A^4)\cos 2\omega t}{4\omega^2}$$

$$+ \frac{2000A^4\cos 4\omega t}{16\omega^2} - (5A^2 + 6000A^4)\frac{t^2}{2} + C_1 t + C_2$$

If the constants of integration C_1 and $C_2 = 0$ so that the motion x_1 and x_2 are periodic,

$$\therefore \quad x_2 = \frac{-5}{\omega^2}\cos\omega t + \frac{1.5A}{\omega}\sin\omega t + \frac{(2.5A^2 + 4000A^4)}{2\omega^2}\cos 2\omega t$$

$$+ \frac{125A^4}{\omega^2}\cos 4\omega t - (2.5A^2 + 3000A^4)t^2$$

This is the second approximate steady state vibration.

Problems

1. A non-linear spring for a single degree of freedom system is given by

$$k(x) = 100x + 1000x^3$$

C for viscous damping is 3 kg sec/cm. A harmonic force of 16 kg amplitude acts on the mass. Find the steady state response.

2. Find the singular points for the following differential equation and explain whether they are stable or unstable:

$$m\ddot{x} + \lambda x + \mu x^3 = 0 \quad \lambda < 0, \mu < 0$$

3. The supporting end of a simple pendulum is given a motion as shown in Fig. 9.1 P. Show that the equation of motion is

$$\ddot{\theta} + \left(\frac{g}{l} - \frac{\omega^2}{l}y_0\cos 2\omega t\right)\sin\theta = 0$$

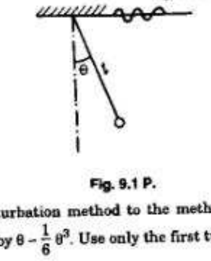


Fig. 9.1 P.

4. Apply the perturbation method to the method simple pendulum with $\sin\theta$ replaced by $\theta - \frac{1}{6}\theta^3$. Use only the first two terms of the series for x and ω.

5. The cord of a simple pendulum is wrapped around a fixed cylinder of radius R such that its length is l when in the vertical position as shown in figure 9.2 P. Find the differential equation of motion.

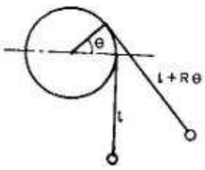


Fig. 9.2 P.

6. Using the iteration method, solve for the period of the linear equation $\ddot{x} + \omega_n^2 x = 0$ with initial conditions $x(0) = A$ and $\dot{x}(0) = 0$.
7. Find the exact solution of the non-linear pendulum equation

$$\ddot{\theta} + \omega_0^2\left(\theta - \frac{\theta^3}{6}\right) = 0$$

where $\dot{\theta} = 0$ when $\theta = \theta_0$,

θ_0 = maximum angular displacement.

8. Find the trajectories of the system given by the equations :

$$\dot{x} = 2x + y \text{ and } \dot{y} = -3x - 2y$$

9. Find the equation of motion of the mass shown in figure 9.3P.

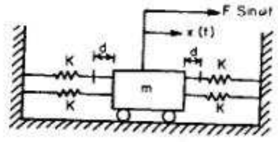


Fig. 9.3 P.

10. Determine the phase plane of a single degree of freedom oscillator

$$\ddot{x} + \omega^2 x = 0$$

11. Find the trajectories of a simple harmonic oscillator.

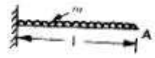
APPENDIX – A

Deflection Formulae

Cantilever with a point load at the free end

$$y_A = \frac{Wl^3}{3EI}$$

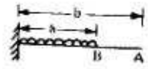
Cantilever with uniformly distributed load



$$y_A = \omega l^4/8EI$$

$$= Wl^3/8EI$$

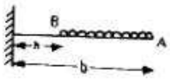
Cantilever partially loaded with a uniformly distributed load



$$y_A = \frac{7\,\omega l^4}{384\,EI} \text{ at } a = b/2$$

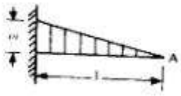
$$y_B = \frac{\omega a^4}{8EI}$$

$$y_A = \frac{\omega a^4}{8EI} + \frac{\omega a^3}{6EI}(b-a)$$



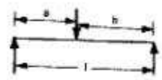
$$y_A = \left[\frac{\omega b^4}{8EI}\right] - \left[\frac{\omega (b-a)^4}{8EI} + \frac{\omega a^3 (b-a)}{6EI}\right]$$

Cantilever with a gradually varying load



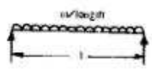
$$y_A = \frac{\omega l^4}{30\,EI}$$

Simply supported beam with a central point load ($a = b$)



$$y_A = \frac{Wa^2 b^2}{3EIl}$$

Simply supported beam with a uniformly distributed load



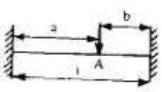
$$y_A = \frac{5}{384}\frac{\omega l^4}{EI}$$

Fixed fixed beam with a centre d

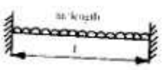
$$y_A = \frac{Wl^3}{192EI}$$

Fixed beam with an eccentric it load

$$y_A = \frac{W a^3 b^3}{3l^3\,EI}$$



Fixed beam with a uniformly dis-uted load



$$y_A = \frac{\omega l^4}{384\,EI}$$

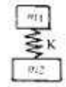
APPENDIX – B

Natural Frequencies of Some Standard Systems

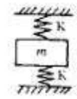
Spring mass system

$$\omega_n = \sqrt{k/m}$$

$$\omega_n = \sqrt{\frac{k\,(m_1 + m_2)}{m_1 m_2}}$$

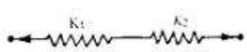


$$\omega_n = \sqrt{\frac{2k}{m}}$$

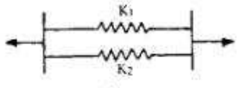


Resultant Stiffness of Springs

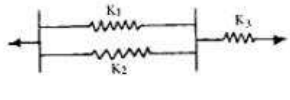
$$k_e = \frac{k_1 k_2}{k_1 + k_2}$$



$$k_e = k_1 + k_2$$

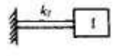


$$k_e = \frac{(k_1 + k_2)\,k_3}{k_1 + k_2 + k_3}$$

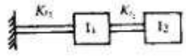


Rotor systems

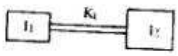
$$\omega_n = \sqrt{\frac{k_t}{I}}$$



$$\omega_n = \sqrt{\frac{1}{2}\left[\frac{k_{t_1}}{I_1} + \frac{k_{t_2}}{I_2}\left(1 + \frac{I_2}{I_1}\right) \pm \sqrt{\left[\frac{k_{t_1}}{I_1} + \frac{k_{t_2}}{I_2}\left(1 + \frac{I_2}{I_1}\right)\right]^2 - \frac{4\,k_{t_1} k_{t_2}}{I_1 I_2}}\right]}$$

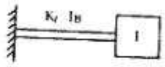


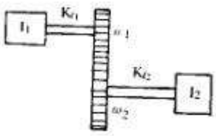
$$\omega_n = \sqrt{\frac{k_t\,(I_1 + I_2)}{I_1 I_2}}$$



Moment of inertia of shaft is included

$$\omega_n = \sqrt{\frac{k_t}{I + \frac{I_s}{3}}}$$





$$\omega_n = \sqrt{\frac{k_{t_1} k_{t_2}\,(I_1 + n^2 I_2)}{I_1 I_2\,(n^2 k_2 + k_1)}}$$

$$n = \frac{\omega_2}{\omega_1} = \text{speed ratio}$$

APPENDIX - C

Matrices

Matrix Definition

A set of numbers (real or complex) presented in the form of a rectangular array having m rows and n columns is called an $m \times n$ matrix.

Generally $m \times n$ matrix is usually written as

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & \dots & a_{mn} \end{bmatrix}$$

In the compact form it can be shown that

$$[A] = [a_{ij}]$$

where $i = 1, 2, 3 \ldots m$

and $j = 1, 2, 3 \ldots n$

The suffix i represents the number of rows and suffix j represents the number of column in the matrix.

Matrix $m \times n$ has m rows and n columns.

In particular $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ is a (2×3) matrix.

Square Matrix

A matrix in which number of rows is equal to number of columns is called a square matrix.

$$[A] = [a_{ij}]$$

In the above matrix $i = j$.

Also if $m \times n$ type matrix shows that $m = n$, it is a square matrix.

Example. The matrix

$$[A] = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 2 & 1 & 0 \end{bmatrix}_{3 \times 3}$$

is a square matrix of order 3. The elements 1, 1, 0 form the diagonal of the matrix.

Unit Matrix

A square matrix whose diagonal elements are 1 is called a unit matrix or identity matrix. It is denoted by I.

Example.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is a unit matrix of order 4.

Diagonal Matrix

It is a square matrix having only the diagonal elements and remaining all other elements being zero.

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Column Matrix

It is matrix which has single column. It is $(m \times 1)$ matrix.

Example

$$[A] = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_m \end{Bmatrix}$$

r $\begin{Bmatrix} 3 \\ 2 \\ 3 \end{Bmatrix}$ is a column matrix of order (3×1)

Row Matrix

It is a $(1 \times n)$ matrix. It has single row.

$$[A] = [a_1, a_2, a_3 \ldots\ldots a_n]$$

r [1 3 6 4] is a row matrix of order (1×4)

Addition of Matrices

Let A and B be two matrices of the same type $m \times n$. Then their ım is defined to be the matrix of the type $m \times n$ obtained by adding the ›rresponding elements of A and B.

Example

$$[A] = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}_{3 \times 3}$$

$$[B] = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}_{3 \times 3}$$

$$[A] + [B] = \begin{bmatrix} 1+1 & 3+2 & 0+3 \\ 2+3 & 1+2 & 3+3 \\ 1+1 & 1+4 & 1+5 \end{bmatrix}_{3 \times 3}$$

$$= \begin{bmatrix} 2 & 5 & 3 \\ 5 & 3 & 6 \\ 2 & 5 & 6 \end{bmatrix}_{3 \times 3}$$

Subtraction of Matrices

Let A and B be two matrices of the same type $m \times n$. Then their substraction is defined to be the matrix of the type $m \times n$ obtained by subtracting the corresponding elements of A and B.

Example

$$[A] = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}_{3 \times 3}$$

$$[B] = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}_{3 \times 3}$$

$$[A] - [B] = \begin{bmatrix} 1-1 & 3-2 & 0-3 \\ 2-3 & 1-2 & 3-3 \\ 1-1 & 1-4 & 1-5 \end{bmatrix}_{3 \times 3}$$

$$= \begin{bmatrix} 0 & 1 & -3 \\ -1 & -1 & 0 \\ 0 & -3 & -4 \end{bmatrix}_{3 \times 3}$$

Matrix Multiplication

The product of two matrices [A] and [B] gives another matrix [C].

$$[A][B] = [C]$$

Example.

$$[B] = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 1 \end{bmatrix} \qquad [A] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 3 & 0 \end{bmatrix}$$

$$[A]\,[B] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0\times1+3\times1+0\times4 & 0\times2+1\times0+0\times1 \\ 0\times1+2\times3+4\times1 & 0\times2+0\times2+1\times1 \\ 2\times1+3\times3+0\times4 & 2\times2+3\times0+0\times1 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 3 & 0 \\ 10 & 1 \\ 11 & 4 \end{bmatrix}_{3 \times 2}$$

$$[A]\,[B] \neq [B]\,[A]$$

Transpose of a Matrix

A matrix $(n \times m)$ is obtained by changing its columns into rows and its rows into columns. It is denoted by A' or A^T.

Example $[A] = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$[A]^T = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 4 & 8 \\ 5 & 6 & 9 \end{bmatrix}$$

Cofactor

If a matrix $[A]$ is given, its cofactors can be determined as

Say $\quad [A] = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$

Cofactors of the elements of the first row of $[A]$

$$\begin{vmatrix} 3 & 1 \\ 5 & -2 \end{vmatrix}, \; -\begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix}, \; \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix}$$

$$3 \times -2 - 5, \quad 2 \times 2, \quad 2 \times 5$$

$$-11, \quad 4, \quad 10$$

Cofactors of the elements of the second row of $[A]$ are

$$-\begin{vmatrix} -2 & -1 \\ 5 & -2 \end{vmatrix}, \; \begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix}, \; -\begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix}$$

$$-9, \quad -2, \quad -5$$

The cofactors of the elements of the third row of $[A]$ are

$$\begin{vmatrix} -2 & -1 \\ 3 & 1 \end{vmatrix}, \; -\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}, \; \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}$$

$$1, \quad -3, \quad 7$$

Adjoint Matrix

The adjoint of a matrix A is the transpose of the matrix formed by factors of A. It is indicated by adj.$[A]$.

Inverse Matrix

The inverse of matrix A can be obtained from the following relation

$$[A]^{-1} = \frac{\text{adj } [A]}{|A|}$$

$|A|$ should be non-zero.

Eigenvalues and Eigenvectors

Let $A = [a_{ij}]_{n \times n}$ be a square matrix of order n. The characteristic ion is $|A - \lambda I| = 0$.

This equation will be of degree n in λ. So it will have n roots. These s will give us the eigenvalues of the matrix A. If λ_1 is an value of A, then the corresponding eigenvalues of A will be given non-zero vectors

$$X = [x_1, x_2 \ldots x_n]$$

satisfying the equation

$$AX = \lambda_1 X \text{ or } (A - \lambda_1 I)X = 0$$

APPENDIX - D

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{x} \, dx = \log x + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \operatorname{cosec}^2 x \, dx = -\cot x + C$$

$$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$$

$$\int x \cos x \, dx = -x \sin x - \cos x$$

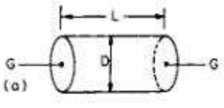
$$\int x \cos nx \, dx = -\frac{x}{n} \sin nx - \frac{\cos nx}{n^2}$$

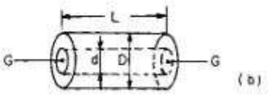
$$\int x \sin nx \, dx = -\frac{x}{n} \cos nx + \frac{\sin nx}{n^2}$$

$$\int uv \, dx = u \int v \, dx - \int \frac{du}{dx} \left(\int v \, dx \right) dx$$

APPENDIX - E

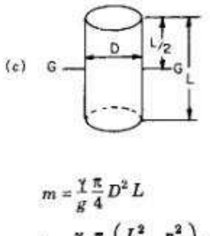
Mass Moment of Inertia of Some Typical Solid bodies.

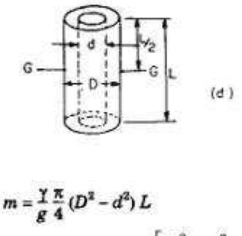




$$m = \frac{\gamma}{g} \frac{\pi}{4} D^2 L \qquad m = \frac{\gamma}{g} \frac{\pi}{4} (D^2 - d^2) L$$

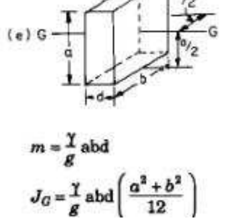
$$J_G = \frac{\gamma}{g} \frac{\pi}{32} D^4 L \qquad J_G = \frac{\gamma}{g} \frac{\pi}{32} (D^4 - d^4) L$$

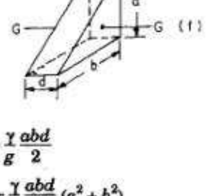




$$m = \frac{\gamma}{g} \frac{\pi}{4} D^2 L \qquad m = \frac{\gamma}{g} \frac{\pi}{4} (D^2 - d^2) L$$

$$J_G = \frac{\gamma}{g} \frac{\pi}{16} \left(\frac{L^2}{3} + \frac{n^2}{4} \right) LD^2 \qquad J_G = \frac{\gamma}{g} \frac{\pi}{16} (D^2 - d^2) \left[\frac{L^2}{3} + \frac{D^2 + d^2}{4} \right] L$$

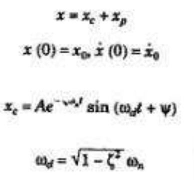
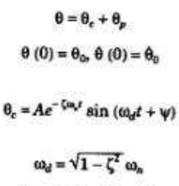




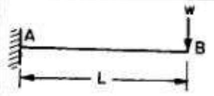
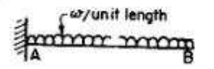
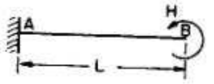
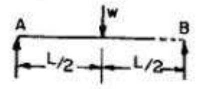
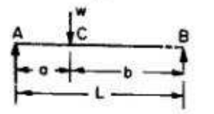
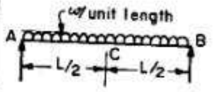
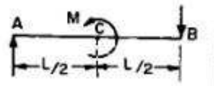
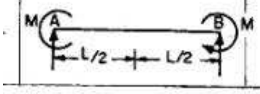
$$m = \frac{\gamma}{g} \text{abd} \qquad m = \frac{\gamma}{g} \frac{abd}{2}$$

$$J_G = \frac{\gamma}{g} \text{abd} \left(\frac{a^2 + b^2}{12} \right) \qquad J_G = \frac{\gamma}{g} \frac{abd}{36} (a^2 + b^2)$$

APPENDIX - F

ITEM	RECTILINEAR SYSTEM	ROTATIONAL SYSTEM
System		
Equation of motion	$m\ddot{x} + c\dot{x} + kx = F \sin \omega t$	$J\ddot{\theta} + c.\dot{\theta} + k_t \theta = T \sin \omega t$
Response	$x = x_c + x_p$	$\theta = \theta_c + \theta_p$
Initial conditions	$x(0) = x_0, \dot{x}(0) = \dot{x}_0$	$\theta(0) = \theta_0, \dot{\theta}(0) = \dot{\theta}_0$
Transient response	$x_c = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \psi)$	$\theta_c = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \psi)$
	$\omega_d = \sqrt{1-\zeta^2}\, \omega_n$	$\omega_d = \sqrt{1-\zeta^2}\, \omega_n$
Steady-state response	$x_p = X \sin(\omega t - \phi)$ $X = \frac{F}{k} k$ $\phi = \tan^{-1} \frac{2\zeta r}{1-r^2}$ $r = \frac{\omega}{\omega_n}$	$\theta_p = \Theta \sin(\omega t - \phi)$ $\Theta = \frac{T}{k_t} k$ $\phi = \tan^{-1} \frac{2\zeta r}{1-r^2}$ $r = \frac{\omega}{\omega_n}$

APPENDIX - G

S. No.	Beam and Loading	Slope	Deflection
1.		$\frac{WL^2}{2EI}$ at B	$\frac{WL^3}{3EI}$ at B
2.		$\frac{wL^3}{6EI}$ at B	$\frac{wL^4}{8EI}$ at B
3.		$\frac{ML}{EI}$ at B	$\frac{ML^2}{2EI}$ at B
		$\frac{WL^2}{16EI}$ at A, B	$\frac{WL^3}{48EI}$ at C
		$\frac{Wab\,(L+b)}{6EIL}$ at A $\frac{Wab\,(L+b)}{6EIL}$ at B $\frac{Wab\,(L-C)}{3EIL}$ at C	$\frac{Wa^2 b^2}{3EIL}$ at C
		$\frac{wL^3}{24EI}$ at A, B	$\frac{5wL^4}{384\,EI}$ at C
		$\frac{ML}{24EI}$ at A, B $\frac{ML}{12EI}$ at C	0 at C
		$\frac{ML}{2EI}$ at A, B	$\frac{ML^2}{8EI}$ at C

APPENDIX - H
Laplace Transforms

$f(t)$	$(f)\ \pounds = \int_0^{\infty} f(t)\, e^{-st}\, dt$
$\delta(t)$	1
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{S^{n+1}}$, $n = 0, 1, \ldots$
	$\frac{\Gamma(n+1)}{S^{n+1}}$, any $n > 0$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, $n = 0, 1, \ldots$
	$\frac{\Gamma(n+1)}{(s-a)^{n+1}}$, any $n > -1$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$t^n \sin at, n > -1$	$\frac{\Gamma(n+1)}{2i\,(s^2+a^2)^{n+1}}\left[(s+ai)^{n+1} - (s-ai)^{n+1}\right]$
$t^n \cos at, n > -1$	$\frac{\Gamma(n+1)}{2\,(s^2+a^2)^{n+1}}\left[(s+ai)^{n+1} + (s-ai)^{n+1}\right]$
$\sin^2 t$	$\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$
$\cos^2 t$	$\frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)$
$\sin at \sin bt$	$\frac{2abs}{[s^2+(a+b)]\,[s^2+(a-b)]}$
$H(r-c), c > 0$	$\frac{e^{-sc}}{s}$
$H(t) - H(t-c), C > 0$	$\frac{1-e^{-cs}}{s}$
$e^{at} H(t-c), c \pounds 0$	$e^{-(s-0)}$

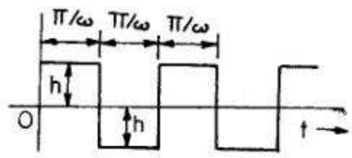
Inverse Laplace Transforms

$\pounds f(t)$	$f(t), t > 0$
$\frac{1}{(s+a)^4}$	$t\, e^{-at}$
$\frac{1}{(s+a)^2}$	$(1-at)\, e^{-at}$
$\frac{1}{(s+a)_n}$, $n = 1, 2, 3, \ldots$	$\frac{1}{(n-1)!} t^{n-1} e^{-at}$
$\frac{1}{(s+a)(s+b)}$	$\frac{1}{(b-a)}(e^{-at} - e^{-bt})$
$\frac{s}{(s+b)(s+b)}$	$\frac{1}{(a-b)}(ae^{-at} - be^{-bt})$
$\frac{1}{s(s^2+a^2)}$	$\frac{1}{a^2}(1 - \cot at)$
$\frac{1}{s^2(s^2+a^2)}$	$\frac{1}{a^3}(at - \sin at)$
$\frac{1}{(s^2+a^2)^2}$	$\frac{1}{2a^2(\sin at - at \cos at)}$
$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a} \sin at$
$\frac{s^2}{(s^2+a^2)^2}$	$\frac{1}{2a}(\sin at + at \cos at)$
$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t \cos at$
e^{-as}	$\delta(t-a)$
$\frac{e^{-as}}{s}$	$H(t-a)$
$\frac{1}{(s+a)^3}$	$\frac{t^2}{2} e^{-at}$
$\frac{1}{(s+a)(s+b)^2}$	$\frac{e^{-bt}}{(b-a)^2} + \left[\frac{t}{b-a} - \frac{1}{(b-a)^2}\right] e^{-at}$
$\frac{e^{-as}}{(s-b)^k}$, $a > 0, k = 1, 2, \ldots$	$\frac{(t-a)^{k-1} e^{b(t-a)} H(t-a)}{(k-1)!}$
$\frac{e^{-as}}{(s-b)^k}$, $a > 0, k = 1, 2, \ldots$	$\frac{(t-a)^{k-1} e^{b(t-a)} H(t-a)}{(k-1)!}$
$\frac{1}{as^2+bs+c}$, $(b^2 - 4ac) < 0$	$\left(\frac{2e^{-\alpha} t}{\beta}\right) \sin\left(\frac{\beta}{2a}\right) t$
$a \neq 0$	$\alpha = \frac{b}{\ }$, $\beta = \sqrt{4ac - b^2}$

APPENDIX - I

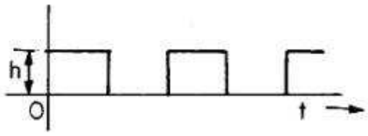
(i) $f(\omega t) = \frac{4h}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin n\omega t$

(i)



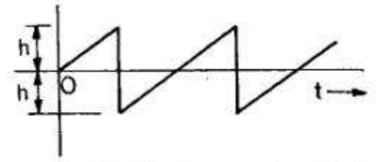
(ii) $f(\omega t) = \frac{h}{2} + \frac{2h}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin n\omega t$

(ii)



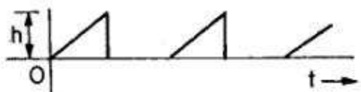
(iii) $f(\omega t) = \frac{2h}{\pi} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\omega t$

(iii)

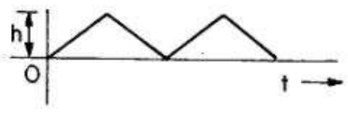


(iv) $f(\omega t) = \frac{h}{4} - \frac{2h}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{\pi^2} \cos n\omega t + \frac{h}{\pi} \sum_{n=1,2,3}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\omega$

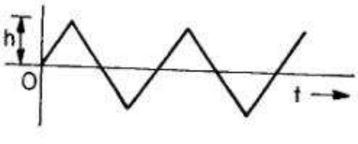
(iv)



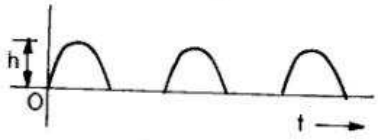
(v) $f(\omega t) = \frac{h}{2} - \frac{4h}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{\pi^2}\cos n\omega t$

(v) 

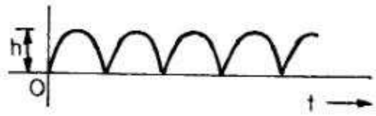
(vi) $f(\omega t) = \frac{8h}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{(-1)}{\pi^2}^{\frac{n-1}{2}}\sin n\omega t$

(vi) 

(vii) $f(\omega t) = \frac{4}{\pi} + \frac{h}{2}\sin \omega t - \frac{2h}{\pi}\sum_{n=2,4,6}^{\infty}\frac{1}{n^2-1}\cos n\omega t$

(vii) 

(viii) $f(\omega t) = \frac{2h}{\pi}\sum_{n=2,4,6}^{\infty}\frac{1}{n^2-1}\cos n\omega t$

(viii) 

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